Manifolds with Exceptional Holonomy

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Acknowledgments

This article is based on the talk I gave at the *Giornata INdAM* in Bologna on 10 June 2015. The purpose of that talk was both to introduce the subject to non-experts, and bring the audience up to date with some of the more exciting developments in this fast-moving field. The last two years have seen yet more striking discoveries.

I would like to express my gratitute to the staff of the Istituto Nazionale di Alta Matematica, both for inviting me on that occasion and for hosting the conference in Rome later that year. The organizational link between the two events is the excuse for including this article in the present volume, and I thank the editors for all the work in putting the volume together. I am aware that this contribution gives only a fleeting presentation of recent topics, and I have omitted some more standard bibliography that can be found in [21, 63, 82].

My own interest in holonomy dates back to my year (1976/77) as masters student in Oxford, when Nigel Hitchin had suggested that I attempt a systematic classification of curvature tensors of metrics enjoying a holonomy group from Berger's list. This was a far-sighted idea at a time in which properties of tensors tended to be computed on a case-by-case basis, and general theory was more obscure. However, having more or less grasped the algebra, I discovered during the course of the year that the problem had already been solved by Dmitry Alekseevsky [4] some time earlier. (It was incidentally an honour to become a close colleague of his later in my career.) This led to a back-up plan involving a diversion into more general G structures and lengthy papers of Guillemin–Sternberg on Spencer cohomology. Both topics influenced my later research, for example in my PhD thesis on quaternioinic geometry, though holonomy won out thanks also to the influence of Alfred Gray.

In the Riemannian context, "exceptional holonomy" means G_2 or $\mathrm{Spin}(7)$, whilst "special holonomy" might (depending on one's interests) include other groups in Berger's list [14]. The development of both fields has been greatly influenced by ideas arising in theoretical physics. Calabi-Yau spaces with holonomy $\mathrm{SU}(3)$ and manifolds with holonomy G_2 play roles in String Theory and M-theory, whilst those with holonomy $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ arise in supersymmetry and relate to so-called special Kähler manifolds and flat symplectic connections.

Real 4-dimensional manifolds, both compact and non-compact, with holonomy group equal to $\mathrm{Sp}(1) = \mathrm{SU}(2)$ provide the starting point of the Ricci-flat theory. Work in four dimensions remains active; a good example is [15]. My own understanding of G_2 structures grew out of familiarity with self-duality arising from the study of instantons and twistor spaces in the 4-dimensional set-up. Since the talk in Bologna, the subject has received a significant impetus from the setting up of the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics[1], and lectures from its workshops informed some of the text.

[1]The author acknowledges support from the Simons Foundation (#488635, Simon Salamon).

1. What is holonomy?

Holonomy is the subgroup generated by *parallel transport* of tangent vectors along all possible loops on a smooth manifold M. The notion makes sense whenever a *connection* is assigned on the tangent bundle TM to the manifold.

More generally, one can consider the holonomy generated by any connection on a vector bundle because of an important property of connections: they *pull back* under smooth mappings, in particular to the pulled-back bundle over the domain of a smooth curve. Holonomy can be studied both locally and globally, and for manifolds with a non-trival fundamental group there is an important distinction. In particular, the holonomy of a flat connection is related to the notion of *monodromy*.

In this article, we shall be discussing the local theory of holonomy associated to a Riemannian manifold with metric tensor

$$\sum_{i,j} g_{ij} dx^i dx^j.$$

This tensor gives rise to Christoffel symbols

$$\Gamma_{ij}^k = \tfrac{1}{2} \sum_m g^{km} \Big(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \Big).$$

The symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$ reflects the fact that this connection has zero torsion. The associated Levi-Civita covariant derivative operator ∇ is defined by setting

$$\nabla_{\partial/\partial x_j} \frac{\partial}{\partial x^k} = \sum_i \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

It is characterized by the fact that (i) it renders the metric constant, and (ii) it is torsion-free. In more abstract notation, $\nabla g = 0$ and

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Often, g and ∇ will not be known explicitly; this is the crux of the matter that leads one into more abstract territory. We begin with an elementary example.

1.1. **The 2-sphere.** The standard metric g on the unit 2-dimensional sphere S^2 is obtained by restricting the standard dot product on $\mathbb{R}^3$. Adopting longitude $x^1 = u$ and latitude $x^2 = v$ as local coordinates on S^2, one can compute the lengths of the corresponding vectors $\partial/\partial u$ (tangent to parallels, circles with v constant) and $\partial/\partial v$ (tangent to meridians, semicircles with u constant). Indeed,

$$g = (\cos v \, du)^2 + (dv)^2,$$

reflecting the fact that meridians have constant length, whereas parallels shrink to zero as one approaches the poles and $v \to \pm\pi/2$.

One computes the table of Christoffel coefficients

$\Gamma^1_{11} = 0$	$\Gamma^1_{12} = -\tan v$	$\Gamma^1_{22} = 0$
$\Gamma^2_{11} = \frac{1}{2}\sin 2v$	$\Gamma^2_{12} = 0$	$\Gamma^2_{22} = 0$

so as to differentiate the vectors

$$e_1 = (\sec v) \frac{\partial}{\partial u}, \qquad e_2 = \frac{\partial}{\partial v}$$

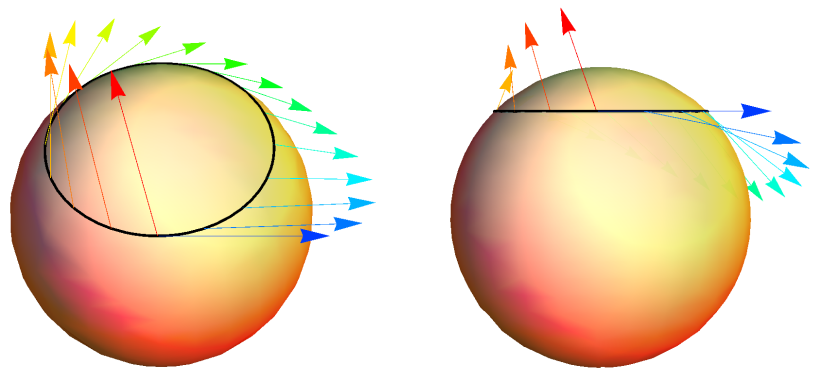


FIGURE 1. Vector field at the latitude of Bologna

that form an orthonormal basis of each tangent space. Consider the unit vector

$$X = e_1 \cos\theta + e_2 \sin\theta$$

defined along a parallel with v constant, where $\theta = \theta(u)$; then $\nabla_{\partial/\partial u} X = 0$ if and only if $\theta = -u \sin v$ up to a constant.

Like any *great* circle, the equator is a geodesic, which means that its unit tangent vector $e_1 = \partial/\partial u$ is invariant under parallel translation. This is consistent with the fact that θ is constant, giving $\nabla_{e_1} e_1 = 0 = \nabla_{e_1} e_2$. Near the north pole, the surface is like a flat plane, so a parallel vector field points in the same direction and rotates through almost 2π relative to the tangent vector to a small circle. Figure 1 displays a parallel vector field X on the circle $v = \pi/4$, roughly the latitude of Bologna. As one moves around the circle, the length of X is preserved, because the metric tensor is itself constant. When one returns to the starting point, the vector has been rotated through $2\pi \sin v \simeq 255^{\circ}$ relative to the tangent vector to the curve. (I confess that I got the calculation wrong in my lecture. As Gerard Watts reminded me, 2π minus this turning angle is the integral of the geodesic curvature, which by Gauss-Bonnet is proportional to the area enclosed by the curve.) By varying the latitude, one can accomplish a rotation through any angle, and it follows that the holonomy group of g on S^2 equals $\mathrm{SO}(2)$.

Now $\mathrm{SO}(2)$ can be identified with $U(1)$, and we can equally well say that the holonomy group is $U(1)$, and preserves a complex structure. The latter is defined infinitesimally by the tensor J given by rotation by $\pi/2$ (counter-clockwise relative to our ordering of the basis): $Je_1 = e_2$ and $Je_2 = -e_1$. This leads naturally to the next subsection.

1.2. **Complex projective space.** The compact complex manifold $\mathbb{CP}^n$ is the set of 1-dimensional subspaces in $\mathbb{C}^{n+1}$. It has a natural Riemannian metric g, called the Fubini-Study metric, which arises from the Hermitian product

$$\langle \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z} \,|\, \mathbf{w} \rangle = \sum_{i=1}^{n+1} \overline{z}_i w_i.$$

A point $[\mathbf{z}]$ of $\mathbb{CP}^n$ then determines a hyperplane, namely the Hermitian complement of $\mathbf{z}$. The construction gives rise to a distance d (or "metric" in the sense of metric space) which is in fact easier to describe than its infinitesimal counterpart.

The distance $d([\mathbf{w}],[\mathbf{z}])$ between two points represented by unit vectors $\mathbf{w},\mathbf{z}\in\mathbb{C}^{n+1}$ equals $2\arccos\sqrt{\rho}$ where

$$\rho=\frac{\langle\mathbf{w},\mathbf{z}\rangle\,\langle\mathbf{z},\mathbf{w}\rangle}{\langle\mathbf{w},\mathbf{w}\rangle\,\langle\mathbf{z},\mathbf{z}\rangle}=\frac{|\,\langle\mathbf{w},\mathbf{z}\rangle\,|^2}{\|\mathbf{w}\|^2\|\mathbf{z}\|^2}.$$

This quantity is the cross ratio of four points, namely $[\mathbf{w}]$, $[\mathbf{z}]$ and the intersections of the complex projective line $\mathbb{CP}^1$ they generate with the associated hyperplanes. It can also be interpreted as a transition probability of pure quantum states [18].

The Fubini-Study metric g is now obtained as the second-order term in the expansion of $d([\mathbf{z}+t\xi],[\mathbf{w}+t\zeta])$ in powers of t. The holonomy group of g is the subgroup $\mathrm{U}(n)$ of $\mathrm{SO}(2n)$, and requiring that the holonomy be $\mathrm{U}(n)$ or a subgroup thereof is one way of defining a *Kähler metric*. Wigner's theorem that that *the isometry group of* $\mathbb{CP}^n$ *is generated by* $\mathrm{SU}(n{+}1)/\mathbb{Z}_{n+1}$ *and complex conjugation*. Freed explains how it can be proved as an application of the holonomy idea [48]. The study of configurations of nine points in $\mathbb{CP}^2$ up to such isometries is the subject of the author's joint paper [59], which contains references to a well-known problem in quantum information concerning the existence of a "Symmetric Informationally Complete Positive Operator Valued Measure" for each n.

The reduction in holonomy to $\mathrm{U}(n)$ at each point of a Kähler manifold M determines an *orthogonal* complex structure J (so $J^2=-1$), and the holonomy condition is the assertion that $\nabla J=0$. There is an associated *symplectic form* ω given by

$$\omega(X,Y)=g(X,JY),$$

the closure of ω being a consequence of the condition $\nabla\omega=0$. Both g and ω can be regarded as objects of type $(1,1)$ relative to J, meaning that they will annihilate a pair of complex vectors of the same type, $(1,0)$ or $(0,1)$.

On a Kähler manifold, there is a very intimate relationship between the complex geometry (defined by J and the associated differential operators ∂ and $\overline{\partial}$) and the Riemannian geometry (defined by the pointwise scalar product g). Locally, ω can be expressed as

$$\omega=i\partial\overline{\partial}f$$

for some real-valued function f, a so-called Kähler potential. For $\mathbb{CP}^n$, one may take $f=\log\|\mathbf{z}\|^2$ on $\mathbb{C}^{n+1}\setminus\{0\}$, and note that the resulting 2-form passes to the quotient.

Any algebraic submanifold M of $\mathbb{CP}^N$ has an induced metric g compatible with both the induced complex and symplectic structure, and is of course Kähler.

Example 1.1. Let $a_0,\dots,a_5$ be distinct complex numbers. The smooth intersection

$$\mathscr{K}=\Big\{\sum_{i=0}^{5}z_i^2=0\Big\}\cap\Big\{\sum_{i=0}^{5}a_iz_i^2=0\Big\}\cap\Big\{\sum_{i=0}^{5}a_i^2z_i^2=0\Big\}$$

of three quadrics in $\mathbb{CP}^5$ is obviously Kähler with holonomy $U(2)$. On the other hand, such a submanifold is a K3 surface: it is simply connected and has $c_1=0$. It is well known that such a 4-manifold has $b_2=22$ and an associated lattice

$$H^2(\mathscr{K},\mathbb{Z})\cong(-\mathrm{E}_8)\oplus(-\mathrm{E}_8)\oplus\mathrm{U}\oplus\mathrm{U}\oplus\mathrm{U}.$$

By Yau's theorem, $\mathscr{K}$ admits a Ricci-flat Kähler metric. Such a metric, whilst not known explicitly, has holonomy group equal to $\mathrm{SU}(2)$, making it *hyperkähler*. The same is true of a quartic hypersurface in $\mathbb{CP}^3$, the smooth intersection of a quadric and cubic in $\mathbb{CP}^4$, and many other complete intersections in products of projective spaces [56], though the *triple* intersection above will be used in the sequel, see §5.1.

1.3. **Hermitian manifolds.** A brief digression is called for in order to describe the relationship between the Riemannian structure (g and the Levi-Civita connection ∇) and an orthogonal almost-complex structure (the tensor J) when the holonomy group does *not* reduce.

The Newlander-Nirenberg theorem tells us that (M,J) is a complex manifold (i.e. has charts with holomorphic transition functions compatible with J) if and only if

$$X,Y\in\Gamma(M,T^{1,0})\quad\Rightarrow\quad [X,Y]\in\Gamma(M,T^{1,0}).$$

It turns out that this is equivalent to the assertion

$$X,Y\in\Gamma(M,T^{1,0})\quad\Rightarrow\quad \nabla_X Y\in\Gamma(M,T^{1,0})$$

[86]. This is significant because the second assertion is apparently weaker, yet it can be used to fully describe the intrinsic torsion of a $\mathrm{U}(n)$ structure.

A consequence of this is that the Riemann curvature tensor R of a Hermitian manifold of real dimension n is constrained by the condition

$$X,Y,Z\in\Gamma(M,T^{1,0})\quad\Rightarrow\quad R_{XY}Z\in\Gamma(M,T^{1,0}).$$

This condition only affects the Weyl tensor W, because J will be compatible with any conformally related metric. When $n=4$ it forces the value of W_+ at each point to lie in a 3-dimensional subspace of what (in the generic case) would be the 5-dimensional space $S^2_0(\Lambda^2_+)$.

At each point, the tensor R belongs to a vector space whose dimension equals

$$\frac{4}{3}n^4+O(n^3)\quad\text{as}\quad n\to\infty,$$

and the constraint above eliminates a subspace of dimension $\frac{1}{6}n^4+O(n^3)$. Thus, asymptotically, the existence of a Hermitian structure "knocks out" one-eighth of R, or rather W.

Example 1.2. Suppose that J is a hypothetical complex structure J on S^6, and consider the compatible Riemannian metric $\hat g$ defined by

$$\hat g(X,Y)=g(X,Y)+g(JX,JY),$$

where g is the standard "round" metric. LeBrun showed that g itself (or, therefore, any conformally flat metric) cannot be compatible with J, so the Weyl tensor $\hat W$ must be non-zero. At each point, it belongs to a subspace

$$S^{2,2}_0\subset\Lambda^3_+\otimes\Lambda^3_-$$

of dimension 84, and the above constraint can be studied for a fixed orbit type.

2. Timeline

In this section, we shall present a selective list of events in the theory of holonomy groups of linear connections.

FIGURE 2. Georges de Rham and Marcel Berger

2.1. **First act.** This can be said to begin with

• 1926: É. Cartan classifies Riemannian symmetric spaces.

These are homogeneous spaces $M = G/H$ with a transitive group of isometries G and Riemann tensor R satisfying $\nabla R = 0$. This means that the holonomy group preserves R, which is then part of the bracket structure of the Lie algbera $\mathfrak{g}$ of G. If G acts effectively on M then the isotropy subgroup H is the holonomy group. Examples of such symmetric space that are neither Hermitian nor quaternion-Kähler are most real and quaternionic Grassmannians, E_6/F_4, $\mathrm{E}_7/\mathrm{SU}(8)$ (see Example 3.3(i)) and $\mathrm{E}_8/\mathrm{Spin}(12)$. Authors wishing a more complete list of inclusions between exceptional groups (and incidentally a bigger class of simple Einstein manifolds) are referred to Wolf's classification of isotropy-irreducible spaces [91].

• 1952: Borel and Lichnerowicz prove that the holonomy group of a Riemannian manifold is a closed subgroup of $O(n)$. De Rham shows that if the holonomy groups acts reducibly then the manifold is locally a product. This feature does not generalize to the pseudo-Riemannian case.

• 1955: Berger lists potential irreducible holonomy groups that do not arise from symmetric spaces. It is important to realize that each is a Lie group H endowed with a *representation* of H on the tangent space $\mathbb{R}^N$ of a hypothetical manifold. All the groups he listed happen to act transitively on the sphere S^{N-1}. Simons gives a direct proof in 1962 that this is the case. The representations of G_2 on $\mathbb{R}^7$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{R}^{4n}$ are studied (in particular, by Bonan and Kraines respectively) using the exterior differential forms that they stabilize. During the period 1968–72, the groups $\mathrm{Spin}(9)$ and $\mathrm{Sp}(n)U(1)$ are eliminated from Berger's list. The former can only occur as the holonomy group of the Cayley plane and its dual symmetric space. Whilst $\mathrm{Sp}(n)U(1)$ never occurs as a Riemannian holonomy group, it highlights the situation in which one fixes a complex structure on a quaternionic manifold, and such geometries have turned out to be of independent significance [60, 49].

We now know that the remaining holonomy groups from Berger's list can all be realized by non-symmetric spaces, though some questions remain regarding the abundance of compact examples.

real dim	holonomy	geometric type
n	$\mathrm{SO}(n)$	generic
$2n$	$\mathrm{U}(n)$	Kähler
$2n$	$\mathrm{SU}(n)$	Ricci-flat Kähler
$4n$	$\mathrm{Sp}(n)$	hyperkähler
$4n$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	quaternion-Kähler
7	G_2	exceptional
8	$\mathrm{Spin}(7)$	exceptional

2.2. **Second act.** Attention focusses on those holonomy groups that produce metrics with zero Ricci tensor. Metrics with exceptional holonomy have this property, but the groups associated to complex structures are more tractable.

• 1977: Yau proves the Calabi conjecture, and thereby the existence of compact manifolds with holonomy equal to $\mathrm{SU}(n)$. The theory is formulated on an underlying complex manifold, the problem being that of finding a suitable Kähler-Einstein metric [92]. The case $n = 1$ corresponds to K3 surfaces. Calabi himself had given constructions of explicit metrics with reduced holonomy $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ on total spaces of vector bundles and coined the name "hyperkähler" [26]. Nonetheless, there are claims that there do not exist compact manifolds with holonomy group $\mathrm{Sp}(n)$ with $n > 1$, and there is considerable pessimism about the existence of metrics with holonomy G_2 or $\mathrm{Spin}(7)$ even locally as computer studies of curvature jets reveal no insight.

• 1983: Beauville discovers two families of compact hyperkähler manifolds (meaning those admitting metrics with holonomy $\mathrm{Sp}(n)$) on Hilbert schemes of points [12]. It is now understood that the language of Hilbert schemes provides a general setting for the description of hyperkähler moduli spaces, in particular monopole moduli spaces [8]. The author has been fascinated by the topology underlying the Beauville examples, and open problems concerning the Betti numbers of low-dimensional hyperkähler manifolds. The apperance of logarithms of Poincaré series in [85] suggests a link with analytic torsion.

• 1987: Bryant establishes local existence of metrics with exceptional holonomy G_2 and $\mathrm{Spin}(7)$. In view of the popular expectation, this result came as a surprise to experts. Bryant's proof used the machinery of exterior differential systems (EDS) that (whilst acknowledging this as Cartan's technique) he had developed with increasing skill to hone in on specialized geometrical problems. As such it was not constructive, though his paper incorporates explicit examples of conical metrics with exceptional holonomy based on manifolds of one dimension less that the author had studied from the point of view of "weak holonomy". We joined forces to describe the first complete examples of metrics with holonomy equal to G_2 (on vector bundles over S^3, S^4 and $\mathbb{CP}^2$), and $\mathrm{Spin}(7)$ (on the spin bundle over S^4) [23, 51]. As Bobby Acharya recently admitted to me, the complexity of the differential forms is well disguised in the notation. The approach has been generalized in [57] to emphasize the importance of "positive-definite connections".

• 1993: By now, it is well understood that Ricci-flat holonomy reductions can be defined by parallel spinors, though the relevance of Killing spinors (and so-called nearly-parallel geometries) had emerged more slowly. The full picture is described by Bär [10], who characterizes those metrics whose *cones* have exceptional holonomy. Later Acharya would highlight the importance of *sine cones* in this regard, see Example 4.7.

2.3. **Third act.** This again concerns again the analytic construction of metrics with reduced holonomy on compact spaces, but now for the exceptional groups. The problem is harder than the case of $\mathrm{SU}(n)$ because of the absence of any complex structures, or indeed any underlying "integrable" non-metric structure.

• 1996: Joyce publishes a proof of the existence of metrics with holonomy G_2 on compact manifolds, after becoming interested in the subject during a train journey in connection with a conference I co-organized in Cortona in 1994. He applies this to many examples, the first (Example 5.2) being a resolution of an orbifold T^7/Γ where Γ is the abelian group $(\mathbb{Z}_2)^3$. Analogous constructions for $Spin\,7$ follow two years later. He is able to mass produce new families of examples by resolving Calabi-Yau orbifolds in increasingly sophisticated ways.

• Work continues in parallel on non-Riemannian linear holonomy. Following Bryant's example [22], a complete classification for torsion-free holonomy groups is provided by Merkulov and Schwachhöfer [87, 74]. Significant results in the Lorentzian case are accomplished in [13, 11].

• New complete metrics with exceptional holonomy are discovered [16] and interpreted in terms of evolution equations [58] and intrinsic torsion [34].

• 2003: Kovalev publishes a construction, suggested by Donaldson, of gluing asymptotically cylindrical spaces (obtained form Fano 3-folds and admitting metrics with holonomy $\mathrm{SU}(3)$) to obtain new compact G_2 manifolds. This is the "twisted connect(ed) sum". Later he will extend this work to $\mathrm{Spin}(7)$. In 2012, Corti, Haskins, Nordström and Pacini introduce new rigour into the procedure, and construct many new compact G_2 manifolds using "semi-Fano' 3-folds.

• 2015: Foscolo and Haskins find new compact nearly-Kähler (Einstein) metrics on S^6 and $S^3\times S^3$, invariant by actions of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ of cohomogeneity one. The models were proposed by Podestà and Spiro [78, 79]. The new techniques involve "matching" rather than gluing, with input from the 5-dimensional hypo set-up [36, 46]. Analysis is needed with regard to solutions of the relevant ODE's.

3. Exceptional holonomy representations

Let us begin with $\mathrm{Spin}(8)$. It has three inequivalent irreducible representations on $\mathbb{R}^8$ that, at the Lie algebra level, are permuted by an outer (triality) automorphism of order 3. Exactly one of these representations factors through $\mathrm{SO}(8)$, and by fixing a vector in the corresponding $\mathbb{R}^8$ one may reduce to $\mathrm{SO}(7)$. The subgroup $\mathrm{Spin}(7)$ that covers this $\mathrm{SO}(7)$ acts irreducibly on the two remaining $\mathbb{R}^8$'s, and this is the relevant holonomy representation (both are equivalent):

$$\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Aut}(\mathbb{R}^8).$$

In this context, the 7-sphere can of course be viewed as the homogeneous space $\mathrm{Spin}(8)/\mathrm{Spin}(7)$.

The subgroup of $\mathrm{Spin}(7)$ fixing a non-zero vector in the irreducible $\mathbb{R}^8$ is the 14-dimensional exceptional Lie group G_2. Moreover,

$$\mathrm{G}_2 \subset \mathrm{SO}(7) \subset \mathrm{Aut}(\mathbb{R}^7),$$

$S^7 \cong \mathrm{Spin}(7)/\mathrm{G}_2$, and

$$\mathrm{Spin}(8)/\mathrm{G}_2 \cong S^7 \times S^7.$$

The subgroup of G_2 fixing a non-zero vector in $\mathbb{R}^7$ is $\mathrm{SU}(3)$ and $S^6 \cong \mathrm{G}_2/\mathrm{SU}(3)$. Recall that the possible (non-symmetric, irreducible) holonomy groups all act *transitively* on the unit sphere in T_xM, in accordance with Simons' result [88].

3.1. **Curvature constraint.** The Riemann curvature tensor $R = R_{ijkl}$ of an n-dimensional manifold is skew in i, j and k, l but symmetric in $(i,j) \leftrightarrow (k,l)$. Moreover, the first Bianchi identity imposes that R belongs to the kernel of the linear mapping

$$S^2(\Lambda^2(\mathbb{R}^n)) \longrightarrow \Lambda^4(\mathbb{R}^n)$$

at each point. The Ricci tensor $R_{jl} = \sum\limits_{i,k} g^{ik} R_{ijkl}$ defines an element of $S^2(\mathbb{R}^n)$. If the holonomy group is H then $\mathfrak{h} \subseteq \mathfrak{so}(n) \cong \Lambda^2$, and one has

$$R \in S^2(\mathfrak{h}).$$

This is the basis of Alekseevsky's description [4]. One can also say that the Levi-Civita connection is an "instanton" for the holonomy structure [80].

Example 3.1. The group G_2 has two inequivalent representations of dimension 77, one isomorphic to $\Lambda^2(\mathfrak{g}_2)/\mathfrak{g}_2$ ($\mathfrak{so}(\mathfrak{g})/\mathfrak{g}$ is always irreducible if $\mathfrak{g}$ is the Lie algebra of a compact simple Lie group [91]). The other arises from the decomposition

$$S^2(\mathfrak{g}_2) \cong V^{77} \oplus S^2_0(\mathbb{R}^7) \oplus \mathbb{R}.$$

As a corollary, a metric with holonomy G_2 is necessarily Ricci flat and has $R \in V^{77}$. The situation for the other exceptional holonomy group is analogous, a metric with holonomy $\mathrm{Spin}(7)$ is Ricci-flat, and R belongs to the subspace

$$W^{168} \subset S^2(\mathfrak{so}(7))$$

generated by Weyl tensors on a 7-manifold. (This particular representation is encoded in a social media identifier SMS168.)

It is well known that G_2 is the stabilizer of a 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$, in which $\mathbb{R}^7$ is viewed as the space of imaginary octonians, with multiplication defined by Figure 3. Forgetting the arrows, this is the matroid characterizing linear independence in the Fano projective plane $\mathbb{P}((\mathbb{Z}/2\mathbb{Z})^3)$, and is one of the "trademarks" of higher mathematics. It was I. M. Gelfand who impressed on me the importance of matroids and combinatorics generally during his visit to Pisa to collect an honorary doctorate in 1981. (At the time, he was equally enthusiastic to hear about my paper with Fran Burstall [25], a conversation that took place in the presence of Soviet security personnel.)

We use the conventions from [82, 83]. In suitable coordinates,

$$\varphi = e^{125} - e^{345} + e^{136} - e^{426} + e^{147} - e^{237} + e^{567}$$

(see below), and so

$$\mathrm{G}_2 = \{g \in \mathrm{GL}(7,\mathbb{R}) : g \cdot \varphi = \varphi\}.$$

Since $49 - 14 = \binom{7}{3}$, the orbit $\mathrm{GL}(7,\mathbb{R}) \cdot \varphi$ is *open* in $\Lambda^3(\mathbb{R}^7)^*$ or *stable* in the sense of [58].

The 3-form φ determines the metric and the 4-form $*\varphi$. The following fundamental result is due to Fernández and Gray.

Proposition 3.2. *Given a* G_2 *structure defined by a 3-form* φ, *the holonomy of its associated metric is contained in* G_2 *if and only if* $d\varphi = 0$ *and* $d * \varphi = 0$.

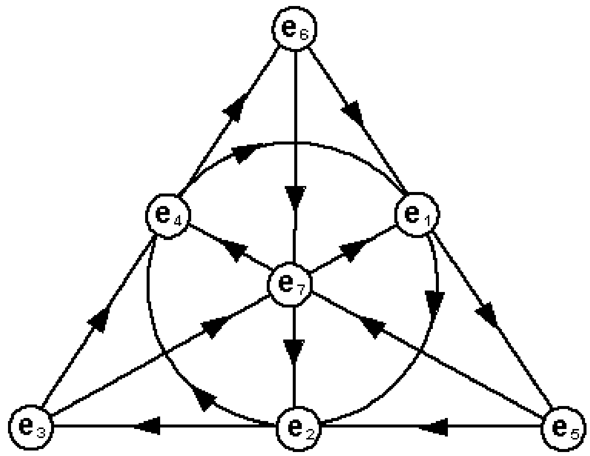


FIGURE 3. Cayley multiplication

Nowadays, this would be seen as an immediate consequence of the decomposition of the space

$$T^* \otimes \mathfrak{h}^\perp \cong \mathbb{R}^7 \otimes \mathbb{R}^7 \cong \mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus \mathbb{R} \oplus S^2_0(\mathbb{R}^7),$$

of intrinsic torsion that contains $\nabla\varphi$, though it was extensive discussions of such theory with Alfred Gray (often carried out in Westbourne Terrace) that kept me enthralled with the problem of exceptional holonomy.

Both φ and $*\varphi$ have open orbits in Λ^3 and Λ^4, and any exterior form of degree 3 or 4 in $\mathbb{R}^7$ can be written a the sum of at most 7 simple (indecomposable) ones. It follows that any 4-form on $\mathbb{R}^8$ can be written as the sum of at most 14 simple forms on $\mathbb{R}^8$. Such an example is the 4-form

$$\Phi = \varphi \wedge e^8 + *\varphi$$

whose stabilizer is $\mathrm{Spin}(7)$.

Comparatively little is known about exterior forms of higher degree in higher dimensions, though we highlight two topics next.

Examples 3.3. (i) A description of 4-forms in $\mathbb{R}^8$ is given in [5, 9]. The trick is to use the fact that the isotropy of the symmetric space $\mathrm{E}_7/\mathrm{SU}(8)$ can be identified with $\Lambda^4(\mathbb{R}^8)$.

(ii) A description of the $\mathrm{Spin}(9)$-invariant 8-form on $\mathbb{R}^{16}$ that arises as the isotropy of the rank-one symmetric space $F_4/\mathrm{Spin}(9)$ (the Cayley plane) can be found in [72]. See also [77].

4. EXPLICIT METRICS IN DIFFERENT DIMENSIONS

Looking back over the early history of the subject, leading up to Bryant's discovery of the local existence of metrics with exceptional holonomy, it is perhaps surprising that one can write down such metrics with relative ease.

Many results in mathematics become self-evident with the passage of time, but it is important to understand that results that seem obvious today were far from obvious when they were first studied. There are many instances in the theory of reduced holonomy and special geometrical structures that attest to this phenomenon. It is a privilege to be involved in the development of a subject and witness theorems taking shape. This first happened for me with the theory of self-duality, which I learnt from lectures of I.M. Singer, before the integrability condition for the twistor space was known.

4.1. **Resolving conical holonomy.** In polar coordinates, the Euclidean metric on $\mathbb{R}^7$ can be expressed as

$$g = dr^2 + r^2 h,$$

where h is the standard metric on the 6-sphere. Of course, the holonomy group of g is the identity, but once we realize that $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$ we can bring G_2 into the picture.

A metric h on a 6-manifold M is *nearly-Kähler* iff the cone $dr^2 + r^2 h$ has holonomy contained in G_2 [10]. Such a manifold has an $\mathrm{SU}(3)$ structure such that ∇J is anti-symmetric, i.e. $(\nabla_X J)(X) = 0$ for all X. The 6-sphere is nearly-Kähler, and one way of defining such a structure is by means of a 2-form ω and a 3-form ψ such that

$$d\omega = \psi, \qquad d * \psi = -\omega \wedge \omega$$

(see [83]). Then

$$\varphi = \omega \wedge dr + r\psi$$

defines a G_2 structure with

$$*\varphi = *\psi \wedge dr + r\,\omega \wedge \omega,$$

and satisfying the condition in Proposition 3.2. If we replace S^6 with one of the other homogeneous nearly-Kähler spaces

$$N = \ S^3 \times S^3, \qquad \mathbb{CP}^3, \qquad \mathbb{F} = \mathrm{SU}(3)/T^2,$$

the conical metric on $\mathbb{R}^+ \times N$ has holonomy group *equal* to G_2.

Another construction has its origins in the so-called Eguchi-Hanson space with holonomy $\mathrm{SU}(2)$. The singular space $X = \mathbb{C}^2/\pm 1$ admits functions $u = z_1^2,\ v = z_2^2,\ w = z_1 z_2$, embedding it in $\mathbb{C}^3$ as the cone

$$uv = w^2,$$

which provides the local model for the singularities in Figure 5.

Let Y denote the (total space of the) cotangent bundle $T^*\mathbb{CP}^1$ of the complex projective line, itself isomorphic to $\mathscr{O}(-2) = L \otimes L$. It follows that, with the origin removed, $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2$ can be identified with the complement

$$\{\alpha \in Y : \alpha \neq 0\}$$

of the zero section. There is a *crepant* resolution

$$\rho\colon Y \to X.$$

The cotangent bundle admits a canonial holomorphic symplectic from $\omega_2 + i\omega_3$ and a complete metric g of the form

$$(r^2+1)^{1/2}(e^1 \otimes e^1 + e^2 \otimes e^2) + (r^2+1)^{-1/2}(e^3 \otimes e^4 + e^4 \otimes e^4),$$

consisting of the sum of a "horizontal" and "vertical" component. The metric g and complex structure also define a closed symplectic form ω_1, so g has holonomy $\mathrm{SU}(2)$. The space Y is the simplest example of one that is *asymptotically locally Euclidean* (ALE).

We can now mimic this construction to find a metric with G_2 holonomy. The next result is one from [23].

Theorem 4.1. *There exist complete metrics with holonomy* G_2 *on the rank 3 vector bundles* $\Lambda^2_- T^*M$ *for* $M = S^4$ *or* $\mathbb{CP}^2$.

The metrics arise from the subgroup $\mathrm{SO}(4)$ of G_2 relative to which

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3 = \langle e^1, e^2, e^3, e^4 \rangle \oplus \langle e^5, e^6, e^7 \rangle.$$

This enables one to identify $\mathbb{R}^3 \cong \Lambda^2_-(\mathbb{R}^4)$ and define a 3-form

$$\varphi = (e^{12} - e^{34}) \wedge e^5 + (e^{13} - e^{42}) \wedge e^6 + (e^{14} - e^{23}) \wedge e^7 + e^{567}.$$

Ignoring universal constants, the associated metric

$$(r^2+1)^{1/2} \sum_1^4 e^i \otimes e^i + (r^2+1)^{-1/2} \sum_5^7 e^i \otimes e^i$$

has holonomy equal to G_2, and is asymptotic to a conical metric on $\mathbb{R}^+ \times \mathbb{CP}^3$ or on $\mathbb{R}^+ \times \mathbb{F}$, as $r \to \infty$.

Although the resulting metrics are not compact, they have been used as testing grounds for various studies, for example of G_2 instantons. The author is involved in the following two projects.

Examples 4.2. Let X denote the total space of $\Lambda^2_- T^*M$, where M is S^4 or $\mathbb{CP}^2$.

(i) Let Σ denote an embedded real surface in M. The pullback of $\Lambda^2_- T^*S^4$ to Σ splits into the direct sum of a rank 1 and a rank 2 vector bundle (the rank 1 one being essentially the twistor lift of Σ). The rank 2 bundle defines a 4-dimensional submanifold of X that is coassociative (meaning that φ restricts to zero) if and only if Σ is *superminimal* (a minimal surface with a horizontal twistor lift) [19, 64]. In the case $M = S^4$, Kovalev and the author have shown that the coassociative submanifold is hyperkähler if Σ has constant Gaussian curvature. If Σ is a totally geodesic 2-sphere, then we recover the Eguchi-Hanson metric that motivated the G_2 construction above.

(ii) The manifold X can be realized as a quotient of an open set of the quaternionic projective plane $\mathbb{HP}^2$ by the action of a circle subgroup (which one depends on whether M is S^4 or $\mathbb{CP}^2$). Such quotients were studied in [7], and the relevance to G_2 metrics observed in [75]. Gambioli, Nagatomo and the author have shown how to generate a closed G_2 3-form on such a quotient [50, 35] from the $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure on $\mathbb{HP}^2$, and there is an analogous theory starting from $\mathrm{Spin}(7)$.

4.2. **Examples from the 4-torus.** Let $W = \Gamma \backslash H$ be the Iwasawa manifold, so here

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\}$$

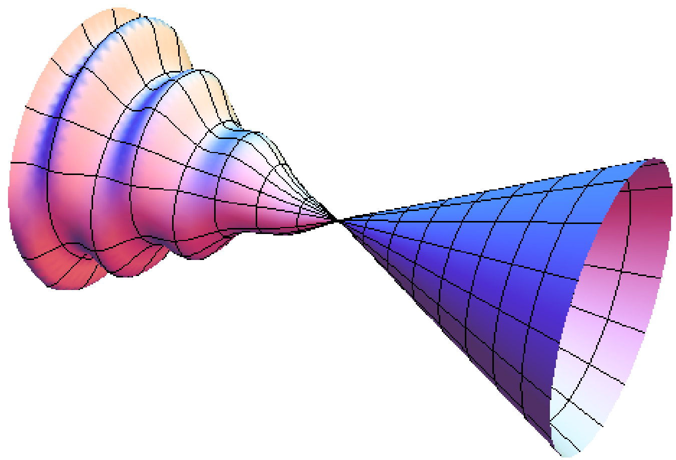


FIGURE 4. The separable solution f in Example 4.5

is the complex Heisenberg group, and Γ is the discrete subgroup for which $z_i\in\mathbb{Z}$. Then W is the compact total space of a T^2 bundle that fibres (via the map to (z_1,z_2)) over T^4. It has a basis of real 1-forms

$$\begin{gathered} e^1=d\lambda,\quad e^2=d\mu,\quad e^3=d\ell,\quad e^4=dm,\\ e^5=dx-\lambda d\ell+\mu dm,\qquad e^6=dy-\mu d\ell-\lambda dm, \end{gathered}$$

notation as in [6, Example 1].

Example 4.3. One can define an $\mathrm{SU}(3)$ structure that extends to a metric

$$\frac{1}{9}dt^2+t^{2/3}\sum_{i=1}^{4}e^i\otimes e^i+t^{-2/3}\sum_{i=5}^{6}e^i\otimes e^i.$$

with holonomy G_2 on $W\times(0,\infty)$ [27, 6]. The corresponding almost-complex structure is J_3 in the notation of [1]. This a "half-complete" example, and it can be written down completely explcitly in local coordinates $(\lambda,\mu,\ell,m,x,y,t)$. We have

$$(g_{ij})=t^{-2/3}\begin{pmatrix} t^{4/3} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & t^{4/3} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & t^{4/3}+r^2 & 0 & x & y & 0\\ 0 & 0 & 0 & t^{4/3}+r^2 & -y & x & 0\\ 0 & 0 & x & -y & 1 & 0 & 0\\ 0 & 0 & y & x & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & t^{2/3} \end{pmatrix},$$

where $r^2=\lambda^2+\mu^2$. One can then check directly that it is Ricci-flat by computer.

This example is generalized by the following result involving a Monge-Ampère evolution equation.

Theorem 4.4 ([6]). *Suppose that a complex surface M has a holomorphic 2-form $\omega_2+i\omega_3$ and a one-parameter family of Kähler forms $\omega=\omega(t)$ and functions $f=f(t)$ such that*

$$\omega''(t)=2i\,\partial\overline{\partial}f(t),$$

where $t\,\omega\wedge\omega=f(t)\omega_2\wedge\omega_2$. Then a rank 3 bundle over M admits a Ricci-flat metric g with holonomy in G_2.

If f is constant on M then $\omega''(t)=0$ and

$$\omega=(p+qt)\omega_0+(r+st)\omega_1,$$

with $(\omega_1,\omega_2,\omega_3)$ a hyperkähler structure and ω_0 an additional closed 2-form, constructible via the Gibbons-Hawking ansatz. Example 4.3 is such a solution in which M is a torus $\mathbb{C}^2/\mathbb{Z}^4$ and

$$\omega_1=\tfrac{1}{2}i(dz_1\wedge d\overline{z}_1+dz_2\wedge d\overline{z}_2),\qquad \omega_2+i\omega_3=dz_1\wedge dz_2.$$

Consider now a non-trivial deformation

$$\omega=\omega_1+i\partial\overline{\partial}\phi$$

where $\phi=\phi(t,z_1)$ with $z_1=\lambda+i\mu$ Then $\partial\overline{\partial}\phi\wedge\partial\overline{\partial}\phi=0$ and $f(t)=\frac{1}{2}\phi''(t)$. There are solutions

$$\phi=\tfrac{1}{3}t^3+A(t)B(\lambda,\mu),$$

with $A''-ctA=0$ and $\Delta B+cB=0$, and the following is copied from [6, Example 3].

Example 4.5. Consider the Airy function

$$\mathrm{Ai}(t)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{u^3}{3}+tu\right)du,$$

the integral being improper. Take $A(t)=\mathrm{Ai}(t)$ and $B(x)=\sin x$, so

$$f(t,x)=t+\tfrac{1}{2}t\,\mathrm{Ai}(t)\sin x.$$

Then

$$t(f\,d\lambda^2+f\,d\mu^2+d\ell^2+dm^2)+f^{-1}(dx-\mathrm{Ai}'(t)\cos\lambda\,d\mu)^2+t^{-2}(dy-\lambda\,d\ell+\mu\,dm)^2+t^2f\,dt^2,$$

defined on $T^2\times\mathbb{R}^4\times(0,\pm\infty)$, has holonomy G_2. The function f is plotted using cylindrical polar coordinates (with $\lambda\in S^1$) in Figure 4.

4.3. **Weak holonomy.** The realization exceptional holonomy can be constructed from classes of Einstein metrics with "nearly parallel" or "weak holonomy" reductions was understood by Gray [52] before studies of Killing spinors and intrinsic torsion took off. The classification of manifolds with intrinsic torsion invariant by various subgroups has been promoted by the author, see for example [31, 28, 33, 34]. See also [3] and references therein.

The theory in §4.2 is based to some extent on the theory of hypersurfaces in spaces with special holonomy. Hitchin studied the corresponding evolution equations, and characterized various structures in terms of stable forms [58].

Let us illustrate this in the case of 7 to 8 dimensions. Any hypersurface of $\mathbb{R}^8$ inherits a G_2 structure φ with $d*\varphi=0$ from a standard $\mathrm{Spin}(7)$ invariant 4-form. In fact, *any* 7-manifold M

(with $w_1 = w_2 = 0$) admits such a "co-calibrated" G_2 structure [42]. If this evolves in time t, the 4-form

$$\Phi = dt \wedge \varphi(t) + *\varphi(t),$$

defines a $\mathrm{Spin}(7)$ structure on $\mathbb{R} \times M$. It is closed provided

$$\frac{\partial}{\partial t}(*\varphi) = -d\varphi$$

a flow which has short-time existence. Therefore, $(0, \varepsilon) \times M$ will always admit a metric with holonomy contained in $\mathrm{Spin}(7)$.

Example 4.6. The homogeneous space $M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ admits a G_2 structure with $d\varphi = *\varphi$. Therefore, $\mathbb{R}^+ \times M^7$ has an explicit conical metric with holonomy $\mathrm{Spin}(7)$. This was the first example that led the author to realize a metric with exceptional holonomy.

The theory for dimensions 5 to 6 was developed in [36]. A metric k on a 5-manifold is *Einstein-Sasaki* if and only if the cone $dy^2 + y^2 k$ has holonomy contained in $\mathrm{SU}(3)$. This is true of $S^2 \times S^3$: its conifold has two Calabi-Yau desingularizations in which the vertex is replaced by S^2 and S^3 respectively.

Example 4.7. The sine cone $d\theta^2 + (\sin\theta)^2 k$ makes

$$\mathbb{E} = (0, \pi) \times S^2 \times S^3$$

nearly-Kähler. To prove this, observe that the conical metric

$$dr^2 + r^2\big(d\theta^2 + (\sin\theta)^2 k\big) = dx^2 + \big(dy^2 + y^2 k\big)$$

has holonomy contained in $\mathrm{SU}(3) \subset \mathrm{G}_2$. This is explained in [2], and is the basis of newer constructions [46].

The nearly-Kähler spaces S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ all admit an action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of co-homogeneity one with generic orbit $S^2 \times S^3$. Desingularizing the sine cone in two ways and deforming the metrics, one recovers the homogeneous nearly-Kähler spaces

$$\begin{array}{llllll} S^6: & S^2 & \leftarrow & \mathbb{E} & \rightarrow & S^3 \\ S^3 \times S^3: & S^3 & \leftarrow & \mathbb{E} & \rightarrow & S^3 \\ \mathbb{CP}^3: & S^2 & \leftarrow & \mathbb{E} & \rightarrow & S^2. \end{array}$$

An analogous construction for $\mathrm{SU}(3)$ has been used to find $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structures on on $\mathrm{G}_2/\mathrm{SO}(4)$ with a closed 4-form [35].

Foscolo and Haskins show that the first two compactifications of $\mathbb{E}$ above can be modified so as to realize new nearly-Kähler metrics on these spaces:

Theorem 4.8 ([47]). *$S^3 \times S^3$ and S^6 each admit a non-homogeneous nearly-Kähler metric (and compatible almost complex structure), so that their cones have metrics with holonomy G_2.*

This discovery of these metrics has thrown the subject open. Cortés and Vásquez [37] show that that only $S^3 \times S^3$ can admit finite quotients that are nearly-Kähler, but that many such quotients do in fact exist.

5. Compact manifolds with exceptional holonomy

Dual to the theory of hypersurfaces is the idea of realizing a metric with special holonomy on a manifold with a circle action. Whilst such a Ricci-flat manifold cannot be compact, the methods are relevant in the quest for new compact examples with exceptional holonomy. When passing to a 7-dimensional G_2 quotient, one has the luxury of starting from either $\mathrm{Sp}(2)\mathrm{Sp}(1)$ or $\mathrm{Spin}(7)$ (see Example 4.2(ii)). The passage from 7 to 6 dimensions is especially fruitful, and a special situation in which the 6-manifold in Kähler was considered in [6].

5.1. **Kummer surface.** We begin by investigating, partly as a diversion, a geometrical model defined by a quadratic line complex. The idea was popular in the 1920's, and assigns a K3 surface to each point of a pseudo-Riemannian 4-manifold [70]. The theory below is derived from [53, Chapter 6] and [43].

Given $\alpha \in \Lambda^2(\mathbb{C}^4)$, the condition $\alpha\wedge\alpha = 0$ is equivalent to asserting that α is a decomposable 2-form. It follows that

$$\mathscr{Q} = \{[\alpha] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) : \alpha \wedge \alpha = 0\}$$

can be identifed with the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ parametrizing the projective lines in $\mathbb{CP}^3$. It is a smooth hypersurface of $\mathbb{CP}^5$ (the "Klein quadric"), and each point $[x] \in \mathbb{CP}^3$ defines the projective plane $\sigma(x) \subset \mathscr{Q}$ parametrizing lines through $[x]$.

Once we have identified $\mathbb{R}^4$ with its dual, the Riemann curvature tensor R in 4 dimensions also defines a quadric

$$\mathscr{R} = \{[\alpha] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) : R(\alpha, \alpha) = 0\}$$

(see §3.1). The intersection $\mathscr{Q} \cap \mathscr{R}$ is smooth precisely because R satisfies the Bianchi identity, and it distinguishes a set of lines in $\mathbb{CP}^3$. For each $[x] \in \mathbb{CP}^3$, the lines of this "quadratic complex" that pass through $[x]$ is a conic

$$C_x = \{[x \wedge u] \in \sigma(x) : R_{xuxu} = 0\}.$$

The quadratic form $R_{x\,\cdot\,x\,\cdot}$ has rank less than 3 if and only if

$$\sum R_{xapq} R_{bxxr} R_{cdsx}\, \varepsilon^{abcd} \varepsilon^{pqrs} = 0,$$

since the left-hand side reduces to a determinant (ε stands for antisymmetrization that is effectively carried out over 3 letters). This cubic identity in R then determines a quartic surface K in $\mathbb{CP}^3$, whose singular locus D (occurring where C_x is a double line) can be shown to consist of 16 singular points: $\mathscr{K}$ is the Kummer surface associated to R.

In order to distinguish the lines in $\sigma(x)$ generated by C_x for $[x] \in K$, one defines a third quadratic form $P \in S^2(\Lambda^2\mathbb{C}^4)^*$, and associated quadric $\mathscr{P} \subset \mathbb{CP}^5$, by setting

$$P_{xyxy} = \sum R_{xyab} R_{xycd}\, \varepsilon^{abcd} = 8(R_{xy12}R_{xy34} + R_{xy13}R_{xy42} + R_{xy14}R_{xy23}),$$

with $x, y \in \mathbb{C}^4$ and $\{1, 2, 3, 4\}$ indicating a standard a basis of $\mathbb{C}^4$. Similar covariants were considered by the author in an early paper [84] on harmonic spaces, inspired by Tom Willmore. Each point of the intersection

$$\mathscr{K} = \mathscr{P} \cap \mathscr{Q} \cap \mathscr{R}$$

is associated to a line in $\mathbb{CP}^3$ that passes through a point $[x]$ of K, and $[x]$ is unique. This defines a mapping $\mathscr{K} \to K$ that is one-to-one over $K \setminus D$.

Proposition 5.1. *$\mathscr{K}$ is a K3 surface and a resolution of K.*

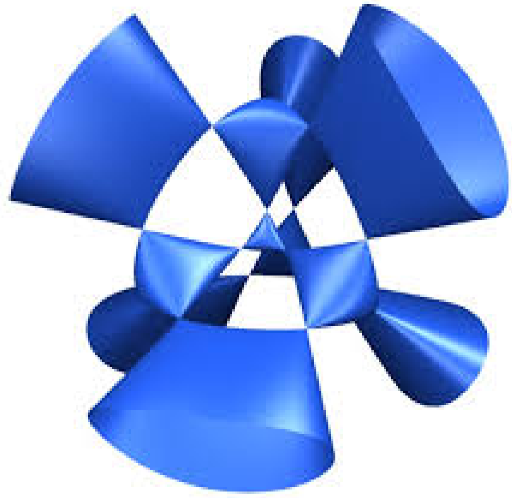


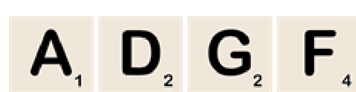
FIGURE 5. Real part of a Kummer surface

A projective line in $\mathbb{CP}^5$ lying on $\mathscr{Q}\cap\mathscr{R}$ corresponds to a pencil of lines in a plane $\sigma(x)$ for some unique $[x]\in K$. There are *two* such pencils for each $[x]\in K\setminus D$, but only *one* for $[x]\in D$. The set of such lines is an abelian variety A equipped with an involution ι, and $\mathscr{K}$ can be identified with $A/\langle\iota\rangle$; each singularity is modelled on $\mathbb{C}^2/\pm1$, just as in Figure 5. Moreover, A is the Jacobian of a genus 2 curve B consisting of those point in $\mathscr{Q}\cap\mathscr{R}$ whose associated lines in $\mathbb{CP}^3$ intersect a fixed line L.

5.2. **Joyce's construction for** G_2**.** The existence of a hyperkähler metric on $\mathscr{K}$ mentioned in Example 1.1 follows analytically because it is possible to resolve each singularity of K using the Eguchi-Hanson space of §4.1. This provides a model for producing *compact* 7-manifolds with holonomy G_2.

Joyce's original idea was to write down a finite subgroup Γ acting on $T^7=\mathbb{R}^7/\mathbb{Z}^7$ preserving the 3-form φ, so T^7/Γ is a G_2 orbifold with amenable singularities [62]. To achieve the G_2-invariance, one works with inclusions

$$\mathrm{SU}(2)_+\subset\mathrm{SO}(4)\subset\mathrm{G}_2\subset\mathrm{F}_4.$$



The presence of F_4 is irrelevant here, but completes the "scrabble" chain and the earlier reference to [91]. Joyce's prototype orbifold is the following.

Example 5.2. Let Γ be the abelian group $(\mathbb{Z}_2)^3$ generated by

$$\begin{aligned}
\alpha(\mathbf{x}) &= (x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7,)\\
\beta(\mathbf{x}) &= (x_1,-x_2,-x_3,x_4,x_5,\tfrac12-x_6,-x_7)\\
\gamma(\mathbf{x}) &= (-x_1,x_2,-x_3,x_4,\tfrac12-x_5,x_6,\tfrac12-x_7)
\end{aligned}$$

FIGURE 6. The Fano Proceedings [76]

While α, β, γ each fix 16 tori T^3, $\beta\gamma, \gamma\alpha, \alpha\beta, \alpha\beta\gamma$ have no fixed points. The singular set of T^7/Γ consists of 12 disjoint 3-tori T^3 each with normal space $\mathbb{C}^2/\pm 1$. Each $\mathbb{Z}_2$ acts within $\mathrm{SU}(2)\times\{e\}$ on $\mathbb{R}^4 \oplus \mathbb{R}^3 = \mathbb{R}^7$ for different $\mathbb{R}^4$'s, so $\Gamma \subset \mathrm{G}_2$.

In the example T^7/Γ, one resolves each singular point by replacing its transverse neighbourhood by an open subset of the ALE space. Each singular T^3 is then surrounded by a "tube" $\mathbb{RP}^3 \times T^3$ outside of which the ALE metric merges with the flat one. In this way, one can define a smooth 7-manifold with a 1-parameter family of *closed* 3-forms φ_t of G_2 type for $t > 0$.

An analytic theorem is needed to deform φ_t into a 3-form with holonomy in G_2. The situation is controlled by the existence of a 3-form ψ_t for which $d *_t \varphi_t = d *_t \psi_t$, where

$$\|\psi_t\|_{L^2} \lesssim t^4, \quad \|\psi_t\|_{C^0} \lesssim t^3, \quad \|d *_t \varphi_t\|_{L^{14}} \lesssim t^{16/7},$$

whilst the metric determined by φ_t has injectivity radius at least of order t and curvature bounded above by t^{-2}.

Theorem 5.3 ([63]). *In this and similar situations, for sufficiently small* t, *there exists a* G_2 *3-form* $\tilde{\varphi} = \varphi_t + d\eta$ *such that* $\nabla\varphi = 0$.

Once one knows that the holonomy lies in G_2, it will *equal* to G_2 if and only if π_1 is finite. Example 5.2 gives rise to a simply-connected manifold admitting a metric with holonomy G_2, and with $b_2 = 12$ and $b_3 = 43$. The latter is the dimension of the moduli space of G_2 metrics up to diffeomorphism.

Joyce further developed this technique in [63]. He was able to resolve more complicated singularties, and realize many diffeomorphism classes of 7-manifolds with holonomy G_2. A partial analysis of relevant finite group actions was carried out in unpublished work by C. Nunan.

5.3. **The 21st century approach.** We now turn to a discussion of more recent "G_2" activity. The start of this was (by coincidence) marked by the 2002 Fano Conference in Turin, which I was fortunate to attend, though it was also the last occasion many of us had to meet Andrei Tyurin.

At the time, I was more interested in Fano manifolds (of dimension $2n+1$) generalizing twistor spaces of quaternion-Kähler manifolds. Although I was aware of Kovalev's work, I did not predict the importance that Fano and related 3-folds would have for stimulating the construction and classification of compact manifolds with exceptional holonomy.

A *Fano* 3-fold is a projective variety with ample anti-canonical bundle $\overline{\mathcal{K}}$. There are 105 deformation types, of which 17 have $b_2 = 1$ (including of course a smooth quadric, cubic, quartic in $\mathbb{CP}^4$). A table of the 88 Fano 3-folds with $b_2 \geqslant 2$ is described by Mori and Mukai in the Fano proceedings [76]; one case had previously been omitted. A Fano 3-fold with $b_2 \geqslant 2$ that is not the blow up of another along a curve is a conic bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$, or fibres over a curve. For an original discussion of Fano 3-folds case by case, by means of their mirrors, see [32].

A *weak Fano* 3-fold Y is a generalization (requiring $\overline{\mathcal{K}}$ to be nef and $\overline{\mathcal{K}}^3 = 2g-2 > 0$) that comes equipped with a resolution

$$\rho\colon Y \to X$$

where X is a (possibly) singular Fano 3-fold. If ρ is semi-small (meaning that no divisor maps to a point) then Y is said to be *semi-Fano*.

Example 5.4. An important case is that in which X is *nodal*, with a finite number of ordinary double points. Then ρ replaces each such point by a rational curve $\mathbb{CP}^1$ with normal bundle $\nu \cong \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. A general quartic X containing a plane Π in $\mathbb{CP}^4$ has 9 nodes on Π, and two small projective resolutions $Y \to X$ with Y semi-Fano.

Fano and semi-Fano 3-folds can be used to construct spaces that are *asymptotically cylindrical Calabi-Yau (ACCY)*. By blowing up a semi-Fano 3-fold Y along a curve (the base locus of a pencil of anti-canonical divisors), one obtains a smooth 3-fold Z with a morphism

$$f\colon Z \to \mathbb{CP}^1$$

such that $f^{-1}(\infty) = \mathscr{K} \in |\overline{\mathcal{K}}|$ is a smooth K3 surface that generates $H^2(Z,\mathbb{Z})$. This ensures that the normal bundle of $\mathscr{K}$ in Z is trivial, and topologically

$$V = Z \setminus \mathscr{K} \;\supset\; \mathscr{K} \times S^1 \times \mathbb{R}^+.$$

By construction, Z admits a holomorphic 3-form with a simple pole along $\mathscr{K}$. The following result is in the spirit of work of Tian and Yau [90] on the existence of complete Ricci-flat Kähler metrics on quasi-projective varieties.

Theorem 5.5 ([67, 54]). *$V = Z \setminus \mathscr{K}$ is ACCY: it possesses a Ricci-flat metric with holonomy $\mathrm{SU}(3)$ that approximates the product metric with holonomy $\mathrm{SU}(2) \times \{e\}$ on the cylinder, with all derivatives bounded by $e^{-\lambda t}$ (for some $\lambda > 0$) as $t \to \infty$.*

The *semi* (as opposed to *weak*) Fano condition gives better control over the deformation theory of Z equipped with an anti-canonical divisor, and consequently the hyperkähler structures induced on $\mathscr{K}$. This is important for solving the matching problem described next.

5.4. **Twisted connect sums.** Take two ACCY 3-folds $V_\pm$, as constructed above. The trick is to identify the 7-manifolds $V_\pm \times S^1$ by gluing along the neck using an isometry r that performs a "hyperkähler rotation" to mimic a switch of S^1 factors:

$$\begin{array}{ccccc}
\mathscr{K}_+ & \times & S^1 \times S^1 & \times & (T, T{+}1)\\
\downarrow r & & \times & & \downarrow - \\
\mathscr{K}_- & \times & S^1 \times S^1 & \times & (T, T{+}1)
\end{array}$$

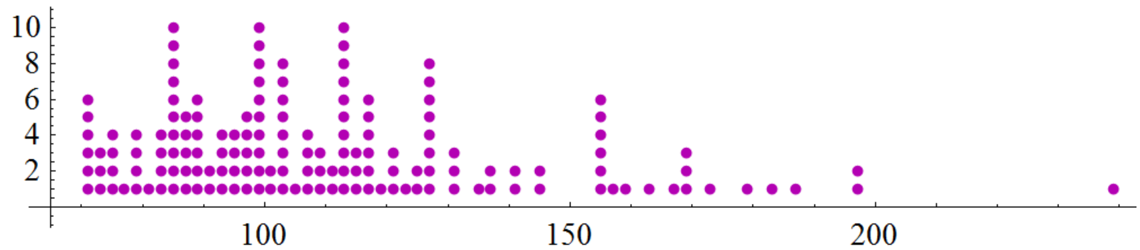


FIGURE 7. G_2 manifolds plotted by b_3

This yields a compact simply-connected 7-manifold M. By the Torelli theorem, the isometry r is determined by the map

$$r^*\colon H^2(\mathscr{K}_-,\mathbb{Z}) \to H^2(\mathscr{K}_+,\mathbb{Z})$$

between the K3 lattices, see Example 1.1.

The theory has been developed by Corti, Haskins, Kovalev, Lee, Nordström and Pacini [38, 39, 69, 68]. The gluing ensures that the underlying G_2 structures on each

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3$$

are identified, and M acquires a *closed* G_2 form φ satisfying

$$\|\nabla^k(d*\varphi)\| = O(e^{-\lambda T}), \quad \forall k.$$

For sufficiently large T, φ can be perturbed (in its cohomology class) as in Joyce's theory, so that $\nabla\tilde{\varphi} = 0$.

The enumeration of examples requires "building-block" 3-folds $Z_\pm$ for ACCY's to glue together. There is at least one for almost all Fano or semi-Fano 3-folds. Now, $H^2(Z_\pm,\mathbb{Z})$ defines a lattice $N_\pm$ in $\mathrm{Pic}(\mathscr{K}) \subseteq H^2(\mathscr{K},\mathbb{Z})$ (cf. Example 1.1). The most straightforward case is that in which N_+, N_- are "orthogonal', which is easy to arrange if $\mathrm{rk}N_+ + \mathrm{rk}N_- \leqslant 11$.

There is no problem matching the 17 Fano 3-folds with $b_2 = 1$. The $153 = \binom{18}{2}$ choices give at least 82 topologically distinct smooth manifolds M with holonomy G_2, plotted by $b_3(M) = 23 + b$ in Figure 7. Here b is even and $32 \leqslant b \leqslant 174$ (with almost all values realized) or $b = 216$. More generally, 5401 of the $5564 = \binom{106}{2}$ pairs taken from all 105 deformation classes can be matched and have $b_2 = 0$. But one finds at most 560 diffeomorphism classes (which are determined by b_3 and the divisibility of p_1). Using one of 75 known ACCY 3-folds of "non-symplectic type" defined directy from a K3 surface [68] is one way of generating G_2 manifolds with $b_2 > 0$.

Toric Fano 3-folds correspond to 3-dimensional reflexive polytopes. There are 4 319 of these, including 18 from smooth Fano 3-folds and 82 terminal (with nodal singularities).

Theorem 5.6 ([39]). *There are 526 130 isomorphism classes of smooth toric semi-Fano 3-folds (1 009 with nodal base), and 435 459 are rigid.*

Of these, 39,584 pairs satisfy $\mathrm{rk}N_+ + \mathrm{rk}N_- \leqslant 11$ but give no new b_3's. Restricting one summand to semi-Fano's of rank 1 and 2, one can generate at least

$$246\,446.(17{+}36{+}150) = 50\,027\,726$$

types of metrics with holonomy G_2 on 7-manifolds with $b_2 = 0$ and $55 \leqslant b_3 \leqslant 287$. This bound may be exceeded by matching in the weak Fano category and using methods of [17].

The analysis above suggests that a given smooth 7-manifold could admit many different types of metrics with holonomy G_2, which may or may not define G_2 structures of the same homotopy type, and may or may not be deformation equivalent. This leads to some exciting questions that are outlined below.

5.5. **Concluding remarks.** Whilst there is an expectation that many of the constructions for G_2 holonomy can be carried over in one dimension higher (and have been [61, 65, 29]), it is true to say that to date the $\mathrm{Spin}(7)$ theory is less advanced. There has been less motivation from string theorists to develop the theory.

The defining structures are rather different: a 4-form Φ defining a $\mathrm{Spin}(7)$ structure is far from being stable, since the dimension of $\mathrm{GL}(8,\mathbb{R})/\mathrm{Spin}(7)$ is 43. On the other, its closure is enough to guarantee the reduction of holonomy. Any oriented spin 7-manifold admits a G_2 structure, but there is a well-known constraint in the 8-dimensional case involving the Euler characteristic χ and Pontrjagin numbers:

Lemma 5.7. *If a closed 8-manifold has a* $\mathrm{Spin}(7)$ *or* $\mathrm{Sp}(2)\mathrm{Sp}(1)$ *structure then* $8\chi = 7p_1^2 - 4p_2$.

This simple result is another that unites two special structure groups in 8 dimensions, though holonomy reductions impose stronger constraints since $\frac{1}{24}(7p_1^2 - 4p_2)$ can also be expressed as $\sigma - 16\widehat{\mathrm{A}}$ where σ is the signature and $\widehat{\mathrm{A}}$ the A-hat genus. The latter is the index of the Dirac operator when the closed 8-manifold W is spin, and must equal 1 if W admits a metric with holonomy *equal* to $\mathrm{Spin}(7)$. In this case, W is simply-connected and its Betti numbers satisfy

$$b_4^+ - 2b_4^- + b_3 - b_2 = 25.$$

This provides a useful constraint that can be checked when using the techniques of Example 5.2 to resolve orbifolds T^8/Γ with a $\mathrm{Spin}(7)$ structure.

Example 5.8. $\mathbb{HP}^2$ and $\mathrm{G}_2/\mathrm{SO}(4)$ both have $\sigma = 1$. Since they carry metrics of positive scalar curvature, they have $\widehat{\mathrm{A}} = 0$ and cannot admit a metric with holonomy $\mathrm{Spin}(7)$ or a subgroup thereof. Both admit $\mathrm{Spin}(7)$ structures. For example, the positive spin representation of $\mathrm{Spin}(8)$ decomposes as

$$\Delta_+ \cong S^2H \oplus \Lambda_0^2 E$$

relative to its subgroup $\mathrm{Sp}(2)\mathrm{Sp}(1)$, in standard notation. The rank 5 vector bundle with fibre $\Lambda_0^2 E$ admits a family of nowhere-zero sections over $\mathbb{HP}^2$ (cf. [42, Remark 2.6]), each of which defines a reduction to a $\mathbb{Z}_2$ quotient of $\mathrm{Sp}(1)^3$.

There remain fascinating questions concerning compact manifolds with holonomy $\mathrm{Spin}(7)$, yet the proposition above has important consequences for G_2. Let M be a closed connected 7-manifold with a spin structure (i.e. $w_1 = 0 = w_2$). Any such manifold admits a G_2 structure (in the non-holonomy sense) and can be represented as the boundary of a manifold W^8 with a $\mathrm{Spin}(7)$ structure. This allowed Crowley and Nordström to define an invariant

$$\nu(M) = \chi(W) - 3\sigma(W) \mod 48$$

of the G_2 structure, helping to understand the set $\pi_0(\mathcal{G}_2(M)/\mathrm{Diff}(M))$ of deformation classes of such structures. Here $\mathcal{G}_2(M)$ denotes the "naïve" set of G_2 structures, each of which can be identified with a unit section (up to sign) of the spin bundle, and the set $\pi_0(\mathcal{G}_2(M))$ of its homotopy classes is parametrized by $\mathbb{Z}$.

Example 5.9 ([42]). There is a "parity check" for the invariant, namely $\nu(M) = 1 + b_1 + b_2 + b_3 \mod 2$. The standard structure on $S^7 = \mathrm{Spin}(7)/G_2$ has $\nu = 1$. The canonical bundle $\mathcal{K} = \mathscr{O}(-4)$ over $\mathbb{CP}^3$ has a natural $\mathrm{SU}(4)$ structure and the G_2 structure induced on its unit circle bundle (isomorphic to $\mathrm{Spin}(7)/(\mathrm{G}_2 \times \mathbb{Z}_4)$) has $\nu = 7$. The G_2 structure induced on S^7 by the construction [23] over S^4 has $\nu = -1$.

The definition of ν makes it easier to formulate questions raised above:

- How many deformation classes of G_2 structures are there on a given smooth manifold?
- Which (if any) admit a metric with holonomy G_2?
- In each such class, is the moduli space of holonomy G_2 metrics connected?

It follows from the definition of ν (and the fact that reversing the sign of the 3-form reverses the sign of ν) that there are at least 24 deformation types. Concerning the second question, it turns out that manifolds constructed from twisted connected sums have $\nu = 24$, but a refinement of the Eells-Kuiper invariant can distinguish matchings [42].

Thanks to additional results of Crowley and Nordström, the existence is now known of pairs of metrics with holonomy G_2 that:

- exist on homeomorphic non-diffeomorphic manifolds [41],
- are disconnected within the same homotopy class of G_2 structures [40].

The non-diffeomorphic examples are formed by pairwise matchings from a triple of Fano 3-folds with $b_2 = 2$ obtained by blowing up $\mathbb{CP}^3$ (or a quadric in $\mathbb{CP}^4$) along a curve. The second result on G_2 moduli is accomplished by lifting ν to an integer invariant defined (using the Atiyah-Patodi-Singer η-invariant for manifolds with boundary) that is locally constant for a family of metrics with *holonomy* G_2. By contrast, any two homotopic G_2 structures with closed 4-form are always connected within that category.

Let us close by mentioning the following topics, the subject of parallel investigations:

• Calibrated submanifolds in manifolds of exceptional holonomy. This means associative 3-folds or coassociative 4-folds in G_2 manifolds, and Cayley 4-folds in $\mathrm{Spin}(7)$ manifolds. The last two give rise in principle to fibrations by K3 surfaces, and generalize the Strominger-Yau-Zaslow approach to mirror symmetry [89]. The twisted connect sum can produce G_2 manifolds with a finite number of rigid compact associative submanifolds diffeomorphic to $S^1 \times S^2$ [38]. The weakening of the Fano condition is crucial here, since the associatives arise from complex curves C in a semi-Fano manifold that satisfy $\mathcal{K} \cdot C = 0$. Moreover, it is possible to glue two K3 fibrations so as to obtain a *co*associative fibration of a manifold with G_2 holonomy over a 3-dimensional sphere [66], a situation that is studied in [44].

• Analogues of Ricci flow for metrics with exceptional holonomy. The Laplacian flow for G_2 structures was introduced by Bryant [20, 24]. When restricted to a closed G_2 3-forms, this takes the form

$$\frac{\partial \varphi}{\partial t} = -d * d * \varphi,$$

and can be regarded as a gradient flow for the volume functional defined by Hitchin [58]. Lotay and Wei have established convergence to holonomy G_2 starting from a closed 3-form with small torsion on a compact manifold (involving a C^9 estimate), in analogy to Theorem 5.3. Homogeneous cases of the Laplacian flow have been examined in [71, 45].

• A Lie-theoretic account of instantons defined by a G structure was given in [80]. Instantons over manifolds with holonomy G_2, in both the compact and the explicit setting are now being studied [81, 30, 73]. The current volume contains a paper on the last topic, and some recent conjectures appear in [55].

Examples described earlier relating structures in dimensions $5, 6, 7, 8$ suggest that one should not consider one dimension in isolation. As in the classification of Fano varieties, there are likely to be underlying theories that transcend dimensions.

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