Notes 10 – The vector cross product

Whilst the scalar product is defined for two vectors of arbitrary length (this was the basis of matrix multiplication), the vector product is special to $\mathbb{R}^3$. (There is a similar operation only in $\mathbb{R}^7$!) The vector product of two vectors is another vector, though combined with the scalar product it is closely related to the determinant of a 3×3 matrix.

L10.1 The cross product as a determinant. As with the scalar product, we first define the vector product in coordinates. Let

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Definition. *The cross product of* $\mathbf{u}$ *with* $\mathbf{v}$*, written* $\mathbf{u}\times\mathbf{v}$ *or* $\mathbf{u}\wedge\mathbf{v}$*, is the vector*

$$\begin{vmatrix} b & y \\ c & z \end{vmatrix}\mathbf{i} - \begin{vmatrix} a & x \\ c & z \end{vmatrix}\mathbf{j} + \begin{vmatrix} a & x \\ b & y \end{vmatrix}\mathbf{k} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} \tag{1}$$

Symbolically, we may write

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} a & x & \mathbf{i} \\ b & y & \mathbf{j} \\ c & z & \mathbf{k} \end{vmatrix} \tag{2}$$

where $\{\mathbf{i}=\mathbf{e}_1,\ \mathbf{j}=\mathbf{e}_2,\ \mathbf{k}=\mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. It is easy to verify that:

(i) $\mathbf{u}\times\mathbf{u} = 0$,

(ii) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$,

(ii) $(a\mathbf{u} + b\mathbf{v})\times\mathbf{w} = a\mathbf{u}\times\mathbf{w} + b\mathbf{v}\times\mathbf{w}$.

In terms of the canonical basis, we have $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$. This gives one another way to compute a cross product. For example,

$$(\mathbf{i}+\mathbf{j})\times(\mathbf{j}+\mathbf{k}) = \mathbf{i}\times\mathbf{j} + \mathbf{i}\times\mathbf{k} + \mathbf{j}\times\mathbf{j} + \mathbf{j}\times\mathbf{k} = \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

More generally,

Lemma. *If* $\mathbf{u}, \mathbf{v}$ *are both nonnull then* $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$*, where* θ *is the angle (with* $0 \leqslant \theta \leqslant \pi$*) measured from* $\mathbf{u}$ *to* $\mathbf{v}$*.*

Proof. From (1),

$$\begin{aligned} |\mathbf{u}\times\mathbf{v}|^2 &= (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2 \\ &= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax)^2 - (by)^2 - (cz)^2 - 2(bycz + axcz + axby) \\ &= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2. \end{aligned}$$

We know from the theory of the dot product that the last line equals

$$|\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2\theta) = |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta,$$

as required. QED

Corollary. *The norm of* $\mathbf{u}\times\mathbf{v}$ *equals the area of a parallelogram with adjacent sides given by* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Observe that $\sin\theta = 0$ iff $\mathbf{u}, \mathbf{v}$ are proportional. Not forgetting the possibility that one of the vectors is null, we can state the

Corollary. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ *iff* $\mathbf{u}, \mathbf{v}$ *are **not** linearly independent (meaning that one is a possibly-zero multiple of the other).*

L10.2 The triple product. Taking the dot product of (2) with $\mathbf{w} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ gives

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}.$$

This is called the *triple product* and sometimes denoted $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$.

Exercise. From the corresponding properties of 3×3 determinants, we have

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = [\mathbf{v}, \mathbf{w}, \mathbf{u}] = [\mathbf{w}, \mathbf{u}, \mathbf{v}],$$

whereas $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$ if any two of the vectors are equal.

Suppose that $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$. Since

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = [\mathbf{u}, \mathbf{v}, \mathbf{u}] \quad \text{and} \quad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = [\mathbf{u}, \mathbf{v}, \mathbf{v}]$$

are both zero, $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. So if $\mathbf{n}$ is a *unit* vector orthogonal to both $\mathbf{u}, \mathbf{v}$ then $[\mathbf{u}, \mathbf{v}, \mathbf{n}] \neq 0$. If we choose (the sign of) $\mathbf{n}$ so that $[\mathbf{u}, \mathbf{v}, \mathbf{n}] > 0$, we can state the

Lemma. $\mathbf{u}\times\mathbf{v} = |\mathbf{u}||\mathbf{v}|(\sin\theta)\mathbf{n}$*.*

If $\mathbf{u}, \mathbf{v}$ are orthogonal unit vectors (so that $\mathbf{u}\cdot\mathbf{v} = 0$), it follows that $\{\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v}\}$ is an *orthonormal* basis of $\mathbb{R}^3$. This makes it easy to construct an ON basis containing a given unit vector $\mathbf{v}_1$.

Consider the parallelopiped $\mathscr{P}$ whose edges are determined by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Regard the parallelogram face generated by $\mathbf{u}$ and $\mathbf{v}$ as the base of $\mathscr{P}$ resting on the table. If ϕ is the angle that $\mathbf{w}$ makes with the base plane then the height of $\mathscr{P}$ is $|\mathbf{w}|\cos\phi = |\mathbf{w}\cdot\mathbf{n}|$. To compute the volume of $\mathscr{P}$, we multiply this height by the area $|\mathbf{u}\times\mathbf{v}|$ of the base.

Corollary. *The volume of* $\mathscr{P}$ *equals* $\Big|[\mathbf{u}, \mathbf{v}, \mathbf{w}]\Big|$.

Exercise. What is the volume of the tetrahedron formed with edges $\mathbf{u}, \mathbf{v}, \mathbf{w}$ emanating from a common vertex?

It follows from the volume formula that three vectors are coplanar (and so linearly dependent) iff their triple product is zero. More formally,

Proposition. $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$ *iff the set* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is not linearly independent.*

Proof. If any one of the three vectors is null, the triple product is obviously zero. If any two of the three vectors are proportional then their cross product is null and the triple product is zero. If one vector is a LC of the other two, then its dot product with the cross product of the two is zero, and the triple product is zero.

Conversely, if the triple product is zero then the dot product of $\mathbf{u}\times\mathbf{v}$ with $\mathbf{w}$ is zero. If $\mathbf{u},\mathbf{v}$ are LI then this means that $\mathbf{w}$ is perpendicular to $\mathbf{n}$ and therefore a LC of $\mathbf{u}$ and $\mathbf{v}$. Thus, the three vectors are not LI. QED

Example. A direct calculation using (1) yields

$$(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{x}\times\mathbf{y}) = (\mathbf{u}\cdot\mathbf{x})(\mathbf{v}\cdot\mathbf{y}) - (\mathbf{u}\cdot\mathbf{y})(\mathbf{v}\cdot\mathbf{x}).$$

The vector $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is orthogonal to $\mathbf{u}\times\mathbf{v}$ and therefore a LC of $\mathbf{u},\mathbf{v}$. It is also orthogonal to $\mathbf{w}$ and the two vectors

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}, \qquad (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

must therefore be proportional. In fact they are *equal*, because taking the dot product of both with $\mathbf{v}$ gives the same result

$$[\mathbf{u}\times\mathbf{v},\ \mathbf{w},\ \mathbf{v}] = (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{v}) = (\mathbf{u}\cdot\mathbf{w})|\mathbf{v}|^2 - (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})$$

(and in general this will be nonzero). Note that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

and that the cross product is not *associative* (parentheses are essential).

L10.3 The inverse of a 3×3 matrix. Let $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3 \in \mathbb{R}^{1,3}$ be three given row vectors, and consider the matrices

$$A = \begin{pmatrix} \leftarrow & \mathbf{v}_1 & \rightarrow \\ \leftarrow & \mathbf{v}_2 & \rightarrow \\ \leftarrow & \mathbf{v}_3 & \rightarrow \end{pmatrix}, \qquad C = \begin{pmatrix} \leftarrow & (\mathbf{v}_2\times\mathbf{v}_3) & \rightarrow \\ \leftarrow & (\mathbf{v}_3\times\mathbf{v}_1) & \rightarrow \\ \leftarrow & (\mathbf{v}_1\times\mathbf{v}_2) & \rightarrow \end{pmatrix}.$$

Then

$$AC^\top = \begin{pmatrix} [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] & [\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_1] & [\mathbf{v}_1,\mathbf{v}_1,\mathbf{v}_2] \\ [\mathbf{v}_2,\mathbf{v}_2,\mathbf{v}_3] & [\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_1] & [\mathbf{v}_2,\mathbf{v}_1,\mathbf{v}_2] \\ [\mathbf{v}_3,\mathbf{v}_2,\mathbf{v}_3] & [\mathbf{v}_3,\mathbf{v}_3,\mathbf{v}_1] & [\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_2] \end{pmatrix} = \delta I_3,$$

where $\delta = [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] = \det A$.

Corollary. *If* $\det A \neq 0$ *then the inverse of* A *equals* $\frac{1}{\delta}C^\top$.

We have not actually verified that $C^\top A = \delta I_3$ but this is also true.

Exercise. Verify that C is the matrix $\widetilde{A}$ of cofactors described at the beginning of the course.

Suppose that the determinant δ is nonzero, so that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is LI and so a *basis* of $\mathbb{R}^3$. The columns of A^{-1} are

$$\mathbf{w}_1 = \frac{1}{\delta}\mathbf{v}_2\times\mathbf{v}_3, \qquad \mathbf{w}_2 = \frac{1}{\delta}\mathbf{v}_3\times\mathbf{v}_1, \qquad \mathbf{w}_3 = \frac{1}{\delta}\mathbf{v}_1\times\mathbf{v}_2,$$

and we obtain the

Lemma. *The coefficients of an arbitrary vector* $\mathbf{v}$ *relative to the basis are given by*

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{w}_1)\mathbf{v}_1 + (\mathbf{v}\cdot\mathbf{w}_2)\mathbf{v}_2 + (\mathbf{v}\cdot\mathbf{w}_3)\mathbf{v}_3 = \sum_{p=1}^{3}(\mathbf{v}\cdot\mathbf{w}_p)\mathbf{v}_p.$$

If the basis is ON, then $\mathbf{w}_i = \mathbf{v}_i$, and we recover a familar formula. In general, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a different basis of $\mathbb{R}^3$, sometimes called the *reciprocal basis.*

L10.4 Further exercises.

1. Let $\mathbf{v}_1 = \mathbf{i} + a\mathbf{k}$, $\mathbf{v}_2 = a\mathbf{j} + \mathbf{k}$, $\mathbf{v}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, with $a \in \mathbb{R}$. Compute $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ without using any general formula.

   (i) For which values of a is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ LI?

   (ii) Find b, c for which $(\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3 = b\mathbf{v}_1 + c\mathbf{v}_2$.

2. Does there exist a *unique* triple (a, b, c) for which $\mathbf{i}+a\mathbf{j}+b\mathbf{k}$ is orthogonal to $\mathbf{i}+2\mathbf{j}+5\mathbf{k}$ and parallel to $c\mathbf{i} + \mathbf{j} + 2\mathbf{k}$?

3. Compute the determinants of the five matrices

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & 5 \\ 4 & 2 & 3 \end{pmatrix}, \quad AB^2, \quad A + B, \quad AB + A^2.$$