Notes 12 – Lines and planes in space

The study of one and more lines and planes in space provides geometrical illustrations of the theory of linear systems.

L12.1 Parametric form. Recall the parametric form of the equation of a line parallel to $\mathbf{p} = (p, q, r)$ and passing through $\mathbf{v}_0 = (x_0, y_0, z_0)$. It is

$$\mathbf{v} = \mathbf{v}_0 + t\mathbf{p},$$

or

$$(x, y, z) = (x_0 + tp,\ y_0 + tq,\ z_0 + tr).$$

Each value of t gives us a point on the line; when $t = 0$ we obtain the assigned point.

Another way to specify a line is to provide two points on it, say P_0, P_1 with respective position vectors

$$\mathbf{v}_0 = (x_0, y_0, z_0), \qquad \mathbf{v}_1 = (x_1, y_1, z_1).$$

In this case, we recover $\mathbf{p}$ by subtraction:

$$\mathbf{p} = \mathbf{v}_1 - \mathbf{v}_0 = (x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0);$$

we can write the equation as

$$\mathbf{v} = \mathbf{v}_0 + t(\mathbf{v}_1 - \mathbf{v}_0) = (1 - t)\mathbf{v}_0 + t\mathbf{v}_1.$$

When $t = 0$ we obtain $\mathbf{v}_0$, and when $t = 1$ we obtain $\mathbf{v}_1$. Points on the segment P_0P_1 are those for which $0 \leqslant t \leqslant 1$.

Exercise. Find the point of intersection (if any) of the two lines described parametrically by

$$(x, y, z) = (1{+}3t,\ 2{+}5t,\ 3 + 8t), \qquad (x, y, z) = (3{-}t,\ 5{-}2t,\ 8{-}3t).$$

In a problem like this, *it is essential to use different symbols for the respective parameters.* So our first step is to re-write the first line as $(x, y, z) = (1{+}3s,\ 2{+}5s,\ 3 + 8s)$. Now we can find a potential point of intersection by equating the respective expressions:

$$(1{+}3s,\ 2{+}5s,\ 3 + 8s) = (3{-}t,\ 5{-}2t,\ 8{-}3t), \tag{1}$$

and solve for either s or t. (There is no reason for such a solution to have $s = t$.) This gives a system of three equations in two unknowns, which when rearranged becomes

$$\left\{\begin{array}{rcl} 3s + t &=& 2\\ 5s + 2t &=& 3\\ 8s + 3t &=& 5, \end{array}\right. \qquad \text{with augmented matrix} \quad A^+ = \left(\begin{array}{cc|c} 3 & 1 & 2\\ 5 & 2 & 3\\ 8 & 3 & 5 \end{array}\right).$$

Since $r(A) = r(A^+) = 2$, there is a solution. Indeed,

$$A^+ \sim \left(\begin{array}{cc|c} 3 & 1 & 2\\ 5 & 2 & 3\\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 1 & 2\\ 0 & \frac{1}{3} & -\frac{1}{3}\\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} \boxed{1} & 0 & 1\\ 0 & \boxed{1} & -1\\ 0 & 0 & 0 \end{array}\right).$$

giving $s = 1$ and $t = -1$ (with hindsight, an obvious solution!). Substituting back into (1), the point of intersection is $(x, y, z) = (4, 7, 11)$.

L12.2 Configuration of two lines. Further to the last example, if we are given two distinct lines in space, there are three mutually exclusive possibilities:

(i) the lines are parallel,

(ii) the lines intersect in a single point,

(iii) the lines are not parallel and do not intersect.

In the last case the lines are said to be *skew* (e sghembe in Italian).

All three situations can be visualized by imagining lines extending the various edges of a cube (think of the intersections of the walls and ceiling of a room!). In case (ii), the two lines lie on a unique plane π. If $\mathbf{p}_1, \mathbf{p}_2$ are vectors parallel to the lines then this plane has normal vector

$$\mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2,$$

and contains the point of intersection.

Exercise. In the previous exercise, $\mathbf{p}_1 = (3,5,8)$ and $\mathbf{p}_2 = (-1,-2,-3)$. Thus

$$\mathbf{n} = -(3,5,8) \times (1,2,3) = (1,1,-1).$$

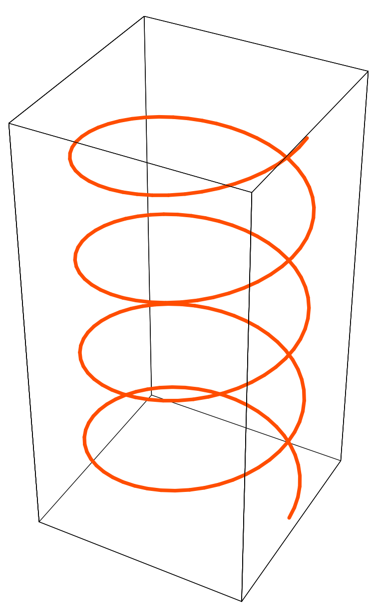
A point on π is $(4,7,11)$, so the plane's equation is

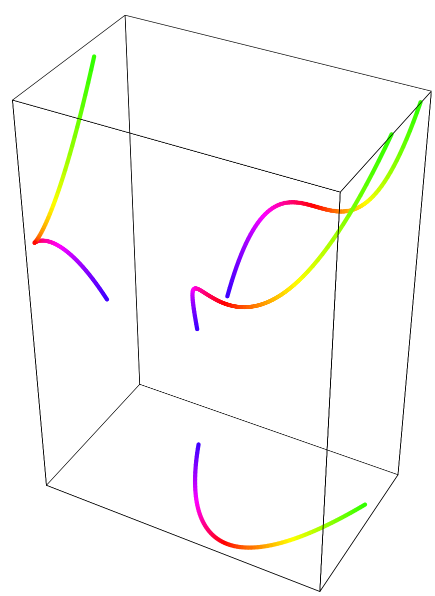
$$1(x-4) + 1(y-7) - 1(z-11) = 0 \quad \Rightarrow \quad x + y - z = 0.$$

L12.3 Space curves. The parametric equation of a straight line has the form

$$(x,y,z) = (f_1(t_1),\ f_2(t),\ f_3(t)),$$

where each f_i is a polynomial in t of degree at most 1. When we take more general functions, we obtain a curve in space. The study of such curves is not part of this course, but to illustrate the point we provide two celebrated examples.





The curve

$$(x, y, z) = (\cos t,\ \sin t,\ \frac{1}{2\pi}t)$$

defines a *helix.* As t increases from 0 to 2π, the point (x, y) describes a unit circle in the plane, since

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

But at the same time, the 'height' z increases from 0 to 1. See the left-hand figure.

The curve

$$(x, y, z) = (t,\ t^2,\ t^3)$$

is manufactured by taking $f_i(t) = t^i$ to be a monomial. It is called the *twisted cubic,* and is illustrated in the middle of the right-hand figure together with its three projections to the coordinate planes at the sides and bottom. These projections are the plane curves whose equations

$$z^2 = y^3 \quad \text{(left)}, \qquad z = x^3 \quad \text{(back)}, \qquad y = x^2 \quad \text{(bottom)}$$

are obtained by eliminating t. The third is a parabola, the second a plane cubic and the first is *'semi-parabola'* that has a *cusp* at the origin.

L12.4 Configurations of three planes. Consider the linear system

$$\left\{\begin{array}{rcl} a_1x + b_1y + c_1z &=& d_1 \\ a_2x + b_2y + c_2z &=& d_2 \\ a_3x + b_3y + c_3z &=& d_3. \end{array}\right. \tag{2}$$

Each equation determines a plane in $\mathbb{R}^3$. The rows of the coefficient matrix A are the three normal vectors

$$\mathbf{n}_i = (a_i, b_i, c_i), \qquad i = 1, 2, 3.$$

Let us assume that no two of the planes are parallel; equivelently no two of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are linearly dependent, and $r(A) \geqslant 2$.

There are three possibilities:

(i) $r(A) = 2 = r(A^+)$,

(ii) $r(A) = 2$, $r(A^+) = 3$,

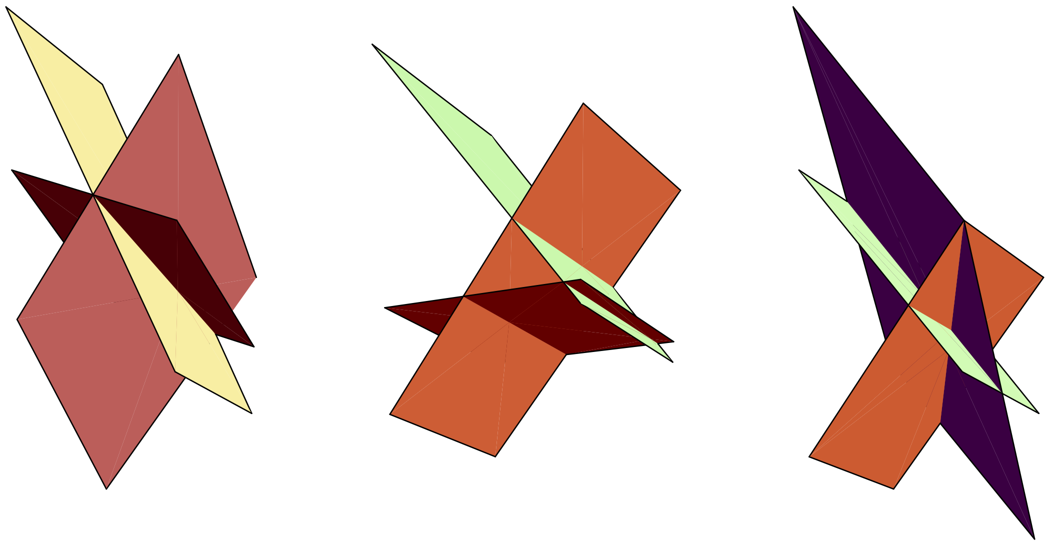
(iii) $r(A) = 3 = r(A^+)$,

which can be matched up with the three geometrical situations illustrated overleaf.

In case (i), the system (2) has ∞^1 solutions, and the three planes intersect in a common line.

For (ii), (2) is inconsistent, and *no* point lies on all three planes. Any two planes interesct in a line, but no two of these lines can intersect. Thus the three lines are parallel and the configuration incorporates a prism.

In (iii), there is a unique solution by (RC2). This corresponds to the three planes intersecting in a single point. Of course, this occurs for ther coordinate planes and axes, but in the general situation the planes will not be mutually orthogonal.



(i) (ii) (iii)

L12.5 Further exercises.

1. In the xy plane (so $z=0$),

(i) find the point P lying on both lines $x+2y=4$ e $\begin{cases} x=1-3t \\ y=2+2t; \end{cases}$,

(ii) write the equation of the line passing through P parallel to $3x-y=7$,

(iii) write the equation of the line passing through P and the point $(3,1,0)$.

2. Determine whether the lines $\begin{cases} x-y+z=1 \\ 2y-z=0 \end{cases}$ and $(x,y,z)=(1-t,\ 2t-1,\ -1+3t)$ are

(i) skew,

(ii) incident but not orthogonal,

(iii) orthogonal but not incident,

(iv) both lie in the plane $z=2y$.

3. Verify that the lines

$$\begin{cases} x-2y=1 \\ x-z=1 \end{cases} \quad \text{and} \quad \begin{cases} x+y=1 \\ y-z=0 \end{cases}$$

are incident, and find the plane that they both belong to.

4. Find the equations of planes that contain:

(i) the points $(1,2,1)$, $(1,3,-1)$, $(0,2,-2)$,

(ii) the point $(1,2,1)$ and the line $\begin{cases} x=1+2t \\ y=3+t \\ z=0 \end{cases}$.

In each case, write down a vector $\mathbf{n}$ orthogonal to the plane determined.