Special Holonomy

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Topics in Geometry
2 December 2022

refers to transformations of the fibres of a bundle $V \to M$, equipped with a connection, as one travels around loops in the base manifold M.

If V is a vector bundle then a connection can be defined as a 'covariant derivative'

$$\begin{aligned} \nabla : \Gamma(M, V) &\longrightarrow \Gamma(M, T^*M \otimes V) \\ \nabla_X : \Gamma(M, V) &\longrightarrow \Gamma(M, V), \end{aligned}$$

($X \in \Gamma(M, TM)$ is a vector field) satisfying

$$\begin{aligned} \nabla(fs) &= f\nabla s + df \otimes s \\ \nabla_X(fs) &= f\nabla s + (Xf)s, \end{aligned}$$

for f a smooth function. Any connection has an associated curvature $F = d_\nabla \circ \nabla$ that measures the obstruction to finding constant sections ($\nabla s \equiv 0$).

Today's assumptions

- M is a smooth manifold of real dimension d.
- M is connected, simply-connected and oriented.
- It has a Riemannian metric $g \in \Gamma(S^2 T^* M)$.
- $V = TM \overset{g}{\cong} T^* M$.
- ∇ is the Levi-Civita connection determined by g.

If $V = TM$, one can also define the torsion of a connection. If this vanishes then

$$\nabla \alpha = 0 \implies d\alpha = 0$$

for any 1-form $\alpha \in \Gamma(T^* M)$. The Levi-Civita connection is characterized by having $\nabla g \equiv 0$ and zero torsion (so $[X, Y] = \nabla_X Y - \nabla_Y X$) [1].

[1] Find the formula for ∇ using Christoffel symbols or otherwise

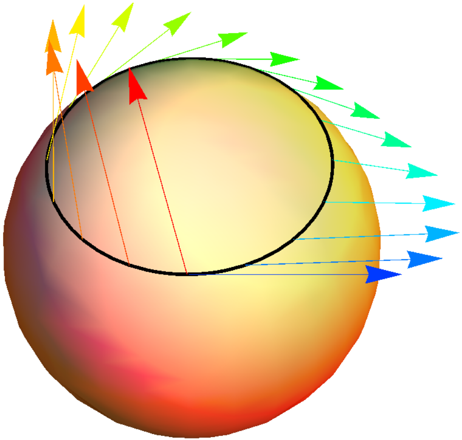
Parallel transport

If $\gamma\colon [0,1] \to M$ is a smooth path from x to y, then both TM and ∇ can be pulled back to $[0,1]$. Any vector $X \in T_xM$ can be extended to a constant field X along γ ($\nabla_{\dot\gamma(t)}X \equiv 0$). In this way, we obtain a linear mapping

$$\Pi\colon T_xM \longrightarrow T_yM.$$

When $x = y$ these linear maps generate the holonomy group Hol_x. As we vary the basis of T_xM and the point x, we obtain a well-defined conjugacy class of a subgroup Hol_x inside $SO(d)$, where $d = \dim M$.

Remark. There is a related concept of monodromy for flat connections ($F \equiv 0$) on a bundle V, giving rise to representations of $\pi_1(M)$.



Tensors

In practice, holonomy is detected by the existence of a parallel tensor Φ ($\nabla\Phi \equiv 0$, meaning Φ is preserved under parallel transport).

What is a tensor? Its value at each point belongs to

$$T_x \otimes \cdots \otimes T_x \otimes T_x^* \otimes \cdots T_x^*,$$

more typically in $T_x \otimes T_x^* \cong \mathrm{Hom}(T_x, T_x)$, or $\Lambda^k T_x^*$ with $k = 2, 3, 4$, or $S^k T^*$ with $k = 2, 3$. For example,

$$g \in \Gamma(M^d, S^2 T^*), \quad J \in \Gamma(M^{2n}, T^*M \otimes TM), \quad \varphi \in \Gamma(M^7, \Lambda^3 T^*M).$$

Proposition. If the stabilizer of Φ is conjugate to G and $\nabla\Phi \equiv 0$ then $Hol \subseteq G$.

This implies that $F \cdot \Phi \equiv 0$, and F is a 2-form with values in $\mathfrak{hol}$.

Any surface in $\mathbb{R}^3$, like the sphere S^2, with its induced metric. In this case, ∇ is just orthogonal projection of ordinary differentiation.

Any abstract surface with local coordinates u, v:

$$g = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

The 2-sphere has $E = \cos^2 v,\ F = 0,\ G = 1.$

We can always find coordinates such that $E = G$ and $F = 0$, and set $z = u + iv$. This makes M into a complex 1-dimensional manifold (a Riemann surface)[2].

The torus $T^d = \mathbb{R}^d/\mathbb{Z}_d \cong S^1 \times \cdots \times S^1$ has a flat metric.

The d-dimensional sphere is a coset space $SO(d+1)/SO(d)$:

$$g\,SO(d) \longmapsto ge_1 \in S^d \subset \mathbb{R}^{d+1}.$$

[2] Find such a coordinate change on S^2

A Lie group G is a smooth manifold with a smooth group action. We can identify $T_e G$ with the Lie algebra $\mathfrak{g}$.

Any connected compact abelian Lie group is a torus T^d, and its tangent space at any point is naturally $\mathbb{R}^d$.

Any compact Lie group G has a maximal torus T^r, and any two are conjugate. Its dimension r is the rank of G.

Up to finite covers, any compact simple Lie group is included in the list

$$A_n = SU(n+1), \quad B_n = SO(2n+1), \quad C_n = Sp(n), \quad D_n = SO(2n),$$
$$G_2, \qquad F_4, \qquad E_6, \qquad E_7, \qquad E_8.$$

The exceptions have respective dimensions $14, 52, 78, 133, 248$. Moreover,

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1), \quad \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2), \quad \mathfrak{sp}(2) \cong \mathfrak{so}(5), \quad \mathfrak{su}(4) \cong \mathfrak{so}(6).$$

Two families

- $SO(d) = \{X \in \mathbb{R}^{d,d} : X^\top X = I,\ \det X = 1\}$ is closed & bounded, so compact.
  It is the stabilizer of the dot product on $\mathbb{R}^d \cong (\mathbb{R}^d)^*$, and
  $\mathfrak{so}(d) = \{X \in \mathbb{R}^{d,d} : X + X^\top = 0\} \cong \Lambda^2(\mathbb{R}^d)$.

- $SU(n) = \{X \in \mathbb{C}^{n,n} : XX^* = I,\ \det X = 1\}$
  $\mathfrak{su}(n) = \{X \in \mathbb{C}^{n,n} : X + X^* = 0,\ tr\, X = 0\}$.
  If we regard $\mathbb{R}^{2n}$ as the real vector space underlying $\mathbb{C}^n$, its complexification is

$$\mathbb{C}^{2n} = \mathbb{C}^n \oplus \overline{\mathbb{C}^n} = \Lambda^{1,0} \oplus \Lambda^{0,1} = [\![\Lambda^{1,0}]\!].$$

  Moreover

$$\begin{aligned} \Lambda^2(\mathbb{C}^{2n}) &\cong \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \\ \Lambda^2(\mathbb{R}^{2n}) &\cong [\![\Lambda^{2,0}]\!] \oplus \langle\omega\rangle \oplus [\Lambda^{1,1}_0], \end{aligned}$$

  and $\mathfrak{su}(n) \cong [\Lambda^{1,1}_0]$.

If G, H are Lie groups, with H compact, then one can define a Riemannian metric on $M = G/H$ using a splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

and choosing an H-invariant inner product on $\mathfrak{m} \cong T_o M$. This is unique up to scale if H acts irreducibly on $\mathfrak{m}$.

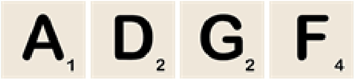
Lemma. If $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ then $Hol \subseteq \mathfrak{h}$.

In this case M is a Riemannian symmetric space, and its curvature tensor can be identified with the Lie bracket $\Lambda^2 \mathfrak{m} \to \mathfrak{h}$. These spaces were classified by Élie Cartan.

Example: For the sphere $S^d = SO(d+1)/SO(d)$,

$$\mathfrak{so}(d+1) \cong \Lambda^2(\mathbb{R}^d \oplus \mathbb{R}) \cong \Lambda^2 \mathbb{R}^d \oplus \mathbb{R}^d \cong \mathfrak{so}(d) \oplus \mathbb{R}^d.$$

Contrasting examples



- $S^5 \cong \dfrac{SU(3)}{SU(2)}$ is not symmetric[3].
- $M^5 \cong \dfrac{SU(3)}{SO(3)}$ is a symmetric space[4].
- $M^{35} = G_2 \times SO(3)$ is a subgroup of F_4, but $\dfrac{F_4}{G_2 \times SO(3)}$ is not symmetric.
- $\mathbb{OP}^2 = \dfrac{F_4}{Spin(9)}$ is is a symmetric space with holonomy $Spin(9)$.

[3] Show that it fibres over $\mathbb{CP}^2$

[4] Show that it parametrizes special Lagrangian subspaces of $\mathbb{C}^3$

Lie group topology

Any compact simple Lie group $G = \dfrac{G \times G}{G}$ with its biinvariant metric and $Hol = G$.

Examples.

- $SU(2)$ is isometric to S^3.
- $SO(3) \cong SU(2)/\mathbb{Z}_2$, and is therefore homeomorphic to $\mathbb{RP}^3$.
- $M^8 = SU(3)$ has a parallel (and so harmonic) tensors

$$\gamma \in \Gamma(S^3, \Lambda^3 T^* M), \quad \delta \in \Gamma(M, S^3 T^* M).$$

Theorem (Hopf). Any connected Lie group has the same real cohomology as a product of r odd-dimensional spheres[5].

[5] Find the dimension of the spheres for all the exceptional groups

are Riemannian symmetric spaces with a compatible complex (and Kähler) structure. A Kähler manifold has $\nabla J \equiv 0$ and $Hol \subseteq U(n)$.

The prime HSS is

$$\mathbb{CP}^n = \frac{U(n+1)}{U(1) \times U(n)} = \frac{SU(n+1)/\mathbb{Z}_{n+1}}{U(n)}$$

with the Fubini-Study metric, obtained via

$$\begin{array}{ccccc} S^{2n+1} & \subset & \mathbb{C}^{n+1} & \ni & z \\ \downarrow & & & & \\ \mathbb{CP}^n & & & \omega = i\partial\overline{\partial}\log(\|z\|^2) & \end{array}$$

Any complex submanifold of $\mathbb{CP}^n$ is also Kähler. In particular, a smooth quadric in $\mathbb{CP}^n$ has a transitive action by $SO(n+1)$ and is also a Hermitian symmetric space[6].

[6] Explain this statement further

If M_1 and M_2 are Riemannian manifolds of dimension d_1, d_2, then the product metric on $M_1 \times M_2$ has $Hol \subseteq SO(d_1) \times SO(d_2)$.

Conversely, de Rham proved that if the holonomy group is a product then the manifold is (at least locally) a product.

Consider the following example:

$$\mathbb{R}^4 \setminus \mathbb{R} \;\cong\; S^2 \times (\mathbb{R} \times \mathbb{R}^+) = S^2 \times \mathbb{C}^+.$$

On the left, we have the flat metric g_E. On the right we have the product of two surfaces with Gaussian curvatures $+1, -1$, and so a Kähler metric g_K. The two metrics g_E, g_K are conformally equivalent[7].

[7] Prove this by finding the function f for which $g_K = f\, g_E$

Berger's list

Theorem (Berger, 1955). If M^d is neither a Riemannian product nor a symmetric space then its holonomy group is one of

- $SO(d)$, the generic case
- $U(n)$ with $d = 2n$, the Kähler case
- $SU(n)$ with $d = 2n$, the Calabi-Yau case*
- $Sp(n)Sp(1)$ with $d = 4n$, the quaternion-kähler case
- $Sp(n)$ with $d = 4n$, the hyperkähler case*
- G_2 with $d = 7$, an exceptional case*
- $\mathrm{Spin}(7)$ with $d = 8$, another exceptional case*

*The metrics necessarily have zero Ricci tensor. It was not known at the time whether compact examples exist, or whether the exceptional cases could arise even locally.

ANNALS OF MATHEMATICS
Vol. 76, No. 2, September, 1962
Printed in Japan

ON THE TRANSITIVITY OF HOLONOMY SYSTEMS

BY JAMES SIMONS

(Received October 13, 1961)

Introduction

Several years ago M. Berger [1] gave a classification of possible candidates for the holonomy groups of manifolds having affine connections with zero torsion. His proofs depended strongly on the classification of simple Lie groups, and his technique consisted in eliminating various groups using the Bianchi identities and the theorem of Ambrose-Singer [2].

The most striking of his results is the list he determines of possible holonomy groups of a riemannian manifold. These groups all turn out to be transitive on the unit sphere in the tangent space of the manifold, except in the case that the manifold is a symmetric space of rank ≥ 2. It is natural to ask for an intrinsic proof of this rather startling fact, one which avoids the classification theorems.

Our object here is to give a purely algebraic generalization of the notion

More examples of reduced holonomy

Kähler. A complex manifold M is Kähler if it has a metric g such that $\nabla J \equiv 0$. This is equivalent to asserting that the associated 2-form $\omega = J \cdot g$ is closed. If $d = 2n$,

$$stab(g) \cong O(2n), \quad stab(J) \cong GL(n, \mathbb{C}), \quad stab\,\omega \cong Sp(2n, \mathbb{R})$$

inside $GL(2n, \mathbb{R})$, and the intersection of any two of these groups is $U(n)$. So a Riemannian manifold is Kähler if and only if $H \subseteq U(n)$.

Calabi-Yau. This is the case $H \subseteq SU(n)$ with $n \geqslant 2$. It means that M is Kähler and there exists a parallel form $\Omega \in \Gamma(\Lambda^{n,0} T^* M)$.

Hyperkähler. This is the case $H \subseteq Sp(n) \subset U(2n)$ and there are parallel complex structures I, J, K with $IJ = K = -JI$. Equivalently a triple of closed 2-forms $\omega_1, \omega_2, \omega_3$ of the correct algebraic type. Note that $Sp(1) = SU(2)$ so CY=HK in real dimension 4.

Theorem (Yau, 1977). Let M be a compact complex manifold (so J is fixed) of real dimension $d = 2n$ with zero first Chern class $c_1 \in H^2(M, \mathbb{Z})$. If M has a Kähler metric g (and associated closed 2-form ω), then there exists a unique metric g_{CY} such that $\omega_{CY} = \omega + i\partial\overline{\partial}\phi$ and $Hol \subseteq SU(n)$.

Let S_d denote a smooth hypersurface of $\mathbb{CP}^N$ of degree d. Then a smooth complete intersection

$$S_{d_1} \cap \cdots \cap S_{d_k} \subset \mathbb{CP}^N, \qquad \sum d_j = N + 1$$

has a metric with holonomy $SU(N-k)$. Examples:

- a cubic curve in $\mathbb{CP}^2$, a complex torus $\mathbb{C}/\Lambda$.
- a quartic surface in $\mathbb{CP}^3$, a K3 surface
- $S_2 \cap S_3$ in $\mathbb{CP}^4$
- $S_2 \cap S_2' \cap S_2''$ in $\mathbb{CP}^5$, the resolution of a Kummer surface[8].

[8] Learn about this from the final chapter of Griffiths & Harris

Explicit metrics with holonomy $SU(2)$

There is such a metric on the cotangent bundle $T^*\mathbb{CP}^1$ that resolves $\mathbb{C}^2/\mathbb{Z}_2$. It can be constructed from a hyperkähler moment map $\mu\colon \mathbb{C}^4 \to \mathbb{R}^3$ as a quotient $\mu^{-1}(\lambda)/U(1)$.

Here is a more explicit example. Let M^3 be the S^1 bundle over T^2 formed as a discrete quotient of the Heisenberg group with elements $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.

It has a global basis of 1-forms $e^1 = dx$, $e^2 = dy$, $e^3 = dz - x\,dy$, so $de^3 = -e^{12}$. Now consider $M^3 \times (0,\infty)$ with $e^4 = dt$, and set

$$\omega_1 = f(t)^2 e^{12} + f(t)^{-2} e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23}.$$

Then $\omega_i \wedge \omega_j = 2\delta_{ij} e^{1234}$, and $d\omega_i = 0$ for $i = 1,2,3$ provided $f(t) = (2t)^{1/4}$. Thus

$$g = (2t)^{1/2}(e^1 \otimes e^1 + e^2 \otimes e^2) + (2t)^{-1/2}(e^3 \otimes e^3 + dt^2)$$

is a hyperkähler metric[9].

[9] ?Check the calculations!

Let M be an oriented Riemannian manifold of dimension d. Then there is a unit volume form $\upsilon \in \Lambda^d T^*M$. We can define $*\colon \Lambda^k T^*M \longrightarrow \Lambda^{d-k} T^*M$ by

$$\alpha \wedge (*\beta) = g(\alpha, \beta)\upsilon, \qquad \alpha, \beta \in \Gamma(\Lambda^k T^*M).$$

If $d = 4$ and $k = 2$, we have $*^2 = 1$ and

$$\Lambda^2 T^*M = \Lambda^2_+ \oplus \Lambda^2_-.$$

Mapping an oriented orthonomal basis $\{e^i\}$ of $T^*_x M$ to

$$\Big(\{e^{12} + e^{34}, e^{13} + e^{42}, e^{14} + e^{23}\},\ \{e^{12} - e^{34}, e^{13} - e^{42}, e^{14} - e^{23}\}\Big)$$

defines a homomorphism $SO(4) \overset{2:1}{\to} SO(3) \times SO(3)$, so that $SO(4)$ is not simple[10].

[10] Use quaternions to show that $SU(2) \times SU(2) \overset{2:1}{\to} SO(4)$

The case of G_2

$SO(d)$ is the stabilizer of a positive-definite bilinear form: $X^\top I X = I$.

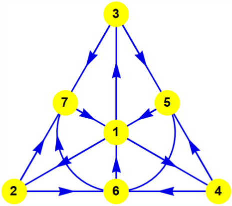
$Sp(2n,\mathbb{R})$ is the stabilizer of a non-degenerate skew-symmetric form: $X^\top J_0 X = J_0$

where $J_0 = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$ for $n = 2$. Equivalently, $Sp(2n,\mathbb{R})$ is the stabilizer of

$$e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots \in \Lambda^2(\mathbb{R}^{2n})^*.$$

By analogy G_2 is the stabilizer of the 3-form

$$\begin{aligned} &(e^1 \wedge e^2 - e^3 \wedge e^4) \wedge e^5 + (e^1 \wedge e^3 - e^4 \wedge e^2) \wedge e^6 + (e^1 \wedge e^4 - e^2 \wedge e^3) \wedge e^7 + e^5 \wedge e^6 \wedge e^7 \\ &= e^{125} - e^{345} + e^{136} - e^{426} + e^{147} - e^{237} + e^{567} \\ &= \varphi \in \Lambda^3(\mathbb{R}^7)^*. \end{aligned}$$



constructed from the inclusion

$$D_2 = SO(4) \subset G_2.$$

So $G_2 = \{g \in GL(7,\mathbb{R}) : g \cdot \varphi = \varphi\}$.

If we accept this result and the fact that $\dim G_2 = 14$ then the orbit

$$GL(7,\mathbb{R}) \cdot \varphi \cong \frac{GL(7,\mathbb{R})}{G_2}$$

is an open subset of $\Lambda^3(\mathbb{R}^7)$. There are only two open orbits.

The 3-form φ determines a conformal class by choosing a volume form υ

$$g(X,Y)\upsilon = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi.$$

But there will be a unique choice of υ for which it is the g's volume form[11]. In our standard form, $\{e^i\}$ is an orthonormal basis for g, and

$$*\varphi = -e^{1267} + e^{3467} - e^{1375} + e^{4275} - e^{1456} + e^{2356} + e^{1234}.$$

[11] Prove this

Suppose that M is a 7-manifold with a smooth 3-form φ of G_2 type at each point. The G_2 structure allows us to decopose the exterior powers of T^*M as follows:

$$\begin{aligned}
\Lambda^2 T_x^* M &= \Lambda_7^2 \oplus \Lambda_{14}^2, && \text{with } \Lambda_{14}^2 \cong \mathfrak{g}_2, \\
\Lambda^3 T_x^* M &= \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3, && \text{with } \Lambda_{27}^3 \cong S_0^2 T_x^* M \\
\Lambda^4 T_x^* M &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, && \text{with } \Lambda_1^4 = \langle *\varphi \rangle \\
\Lambda^5 T_x^* M &= \Lambda_7^5 \oplus \Lambda_{14}^5.
\end{aligned}$$

General theory implies that $\nabla\varphi$ takes values in $T_M^* \otimes \Lambda_7^2 \cong \Lambda_1 \oplus \Lambda_{27} \oplus \Lambda_7 \oplus \Lambda_{14}$. It follows that

$$\nabla\varphi \equiv 0 \quad \Longleftrightarrow \quad d\varphi = 0 \quad \text{and} \quad d * \varphi = 0.$$

Another nilmanifold example

There exists a 6-manifold W (the Iwasawa manifold, a discrete quotient of the complex Heisenberg group) which has a global basis of 1-forms satisfying

$$de^i = \begin{cases} 0 & \text{if } i = 1,2,3,4 \\ e^{13} + e^{42} & \text{if } i = 5 \\ e^{14} + e^{23} & \text{if } i = 6. \end{cases}$$

There exists a metric g with holonomy G_2 on $W \times (0, \infty)$ by setting $e^7 = dt$ and

$$\varphi = t^2(e^{127} + e^{347}) - t^{-2}e^{567} - t(e^{136} + e^{426} - e^{514} - e^{513}).$$

Namely

$$g = t^2 \sum_{i=1}^{4} e^i \otimes e^i + t^{-2} \sum_{i=5}^{6} e^i \otimes e^i + t^4\, dt^2.$$

One can check by computer that g is Ricci-flat.