Notes 11 – The equation of a plane

A plane is the locus of points determined by a single linear equation, and is parametrized by two free variables. We begin with the second point of view.

L11.1 The equation of a plane. What is a plane exactly? It is a flat 2-dimensional surface. As a first example, consider the plane consisting of all points of 'height' $z = 1$. If we use column vectors and set $x = s,\ y = t$, we can say that an arbitrary point of the plane has position vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{1}$$

$$\mathbf{v} \;=\; \mathbf{v}_0 \;+ s\mathbf{p} \;+\; t\mathbf{q},$$

where $\mathbf{p} = \mathbf{e}_1$ and $\mathbf{q} = \mathbf{e}_2$. As the real parameters s, t vary we obtain the plane. We may call either version in (1) the *parametric equation* of the plane.

Setting $s = 0 = t$ in (1) gives us a particular point $(0, 0, 1)$ on the plane with position vector $\mathbf{v}_0$. The whole plane is described by adding to $\mathbf{v}_0$ linear combinations of two fixed vectors $\mathbf{p}$, $\mathbf{q}$ that are parallel to the plane. An abbreviated way of writing the *set* of elements in (1) (with $s, t \in \mathbb{R}$) is

$$\mathbf{v}_0 + \mathscr{L}\{\mathbf{p}, \mathbf{q}\}, \tag{2}$$

and we may think of the plane as a translate of the *subspace* generated by $\mathbf{p}$, $\mathbf{q}$ (which itself corresponds to the 'xy-plane' with equation $z = 0$.

A general plane will have the form (2) for arbitrary choices of $\mathbf{v}_0$, $\mathbf{p}$, $\mathbf{q}$, provided the last two are LI. But a more common description is provided by the

Proposition. *A plane is the set of points (x, y, z) satisfying an equation*

$$ax + by + cz = d, \tag{3}$$

where a, b, c, d are constants with a, b, c not all zero.

Proof. It is obvious (and a consequence of (RC2)) that (3) admits solutions. Write it in vector form as

$$\mathbf{v} \cdot \mathbf{n} = d, \tag{4}$$

where $\mathbf{n}$ stands for (a, b, c), and choose a solution $\mathbf{v} = \mathbf{v}_0$. Then $\mathbf{v}_0 \cdot \mathbf{n} = d$, and

$$(\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} = 0.$$

If $\mathbf{v}_0$, $\mathbf{v}$ are the position vectors of two points P_0, P (the first fixed, the second varying) then our equation is saying that the displacement vector $\overrightarrow{P_0P} = \mathbf{v} - \mathbf{v}_0$ is orthogonal to $\mathbf{n}$. The point P therefore lies in a plane passing through P_0 and perpendicular to $\mathbf{n}$. QED

The vector

$$\mathbf{n} = (a, b, c)$$

is called the *normal* to the plane. Any reasonable surface in $\mathbb{R}^3$ has a normal at each point, but only for the plane is the normal direction constant.

L11.2 Distances. In calculating distances, it is convenient to arrange that $\mathbf{n}$ be a *unit* vector. If this is not already the case, it suffices to divide both sides of the equation by $|\mathbf{n}|$, thus modifying also d.

Lemma. *If $|\mathbf{n}| = 1$, so that $a^2 + b^2 + c^2 = 1$, the distance between a point with position vector $\mathbf{p}$ and the plane (4) equals $|\mathbf{p} \cdot \mathbf{n} - d|$.*

Proof. The distance is the length of the vector $\mathbf{p} - \mathbf{p}_0$, where $\mathbf{p}_0$ is the point of the plane π at the foot of the perpendicular from $\mathbf{p}$ to π. Since $\mathbf{p} - \mathbf{p}_0$ is parallel to $\mathbf{n}$, this distance is the absolute value of

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n} - \mathbf{p}_0 \cdot \mathbf{n}.$$

Since $\mathbf{p}_0 \in \pi$, the last term equals d. QED

Example. To find the distance of $(2,0,0)$ from $x + y + z = 0$, we merely take

$$\mathbf{n} = (1,1,1)/\sqrt{3},$$

and $d = 0$. The distance is $|\mathbf{p} \cdot \mathbf{n}| = 2/\sqrt{3}$.

L11.3 Intersection of two planes. Consider the linear system

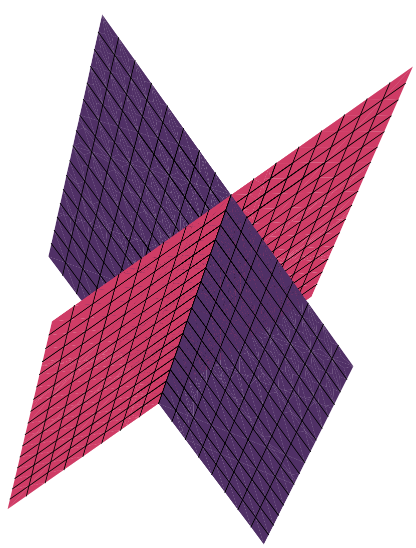
$$\left\{ \begin{array}{rcl} a_1x + b_1y + c_1z & = & d_1, \\ a_2x + b_2y + c_2z & = & d_2. \end{array} \right. \tag{5}$$

Each equation determines a plane π_i with normal vector

$$\mathbf{n}_i = (a_i, b_i, c_i), \qquad i = 1, 2.$$

We shall assume that the two planes are not parallel, equivalently $\mathbf{n}_1, \mathbf{n}_2$ are LI. As the figure shows, their intersection will be a line

$$\ell = \pi_1 \cap \pi_2.$$



We can describe ℓ analytically by solving the linear system of two simultaneous equations.

Since the matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}, \qquad A^+ = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array}\right)$$

associated to (5) both have rank 2, it follows from (RC2) that the general solution of the system will depend on one parameter, and can be written in one of the equivalent ways:

$$(x, y, z) = (x_0 + tp,\ y_0 + tq,\ z_0 + tr),$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \tag{6}$$

$$\mathbf{v} \ = \ \mathbf{v}_0 \ + \ t\mathbf{p}, \qquad \mathbf{p} \neq \mathbf{0}.$$

The last equation asserts that $\mathbf{v} - \mathbf{v}_0$ is parallel to the fixed vector $\mathbf{p}$. The equations therefore determine a straight line ℓ that passes through the point (x_0, y_0, z_0) with position vector $\mathbf{v}_0$ and direction $\mathbf{p}$. Any one of (6) is called the *parametric equation* of the line.

Because ℓ lies in both planes, it is perpendicular to both $\mathbf{n}_1, \mathbf{n}_2$. Thus,

Lemma. $\mathbf{p}$ *is a multiple of* $\mathbf{n}_1 \times \mathbf{n}_2$.

There are therefore two ways to find the equation of ℓ:

(a) Solve the system (5) by super-reducing A^+;

(b) Compute $\mathbf{n}_1 \times \mathbf{n}_2$ and then find a particular solution of (5) perhaps by setting $z = 0$.

We illustrate these two approaches next.

Example. We shall find the parametric equation of the line $\ell : \left\{ \begin{array}{rcl} x + y + z & = & 1, \\ x + 2y + 3x & = & 4. \end{array} \right.$

Method (a). The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \end{array}\right),$$

whence

$$(x, y, z) = (-2 + t,\ 3 - 2t,\ t). \tag{7}$$

Method (b). The direction of ℓ is given by $\mathbf{p} = (1,1,1) \times (1,2,3) = (1,-2,1)$. To find one point $\mathbf{v}_0$ on the line, we set $z = 0$ so that

$$x + y = 1, \quad x + 2y = 4 \quad \Rightarrow \quad y = 3,\ x = -2.$$

Therefore the equation is

$$\mathbf{x} = \mathbf{v}_0 + t\mathbf{p} = (-2, 3, 0) + t(1, -2, 1).$$

By luck, this is the same as (7), though the parametric equation of a line is not unique so the diffrerent methods will not necessarily give identical equations.

L11.4 Further exercises.

1. Given the plane $\pi:\ x+2y-4z=7$, find an orthonormal set $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ of vectors such that $\mathbf{u}$ and $\mathbf{v}$ are parallel to π.

2. Let $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\mathbf{v}_0=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$, $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Find the distances between the following points/planes:

(i) $(0,0,0)$ and $x+y+z+6=0$,

(ii) $(1,2,3)$ and $x=4$,

(iii) $(1,2,3)$ and $x+y+z=0$.

3. Given the planes $\pi_1: x+y+z=1$ and $\pi_2: x+2y-z=0$, let $\ell=\pi_1\cap\pi_2$. Say whether

(i) there exists $a,b,c\in\mathbb{R}$ such that ℓ is given by $(x,y,z)=(1{+}at,\ {-1}{+}bt,\ 1{+}ct)$,

(ii) there exists $p,q,r\in\mathbb{R}$ such that ℓ is given by $(x,y,z)=(p{-}3t,\ q{+}2t,\ r{+}t)$.

4. Find a unit vector orthogonal to

(i) the plane $7x+y+z=5$,

(ii) the plane that contains the points $(1,2,-2)$, $(1,0,3)$, $(-4,4,4)$,

(iii) both the lines $(x,y,z)=(-1,t,2t)$ and $(x,y,z)=(t,1+t,1-t)$.

5. Given the planes

$$\begin{aligned}\pi_1:&\quad ax+y-2z=0,\\ \pi_2:&\quad y+z+2b=0,\\ \pi_3:&\quad 2x+y+2z=1,\end{aligned}$$

find a and b such that the line $\pi_1\cap\pi_2$ is parallel to π_3.

6. Let ℓ be the line $x=2y=z$ and π the plane $x=y+z$. Explain why a line m in π that meets ℓ necessarily has the form $(x,y,z)=(at,bt,ct)$, and find the condition on a,b,c for which m is orthogonal to ℓ.

7. Let ℓ and m be two lines parallel to vectors $\mathbf{p}$ e $\mathbf{q}$ and containing points P and Q. Suppose that $\mathbf{v}=\mathbf{p}\times\mathbf{q}\neq\mathbf{0}$. Show that the (minimum) distance between ℓ and m equals $|\vec{PQ}\cdot\mathbf{v}|/|\mathbf{v}|$.