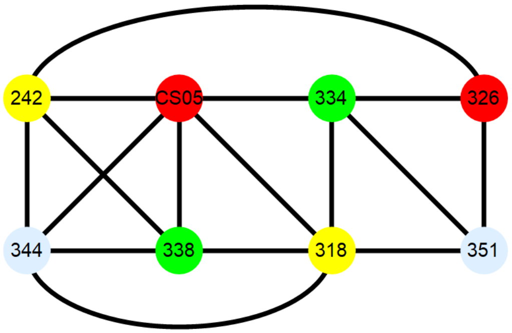
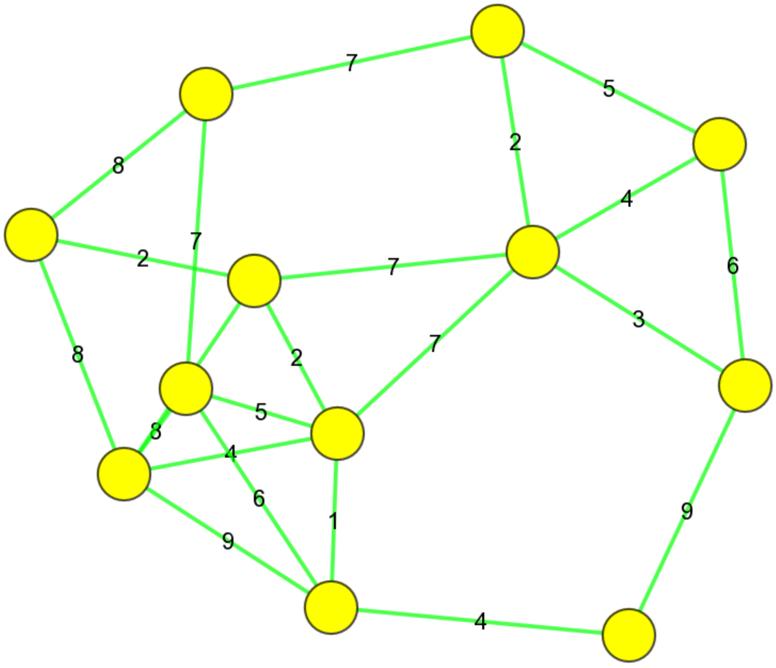


251 DISCRETE MATHEMATICS



Simon Salamon

Preface

This is a concatenated and reorganized version of the 2019/20 lecture notes for the modules 5ccm251a and 6ccm251b.

It is likely that these notes will be updated weel by week as the module is taught online in the winter/spring of 2021. For this reason, it is not recommended that they be printed.

I take this opportunity to thank Naz Mihesi for his ongoing support in running the course.

The module was successfully taught for many years by Simon Fairthorne, and the current material is based on the module as he designed it. These lecture notes are dedicated to his memory.

Simon Salamon

28 December 2020

Contents

1. Arithmetic

Notation. We will use the sets

$$\begin{aligned}\mathbb{N} &= \{0, 1, 2, , 3, \dots\}\\ \mathbb{Z} &= \{0, 1, -1, 2, -2, \dots\}.\end{aligned}$$

1.1. Induction

First principle. Let $\mathscr{P}(n)$ be some statement that makes sense for all $n \geqslant n_0$. (Typically, $n_0 = 0$, 1 or 2.) Suppose that

(1) $\mathscr{P}(n_0)$ is true, and
(2) for all $n \geqslant n_0$, $\mathscr{P}(n)$ is true $\Rightarrow$ $\mathscr{P}(n+1)$ is true.

Then $\mathscr{P}(n)$ is true for all n.

Example. $\mathscr{P}(n)$ is the assertion that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Take $n_0 = 1$.
(1) $\mathscr{P}(1)$ is true because both sides equal 1.
(2) Now suppose that $\mathscr{P}(n)$ is true, and add $2n+1$ to both sides above to give

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1).$$

The right-hand side simplifies to $(n+1)^2$, so this is assertion $\mathscr{P}(n+1)$.
Therefore $\mathscr{P}(n)$ must be true for all $n \geqslant 1$.

Note. Curly $\mathscr{P}$ emphasizes that $\mathscr{P}$ is a statement, not an arithmetical function.

Second principle. Same start as above. Suppose that

(1) $\mathscr{P}(n_0)$ is true, and
(2') for all $n \geqslant n_0$, $\mathscr{P}(k)$ is true for all $n_0 \leqslant k < n \Rightarrow \mathscr{P}(n)$ is true.

Then $\mathscr{P}(n)$ is true for all n.

Example. Take $n_0 = 2$. $\mathscr{P}(n)$ is the assertion "n can be written as a product of (one or more) prime numbers".
(1) 2 is a prime number, so obviously $\mathscr{P}(2)$ is true.
(2') (i) If n is prime, then $\mathscr{P}(n)$ is already true. (ii) If not, then n has a divisor other than 1 and n, so we can write $n = ab$ with $1 < a < n$ and $1 < b < n$. If $\mathscr{P}(k)$ is true for all $k < n$ then $\mathscr{P}(a)$ and $\mathscr{P}(b)$ are both true, which means that a is a prime or a product of primes, and b similarly. The same must be true of ab, and $\mathscr{P}(n)$ is true.
Therefore any integer $n \geqslant 2$ is a product of primes.

Summary. Use the first principle when $\mathscr{P}(n+1)$ appears to depend only on $\mathscr{P}(n)$. The second is needed when $\mathscr{P}(n)$ or $\mathscr{P}(n+1)$ depends on more than one predecessor. But sometimes it becomes necessary to check more than one initial value.

Example. Prove that $a_n = 2^n + (-3)^n$ is a solution of

$$\begin{cases} a_n = 6a_{n-2} - a_{n-1}, & n \geqslant 2 \\ a_0 = 2,\ a_1 = -1. \end{cases}$$

We use the second principle with $\mathscr{P}(n)$ the assertion "$a_n = 2^n + (-3)^n$" for $n \geqslant 0$. Then $\mathscr{P}(0)$ is true, since $2^0 + (-3)^0 = 2$. Now assume that $\mathscr{P}(k)$ is true for all $k \leqslant n$. Then

$$\begin{aligned} a_n &= 6a_{n-2} = a_{n-1} \\ &= 6[2^{n-2} + (-3)^{n-2}] - [2^{n-1} + (-3)^{n-1}] \\ &= 6[2^{n-2} + (-3)^{n-2}] - [2 * 2^{n-2} - 3 * (-3)^{n-2}] \\ &= 4 * 2^{n-2} + 9^{n-2} \\ &= 2^n + (-3)^n, \end{aligned}$$

provided $n \geqslant 2$ (for the second line). The punch line is that we need to check $\mathscr{P}(1)$ separately, which is easily done: $2^1 + (-3)^1 = 2 - 3 = -1$. Thus, $\mathscr{P}(n)$ is true for all n.

Notation. To avoid confusion, we shall often indicate multiplication between actual numbers by $*$ as in common software.

1.2. Divisibility

Notation. Let $m, n \in \mathbb{Z}$. One says that m divides n, abbreviated to $m \mid n$ if there exists an integer q such that $mq = n$. For example,

$$13 \mid 0, \qquad \text{but} \qquad 0 \nmid 13.$$

Here is a formal

Definition. A positive integer $p \geqslant 2$ is a *prime number* if $a \in \mathbb{N},\ a \mid p \Rightarrow a = 1$ or $a = p$.

Let a be any integer, and b a *positive* integer. *There exist integers* q, r *such that*

$$a = qb + r, \qquad 0 \leqslant r < b.$$

We shall take the validity of this statement for granted. It is sometimes called the *Division Algorithm*, and one can imagine a mechanical way of finding the *quotient* q and the *remainder* r. Note that $b \mid a$ if and only if $r = 0$.

Examples.

$$\begin{aligned} 23 &= 4 * 5 + 3 \\ -17 &= (-4) * 5 + 3 \\ 510510 &= 510 * 1001 + 0 \\ 104729 &= 104 * 999 + 833. \end{aligned}$$

One uses notation like

$$a = r \bmod b, \qquad \text{or} \qquad a \equiv r\ (b),$$

or a mixture. We shall adopt the former, so for example

$$104729 = 833 \bmod 999, \quad \text{also} \quad 104729 = 1 \bmod 104.$$

Greatest common divisor. Let a, b be integers. Then $\gcd(a, b)$ is the largest positive integer that divides both a and b. It is undefined when $a = b = 0$. It is the same as $\mathrm{hcf}(a, b)$, the *highest common factor* of a and b, and is sometimes abbreviated to (a, b).

If $\gcd(a, b) = 1$ then a and b are called *coprime*.

Examples.

$$\begin{aligned}
\gcd(24, 15) &= 3 \\
\gcd(6, 0) &= 6 \\
\gcd(-12, -24) &= 12 \\
\gcd(510510, 44) &= 22 \\
\gcd(104729, 10^9) &= 1.
\end{aligned}$$

Proposition. There exist integers x, y such that $\gcd(a, b) = xa + by$.

Later, we shall recall Euclid's algorithm that determines x and y. The proposition has the following consequences:

Corollary 1. If m is *any* divisor of a and b and $n = \gcd(a, b)$ then m divides n.

Proof. This follows immediately from the formula $n = xa + yb$, since m must divide the right-hand side. $\square$

Corollary 2. Let p be a prime number. Then

$$p \mid mn \quad \Rightarrow \quad p \mid m \quad \text{or} \quad p \mid n.$$

Proof. Suppose that $p \nmid m$. Then $(p, m) = 1$, since the only divisors of p are 1 and p, but the latter does not divide m. So we can write $1 = xp + ym$. Thus

$$n = xpn + ymn,$$

and (since p divides both terms on the right-hand side) $p \mid n$. $\square$

1.3. Modular arithmetic

Definiton. We say that a_1 and a_2 are *congruent* (or *equal*) *modulo* n if n divides $a_1 - a_2$. In symbols,

$$a_1 = a_2 \bmod n \quad \Leftrightarrow \quad n \mid (a_1 - a_2).$$

We'll sometimes write $a_1 \equiv a_2$ if n has been fixed in advance.

Because of the division algorithm (with $b = n$) we know that any integer is equal modulo n to some remainder r in

$$R = \{0, 1, 2, \ldots, n-1\}.$$

We can define addition and multiplication on this set by taking remainders modulo n, like on a clockface.

Example. With $n = 7$

$$\begin{aligned} 3+5 &= 1 \bmod 7 \\ 3*5 &= 1 \bmod 7 \\ 6*6 &= 1 \bmod 7 \\ 6 &= -1 \bmod 7 \end{aligned}$$

When we are working modulo n, an element $r \in R$ really represents *all* integers obtained from r by adding or subtracting multiples of n, i.e. it represents the *set*

$$\{r + kn : k \in \mathbb{Z}\} = r + n\mathbb{Z}.$$

In the laguange of abstract algebra, $\mathbb{Z}$ is a ring, $n\mathbb{Z}$ is an *ideal*, and $R = \mathbb{Z}/n\mathbb{Z}$ is the *quotient ring* each of whose elements is a *coset* $r + n\mathbb{Z}$.

Since R is a ring, almost all the usual laws of arithmetic apply: if $a = b \bmod n$ then

$$a + c = b + c, \quad ac = bc, \quad a^2 = b^2, \ \ldots \qquad \bmod n.$$

Beware though that one can have divisors of zero: the statement

$$ab = 0 \quad \Rightarrow \quad a = 0 \text{ or } b = 0 \bmod n$$

is *false* in general. For example, $2 * 3 = 0 \bmod 6$. But it is true if n is a prime number:

Proposition. Suppose that $n = p$ is a prime number, and that p does not divide a. Then a has an inverse modulo p.

Proof. By assumption, $\gcd(a, p) = 1$ since the only factors of p are 1 and p, and $p \nmid a$. By §1.2, we know that $xa + yp = 1$ for some $x, y \in \mathbb{Z}$. It follows that $xa = 1 \bmod n$, and we can suppose that $0 < x < n$. □

Example. To perform a sequence of operations, take remainders at each stage. Compute $15^8 \bmod 16$. Note that $15 = -1 \bmod 16$, so $15^8 = (-1)^8 = 1 \bmod 16$.

Example. Solve $2x = 2 \bmod 16$. This means

$$2x = 2 + 16k,$$

so $x = 1 + 8k$. There are two solutions modulo 16, namely 1 and $9 \equiv -7$.

Let $p \geqslant 2$ be a prime number. Then

$$R^* = R \setminus \{0\} = \{1, 2, \ldots p-1\}$$

is a *group* under multiplication modulo p, and R itself is a *field* (a ring in which multiplication is commutative and has inverses). Let a be an integer that is not a muliple of p. Its remainder modulo p is an element of R^*, whose order (by Cauchy's theorem) divides $p-1$. This implies

Fermat's little theorem. If p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \bmod p$.

We can include the possibility that $p \mid a$ by simply multiplying both sides by a:

$$a^p \equiv a \bmod p, \qquad \forall a \in \mathbb{Z}.$$

Examples. Taking $p = 11$ and $a = 2$ gives

$$2^{10} = 1 \bmod 11,$$

which is easy to check immediately as $2^{10} = 1024$.

Note that

$$8^8 \equiv 1 \bmod 9,$$

because $8^8 \equiv (-1)^8 \bmod 9$, so taking $a = 8$ and $p = 9$ satisfies Fermat's little equation, even though p is not prime. Even better:

Example. Let $n = 561$, which is certainly not prime. Then it is known that

$$a^{561} = a \bmod n, \qquad \text{for } \textit{all } a \in \mathbb{Z},$$

which makes 561 a *Carmichael number* (it is the first).

Proposition. If p is prime, the only solutions of $x^2 = 1 \bmod p$ are $x \equiv 1$ and $x \equiv -1$.

Proof. $x^2 = 1 \bmod p$ means $p \mid (x^2 - 1)$, so

$$p \mid (x-1)(x+1).$$

By an earlier corollary, p must divide at least one of these factors. If $p \mid (x-1)$ then $x \equiv 1$, whereas $p \mid (x+1)$ implies $x \equiv -1$. $\square$

For example, modulo 7, we know that $a^6 \equiv 1$. A solution of $x^2 = a^6$ is $x = a^3$ and we observe that

$$1^3 \equiv 1, \quad 2^3 \equiv 1, \quad 3^3 \equiv -1, \quad 4^3 \equiv 1, \quad 5^3 \equiv -1, \quad 6^3 \equiv -1.$$

1.4. Binary expansions

To find the decimal expansion of an integer, we repeatedly divide by 10, and read the remainders from bottom to top. For example,

$$\begin{aligned} 327 &= 32 * 10 + \boxed{7} \\ 32 &= 3 * 10 + \boxed{2} \\ 3 &= 0 * 10 + \boxed{3} \end{aligned}$$

The same process works in base 2 (binary)

$$\begin{aligned}
39 &= 19 * 2 + \boxed{1} \\
19 &= 9 * 2 + \boxed{1} \\
9 &= 4 * 2 + \boxed{1} \\
4 &= 2 * 2 + \boxed{0} \\
2 &= 1 * 2 + \boxed{0} \\
1 &= 0 * 2 + \boxed{1}.
\end{aligned}$$

Therefore

$$39 = 100111_2,$$

which is correct since $39 = 2^5 + 7 = 100000_2 + 111_2$. On a computer, 6 bits are needed to represent 39.

Recall the concept of *logarithm to base* b. It is the inverse to exponentiation:

$$\text{if } y = b^x \quad \text{then} \quad x = \log_b y.$$

We write

$$\ln y = \log_e y, \qquad \lg y = \log_2 y,$$

where

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7182818\ldots$$

It is easy to show that

$$\lg y = \log_2 y = \frac{\ln y}{\ln 2} \approx 1.44 \ln y.$$

In this course, we shall only use logarithms to base 2. Here are the key properties:

- $\lg(2^x) = x$
- $2^{\lg y} = y$
- lg is strictly increasing: $a < b \Rightarrow \lg a < \lg b$.

Suppose that n is trapped between two powers of 2:

$$\begin{aligned} 2^k \leqslant n < 2^{k+1}, & \qquad \text{so} \\ k \leqslant \lg n < k+1. & \end{aligned}$$

It follows that the "floor" of $\lg n$ equals k:

$$\lfloor \lg n \rfloor = k.$$

Here "floor" means *the largest integer less than or equal to*. Observe that

$$2^{k+1} - 1 = \underbrace{11 \cdots 1}_{k+1}$$

is the largest binary number that can be represented with $k+1$ bits: we need $k+1 = \lfloor \lg n \rfloor + 1$ bits to represent n.

Example. How many bits are needed to represent $n = 8293417$? We must trap n between two powers of 2. For this purpose it is useful to know that

$$10^3 \simeq 2^{10}.$$

We can easily calculate

$$\begin{aligned} 2^{20} &= (2^{10})^2 \\ &= (1024)^2 \\ &= 1048576. \end{aligned}$$

It follows easily that

$$2^{22} < n < 2^{23},$$

and 23 bits are needed. In fact,

$$n = 11111101000110000101001_2.$$

Example. On the piano, 7 octaves are equivalent to 12 perfect fifths. We can write

$$2^7 \approx (3/2)^{12}, \quad \text{so} \quad 2^{19} \approx 3^{12},$$

where $\approx$ means 'approximately equal'. Which side is greater? Musically, the approximation can be resolved by making every semitone correspond to an interval of $2^{1/12}$, so that a "perfect" fifth corresponds to the ratio $2^{7/12} \approx 1.498\ldots$ This is the *equal temperament* system of tuning keyboard instruments, a concept dating back to 1584 or earlier.

2. Recurrence relations

2.1. Recursive functions

In this section, we shall be dealing with functions $f\colon \mathbb{N} \to \mathbb{N}$. We are used to having such functions defined explicitly, such as

$$f(n) = 3^n + (-2)^n - \tfrac{1}{6}n - \tfrac{13}{36}.$$

But one can also define functions in terms of earlier values, using a prescription like

$$f(n) = \begin{cases} 0 & \text{if} \quad n = 0 \\ 3f(n-1)+1 & \text{if} \quad n \geqslant 1. \end{cases}$$

This gives the table

n	0	1	2	3	4
$f(n)$	0	1	4	13	40
$2f(n)$	0	2	8	26	80
$2f(n)+1$	1	3	9	27	81

from which we might guess that

$$f(n) = \tfrac{1}{2}(3^n - 1).$$

This can be proved by induction. But such explicit formulae are often not possible.

Example. Define $g\colon \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \begin{cases} n & \text{if } n = 0, 1 \\ g(n/2)+1 & \text{if } n \geqslant 2 \text{ is even} \\ g(3n+1)+1 & \text{if } n \geqslant 3 \text{ is odd} \end{cases}$$

This function is related to the so-called *Collatz conjecture* or $3n+1$ *problem*. In fact, $g(n)-1$ is the 'stopping time' for the Collatz map

$$f(n) = \begin{cases} n/2 & \text{if } n \geqslant 2 \text{ is even} \\ 3n+1 & \text{if } n \geqslant 3 \text{ is odd;} \end{cases}$$

it equals the number of applications of f needed to reach 1. Note that one can only reach 1 through the sequence $16, 8, 4, 2, 1$ though one can reach 16 from both 32 and 5. It is unknown if the stopping time is finite for every integer $n \geqslant 2$, but it has been checked for all integers up to at least 2^{60}. Of course, $g(2^k) = k+1$ since f keeps halving.

Let us compute $g(7)$; this is done by recording a somewhat tortuous succession of equations so as to arrive at $g(1)$, and then backtracking with the values:

$$
\begin{array}{lrcl@{\qquad\qquad\qquad}rrcrl}
 & g(1) & = & 1 & & & & & \\
 & g(2) & = & g(1)+1 & \Rightarrow & g(2) & = & 2 & \\
 & g(4) & = & g(2)+1 & & g(4) & = & 3 & \\
 & g(8) & = & g(4)+1 & & g(8) & = & 4 & \\
 & g(16) & = & g(8)+1 & & g(16) & = & 5 & \\
 & g(5) & = & g(16)+1 & & g(5) & = & 6 & \\
 & g(10) & = & g(5)+1 & & g(10) & = & 7 & \\
 & g(20) & = & g(10)+1 & & g(20) & = & 8 & \\
 & g(40) & = & g(20)+1 & & g(40) & = & 9 & \\
 & g(13) & = & g(40)+1 & & g(13) & = & 10 & \\
 & g(26) & = & g(13)+1 & & g(26) & = & 11 & \\
 & g(52) & = & g(26)+1 & & g(52) & = & 12 & \\
 & g(17) & = & g(52)+1 & & g(17) & = & 13 & \\
 & g(34) & = & g(17)+1 & & g(34) & = & 14 & \\
 & g(11) & = & g(34)+1 & & g(11) & = & 15 & \\
\uparrow & g(22) & = & g(11)+1 & & g(22) & = & 16 & \downarrow \\
\text{start} & g(7) & = & g(22)+1 & & g(7) & = & 17 & \text{end}
\end{array}
$$

We do not know the answer until we have got all the way to the top (and $g(1)$) and then back down again on the right, to find that $g(7) = 17$.

This set-up is called a *stack*, since the left-hand column resembles a stack of trays in a cafeteria (which is why we started at the bottom): $g(5)$ went in first, and $g(1)$ last. Then we could retrieve $g(1)$ first and $g(5)$ last. This illustrates the principle 'Last In First Out' or LIFO. Later on, we shall meet a different set-up, called a *queue*, in which the first item in is the first to be processed. So 'First In First Out' or FIFO (like most underground lifts).

Here is a table of values of $g(n)$ for $n = 0, 1, 2, \ldots, 104$:

$0, 1, 2, 8, 3, 6, 9, \underline{17}, 4, 20, 7, 15, 10, 10, 18, 18, 5, 13, 21, 21, 8, 8, 16, 16, 11, 24, 11, 112,$
$19, 19, 19, 107, 6, 27, 14, 14, 22, 22, 22, 35, 9, 110, 9, 30, 17, 17, 17, 105, 12, 25, 25, 25, 12, 12,$
$113, 113, 20, 33, 20, 33, 20, 20, 108, 108, 7, 28, 28, 28, 15, 15, 15, 103, 23, 116, 23, 15, 23, 23,$
$36, 36, 10, 23, 111, 111, 10, 10, 31, 31, 18, 31, 18, 93, 18, 18, 106, 106, 13, 119, 26, 26, 26, 26, 26, 88$

2.2. Fibonacci numbers

Leonardo di Pisa (c. 1175–1250) found his famous sequence of numbers in connection with the breeding of rabbits. One starts with a newly-born pair of rabbits, one male one female. The idealized assumption is that at one month they mature and become fertile, and at two months (and at each month thereafter) the female gives birth to another male-female pair. Let $F_n = F(n)$ denote the total number of rabbit pairs in the middle of the nth month, so $F_1 = F_2 = 1$. Then

$$
\begin{array}{rcccc}
F_n & = & \#\{\text{immature pairs}\} & + & \#\{\text{mature pairs}\} \\
 & = & F_{n-2} & + & F_{n-1}
\end{array}
$$

for $n \geqslant 3$. This is because all pairs from a month ago will be mature, and the newly-born immature rabbits are offspring of parents from two months ago.

To extend this relation to $n = 2$, we can set $F_0 = 0$. We then have the *recurrence relation*

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \quad F_1 = 1,$$

which can be solved recursively. The aim of this section is to show that there is a simple formula for the Fibonacci number F_n. For this purpose, define

$$\varphi = \tfrac{1}{2}(1+\sqrt{5}) = 1.6180\ldots, \qquad \psi = \tfrac{1}{2}(1-\sqrt{5}) = -0.6180\ldots$$

In §2.3, we shall show that

Proposition. $F_n = \dfrac{1}{\sqrt{5}}(\varphi^n - \psi^n)$.

Note that φ and ψ are the roots of $x^2 - x - 1 = 0$ or

$$\frac{x}{1} = \frac{1}{x-1},$$

and that φ (the positive root) is the so-called *golden ratio.* We leave proofs of the following statements as exercises.

Corollary 1. The ratio F_{n+1}/F_n tends to φ as $n \to \infty$.

Corollary 2. F_n is the closest integer to $\varphi^n/\sqrt{5}$ for all n.

One can compute the so-called *generating function* for the sequence (F_n) directly from the recurrence relation. Indeed,

Corollary 3. Suppose that $|x| < 1/\varphi$. Then

$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1-x-x^2}. \tag{1}$$

We can check this by setting $\lambda = x + x^2$ and using the binomial expansion

$$\begin{aligned}(1-\lambda)^{-1} &= 1 + \lambda + \lambda^2 + \lambda^3 + \cdots \\ &= 1 + x + x^2 + (x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6) + (x^4 + \cdots) + \cdots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots\end{aligned}$$

In particular, $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{10^n} = \frac{10}{89} = 0.11235955$.

We shall not be concerned with questions of convergence in this course, but Corollary 1 implies that the radius of convergence of the series (1) equals $1/\varphi$.

A curiosity. Since 1 mile equals 1.609...kilometers, Fibonacci's numbers (if you can remember them) give a sufficiently accurate way of converting. For example, 144 km/h is almost exactly 89 mph.

Example. The sum s_n of the first n odd numbers satisfies an obvious recurrence relation:

$$\begin{cases} s_n = s_{n-1} + 2n - 1, \\ s_1 = 1 \end{cases}$$

We already know that the solution is $s_n = n^2$, but the aim will be to solve such relations systematically without knowing the answer by other means.

Definition. A recurrence relation of *order* k will specify

$$a_n = f(n, a_{n-1}, a_{n-2}, \dots, a_{n-k})$$

as a function of the k preceding values and possibly n itself. One also needs to prescribe k initial values of the function $n \mapsto a_n$.

The relation is called *linear* if the right-hand side equals

$$c_0(n) + c_1(n)a_{n-1} + \cdots + c_k(n)a_{n-k},$$

for some functions $c_i(n)$ of n, as in the previous example. Such a linear relation is called *homogeneous* if $c_0(n)$ is absent, and it has *constant coefficients* if $c_1, \dots, c_k$ are independent of n (so constants). The usual relation described the Fibonacci numbers is therefore or order 2, linear, homogeneous with constant coefficients. By contrast,

$$a_n = a_{n-1} * a_{n-2}$$

also has order 2, but is is not linear (so the other qualifications are irrelevant).

2.3. Constant coefficients

Consider a recurrence relation with constant coefficients:

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + c_0(n), \tag{NH}$$

with $c_0(n)$ a non-zero function. The associated homogeneous relation is

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}, \tag{H}$$

without the term at the end. We are likely to consider only $k \leqslant 3$.

Proposition. (i) If (a_n) and (b_n) are sequences solving (H) (with b_n in place of a_n) then $(Aa_n + Bb_n)$ will also solve (H) for any $A, B \in \mathbb{R}$.
(ii) There are k linearly independent solutions to (H).
(iii) If (a_n) and (b_b) solve (NH) then $(a_n - b_n)$ solves (H).

The proposition is also valid for linear relations, and there is an analogy with ordinary differential equations. We omit the proofs here.

For $k = 2$, 'linearly independent' simply means that one solution is not an overall multiple of the other: sequences with $a_n = n^2$ and $b_n = n^2 + 1$ are independent, but $a_n = n^2$ and $b_n = -7n^2$ are not.

(iii) implies that the general solution of (NH) is *any* particular solution to it plus the general solution of (H).

It is known that solutions of (H) are mostly linear combinations of λ^n, where $\lambda \in \mathbb{R}$ is constant. The next example will verify this.

Example. Solve

$$\begin{cases} a_n = -a_{n-1} + 6a_{n-2}, & n \geqslant 2 \\ a_0 = 2,\ a_1 = -1. \end{cases}$$

Try $a_n = \lambda^n$. Substituting into the recurrence relation,

$$\lambda^n = 6\lambda^{n-2} - \lambda^{n-1},$$

and (since we can assume $\lambda \neq 0$),

$$\lambda^2 + \lambda - 6 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda + 3) = 0.$$

Taking $\lambda = 2$ and $\lambda = 3$ gives two independent solutions, and (from (i) and (ii) above) the genral solution is

$$a_n = A * 2^n + B * (-3)^n.$$

The constants A, B are determined by the initial conditions, which give

$$2 = A * 1 + B * 1, \quad -1 = A * 2 + B * (-3) \quad \Rightarrow \quad A = B = 1.$$

The final answer is therefore $a_n = 2^n + (-3)^n$.

Example. For the Fibonacci sequence, the equation is $\lambda^2 = \lambda + 1$ or $\lambda^2 - \lambda - 1 = 0$, which has roots

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}), \qquad \psi = \tfrac{1}{2}(1 - \sqrt{5}),$$

giving a general solution $A\varphi^n + B\psi^n$. Then A, B are found by solving $0 = F_0$ (which implies $B = -A$) and

$$1 = F_1 = A\varphi + B\psi = A(\varphi - \psi) = A\sqrt{5}.$$

This proves the previous proposition.

To summarize, here is the strategy for solving (H):

Substitute $a_n = \lambda^n$
Obtain a polynomial equation of degree k in λ
Find its roots $\lambda_1, \ldots, \lambda_k$
The general solution is $a_n = A_1\lambda_1^n + \cdots + A_k\lambda_k^n$
Find the constants by solving the initial conditions

There are two possible snags. The roots may be complex, though if the original equation is real, they will always come in complex conjugates, and the choice of constants A_i will ensure that all solutions are real. Or, there may be repeated roots, in which case (by (ii)) there must exist additional solutions.

Example. Express the solution of $a_n = -a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$ in closed form. Of course,

$$(a_n) = (0, 1, 0, -1, 0, -1, 0, \ldots),$$

but we are asked for a formula. We have $\lambda^2 + 1 = 0$ so the roots are $\pm i$ where $i = \sqrt{-1}$. So the solution is $Ai^n + B(-i)^n$, with $A + B = 0$ and $i(A - B) = 1$. Then $A = -B = -\frac{1}{2}i$, and the closed formula is

$$a_n = -\tfrac{1}{2}i(i^n - (-i)^n) = -\tfrac{1}{2}(i^{n+1} + (-i)^{n+1}).$$

In the case of repeated roots, let us consider what happens when the roots are λ and $\lambda + \delta$ for $\delta > 0$. We know from (i) that

$$\frac{(\lambda + \delta)^n - \lambda^n}{\delta}$$

must be a solution. If we let $\delta \to 0$ then in the limit this becomes the derivative of λ^n, namely $n\lambda^{n-1}$. So we expect this (or equivalently $n\lambda^n$) to be a second solution. In fact, a repeated root of multiplicity m will allow us to introduce solutions

$$\lambda^n, \quad n\lambda^n, \quad \ldots \quad , n^{m-1}\lambda^n.$$

Example. Find the general solution of $a_n = 2a_{n-1} - a_{n-2}$. Here, $\lambda^2 - 2\lambda + 1 = 0$ or $(\lambda - 1)^2 = 0$, so we get

$$a_n = A * 1^n + B * n * 1^n = A + Bn.$$

2.4. Particular solutions

To solve a non-homogeneous linear equation (NH), proceed as follows. The order is important:

Find the general solution of (H) with constants
Determine the form of a particular solution of (NH)
Substitute to determine any constants in the particular solution
Add the two solutions
Apply the initial values to determine the constants relating to (H)

We shall mostly see assigned functions of the form

$$c_0(n) = p(n) * \mu^n,$$

where $p(n)$ is a polynomial such as 7 or n^2 or $n^3 - n + 7$. Given such a function, one guesses a solution $q(n) * \mu^n$, where $q(n)$ is now an *arbitrary* polynomial of the same degree as $p(n)$. Here are some examples:

$c_0(n)$	guess
7	α
n	$\alpha n + \beta$
2^n	$\alpha 2^n$
$n^2\, 3^n$	$(\alpha n^2 + \beta n + \gamma)3^n$

One needs to substitute into (NH) to find the constants α, β, γ. This will work provided no term in the guess is a solution of (H). In the letter case, one needs to multiply by one or more factors of n. A simple instance follows, though future exercises will clarify this.

Example. Find the general solution of

$$a_n = -a_{n-1} + 6a_{n-2} + 2^n.$$

Had 2 not been a root, we would have tried $\alpha\, 2^n$, but (since 2^n solves (H)) this would have given $0 = 2^n$. So we try $a_n = \alpha\, n\, 2^n$. This gives

$$\alpha n 2^n = -\alpha(n-1)2^{n-1} + 6\alpha(n-2)2^{n-2} + 2^n.$$

Dividing by a^{n-2},

$$4n\alpha = -2\alpha(n-1) + 6\alpha(n-2) + 4;$$

the terms involving n cancel out (as they must), and we are left with

$$0 = 2\alpha - 12\alpha + 4.$$

Thus, $\alpha = 2/5$ and we finish up with

$$a_n = A\, 2^n + B(-3)^n + \tfrac{2}{5}n2^n.$$

Example. Solve

$$a_n = -a_{n-1} + 6a_{n-2} + n, \qquad a_0 = 2, \quad a_1 = -1.$$

The homogenous equations has general solution $A * 2^n + B * (-3)^n$. For a particular solution, we substitute $a_n = \alpha n + \beta$. Since the resulting equation must hold for *all* n, we can separate out the terms involving n and those that do not. This gives two separate equations, which imply that $\alpha = -1/4$ and $\beta = -11/16$. Then we substitute

$$a_n = A * 2^n + B * (-3)^n - \tfrac{1}{4}n - \tfrac{11}{16}$$

to find that

$$2 = A + B - \tfrac{11}{16}, \qquad -1 = 2A - 3B - \tfrac{11}{16}$$

giving $A = 8/5$ and $B = 87/80$.

2.5. Derangements

In this section, we will work with some linear recurrence relations that do not have constant coefficients.

Fix a positive integer n. Recall that a *permutation* of the set $\Omega = \{1, 2, \ldots, n\}$ is a bijective mapping $f\colon \Omega \to \Omega$. There are $n!$ such permutations.

Definition. A *derangement* of Ω is a permutation $f\colon \Omega \to \Omega$ such that no element $i \in \Omega$ has the property that $f(i) = i$. This means that no number 'stays put', or (in the language of analysis) f has no fixed point.

Let d_n denote the number of derangements of $\{1, 2, \cdots, n\}$. It is obvious that $d_1 = 0$ (because there is only the identity permutation), $d_2 = 1$ (only swapping $1, 2$ works) and $d_3 = 2$ (because only the two 3-cycles do not have a fixed point).

Recall that a *cycle* of order k is a permutation of the form

$$f\colon \quad i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1;$$

here $i_1, \ldots, i_k$ are distinct positive integers. This permutation is denoted by the symbol $(i_1\, i_2\, \ldots\, i_k)$, which we regard as a function applied (on the left) to numbers. If $i_j \leqslant n$ for all j then f is an element of order n inside the group of a permutations of $\{1, 2, \ldots, n\}$. Moreover,

$$f = \sigma \circ (1\,2\,\ldots\,k) \circ \sigma^{-1},$$

where σ is any permutation that maps j to i_j for all j. This establishes the well-known fact that *any two k-cycles are conjugate* in a group of permutations.

To compute d_4, we appeal to the description of permutations from group theory. *Any permutation can be expressed (uniquely, up to order of the factors) as a product of disjoiont cycles.*

type of permutation	example	number of them
identity		1
4 cycle*	(1234)	6*
3 cycle	(123)	8
2 cycle	(12)	6
pair of 2 cycles*	(12)(34)	3*
		$4! = 24$

Only the asterisked permutations are derangements, so $d_4 = 9$. One can show that $d_5 = 44$ with a similar table, but then it becomes complicated because of the large variety of cycle decompositions.

Fortunately, we can easily find a way of computing d_n using recurrence relations. It is convenient to define $d_0 = 1$.

Theorem. (i) d_n satisfies the second-order linear homogeneous recurrence relation

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \quad n \geqslant 2.$$

(ii) d_n satisfies the first-order linear non-homogeneous recurrence relation

$$d_n = nd_{n-1} + (-1)^n.$$

(iii) The ratio $d_n/n!$ tends to $1/e$ as $n \to \infty$.

Proof. (i) The definition of d_0 makes the formula work for $n = 2$. Now let f be a derangement of $\{1, 2, \ldots, n\}$ for $n \geqslant 3$. Suppose that $f(1) = k$, so $k \geqslant 2$. There are two subcases:
(a) $f(k) = 1$, so the cycle decomposition contains a 2-cycle $(1\ k)$. Forgetting $1, k,$ what is left of f is a derangement of $n-2$ objects, and there are d_{n-2} of these. Together with the choice of i, this gives $(n-1)d_{n-2}$ possibilities.
(b) $f(k) = j \neq 1$. This time, remove k and define a derangement $\tilde{f}$ of $n-1$ objects by setting

$$\tilde{f}(i) = \begin{cases} j & \text{if} \quad i = 1 \\ f(i) & \text{if} \quad i \neq 1,\, i \neq k. \end{cases}$$

There are $(n-1)d_{n-1}$ possibilities in this subcase.

(ii) Set $a_n = d_n - nd_{n-1}$. We want to show that $a_n = (-1)^n$. Well,

$$\begin{aligned} a_n + a_{n-1} &= (d_n - nd_{n-1}) + (d_{n-1} - (n-1)d_{n-2}) \\ &= d_n - (n-1)(d_{n-1} + d_{n-2}) \\ &= 0, \end{aligned}$$

the last equality by (1). Since $a_1 = -1$, it follows that $a_n = (-1)^n$.

(iii) Rewrite (2) as

$$\begin{aligned} \frac{d_n}{n!} &= \frac{d_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \frac{d_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ & \quad \ldots\ldots \\ &= \frac{d_1}{1!} + \sum_{i=2}^{n} \frac{(-1)^i}{i!}. \end{aligned}$$

with a total of $n-1$ lines. Since $d_1 = 0$, it follows (strictly speaking, by induction) that

$$\frac{d_n}{n!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!},$$

since the first two terms in the summation cancel (by convemtion, $0! = 1$). As $n \to \infty$, the summation tends to e^{-1}. $\square$

Here is a table of values:

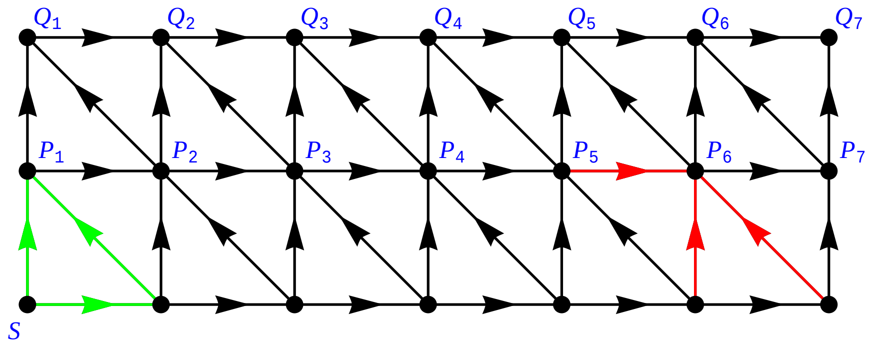
n	1	2	3	4	5	6	7	
d_n	0	1	2	9	44	265	1854	$\cdots$
$n!$	1	2	6	24	120	720	5040	$\cdots$
$d_n/n!$	0	0.5	0.333	0.375	0.367	0.368	0.368	$\cdots$

Here the ratio is given to three significant figures, so the probability of 'no snap' playing with two decks of 6 shuffled cards is already a good approximation to $1/e$.

2.6. Other counting applications

This section highlights two situations in which recurrence equations also occur naturally.

Example. Consider the system of one-way roads illustrated:



Let a_n denote the number of different routes from the starting point S to P_n. (If $n \geqslant 7$ the diagram needs extending to the right in the obvious fashion.)

To illustrate the method, we first take $n = 6$. One can reach P_n in one step from three directions, shown in red. Namely, travelling north or north-west from the bottom row, or travelling east from P_5. There is only one route to any point on the bottom row, so $a_6 = a_5 + 1 + 1$, since there are a_5 routes to P_5 which can be followed by the one step eastwards. The same argument shows us that

$$a_n = a_{n-1} + 2.$$

The green steps show that there are 2 routes to P_1, so $a_1 = 2$. The solution of this recurrence relation is obviously $a_n = 2n$.

A similar argument can now be used to count the number b_n of routes from S to Q_n in the top row. Again, one should consider the *immediate* predecessors of Q_n, which are P_n, P_{n+1}, Q_{n-1}, provided $n \geqslant 2$. These furnish a_n, a_{n+1}, b_{n-1} routes, so

$$b_n = a_n + a_{n+1} + b_{n-1} = b_{n-1} + 4n + 2, \qquad n \geqslant 2.$$

One can arrive at Q_1 from either P_1 or P_2, so $b_1 = a_1 + a_2 = 6$. The homogeneous relation (H) has general solution $b_n = C =$ constant, so for a particular solution of (NH) we try

$$b_n = An^2 + Bn,$$

giving

$$\begin{aligned} An^2 + Bn = A(n-1)^2 + B(n-1) + 4n + 2 \quad &\Rightarrow \quad 0 = -2An + A - B + 4n + 2 = 0 \\ &\Rightarrow \quad A = 2,\ B = 4. \end{aligned}$$

We also have $6 = b_1 = A + B + C$, so $C = 0$. Therefore

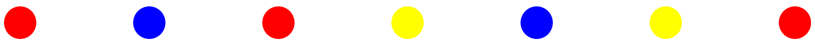
$$b_n = 2n^2 + 4n.$$

Example. A gardener has to plant a row of $n \geqslant 2$ rose bushes, which come in three varieties (red, artificially blue, yellow), observing the following rules:

1. the first bush must be red;
2. the last (nth) bush must be red;

3. no two colours can be adjacent.

We seek the number r_n of different ways of planting the bushes.



The lowest possible value of n is 3 to avoid the two reds together. For $n = 3$ we just need to choose the middle colour, so $r_3 = 2$. More generally, once we know the colour of the kth bush then there are two choices of colour for bush $k+1$. So without condition 2., there are

$$1 * \underbrace{2 * 2 * \cdots * 2}_{n-1} = 2^{n-1}$$

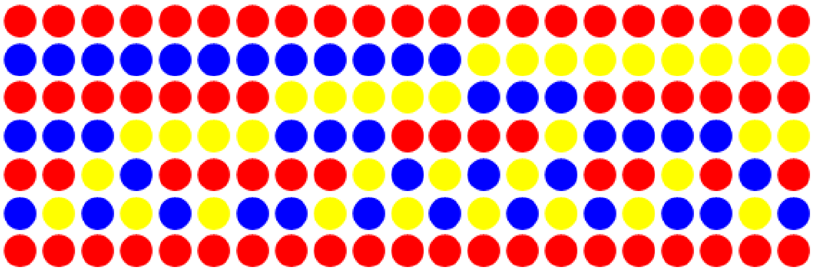
choices. With condition 2., we must insist that bush $n-1$ is not red, after which there is no more choice. Therefore

$$b_n = 2^{n-2} - b_{n-1}.$$

The solution is

$$r_n = \tfrac{1}{3} * 2^{n-1} - \tfrac{2}{3}(-1)^n, \qquad n \geqslant 3.$$

When $n = 7$ (as in the picture), there are 22 ways of planting, illustrated below with each row now vertical.



3. Arithmetical algorithms

3.1. First concepts

Definition. An *algorithm* is a finite set of unambiguous instructions that when executed terminate in a finite number of steps.

Named after Muhammad ibn Musa al-Khwarizmi (c. 780–850). A more formal specification (beyond the scope of this course) takes one into the area of recursive function theory, Turing machines and mathematical logic.

Example. Consider the factorial function $\mathbb{N} \to \mathbb{N}$. There are two distinct processes that can be used to compute n!

Firstly, by *iteration*. This is exemplified by the following SAGE code:

```
def fac(n):
    x = 1
    for i in range(2,n+1):
        x = x*i
    return x
```

In this code, the command range(p, q) lists integers from p to $q-1$ (rather than q). The penultimate line has the effect of replacing x by x times i. Note that the program correctly outputs fac(0) = fac(1). By regarding i more as a variable than a counter, we can replace the 'for/do loop' by a conditional:

```
def fac(n):
    x = 1
    i = 1
    while i < n+1:
        x = x*i
        i = i+1
    return x
```

The time needed to carry out the computation is estimated by counting the number $n-1$ of multiplications (the most 'expensive' operation). If we suppose that each multiplication takes one unit of time, then the total time $T = n-1$ satisfies

$$T = \Theta(n).$$

This equation is shorthand for saying that, for sufficiently large n, there exist constants $0 < c_1 < c_2$ such that

$$c_1 n \leqslant T \leqslant c_2 n.$$

Equivalently, $T = O(n)$ *and* $n = O(T)$, so both T/n and n/T are bounded as $n \to \infty$. Observe that it does not matter whether a unit of time is one millisecond or one minute.

Alternatively, one can use the recurrence relation $a_n = na_{n-1}$; this gives the definition of factorials by *recursion*:

```
def fac(n):
    if n == 0:
        return 1
    else:
        return n*fac(n-1)
```

This approach starts by asserting that $0! = 1$, which is true for convention, or rather convenience, as it seems to work in many formulae. Let t_n denote the number of times multiplication is used to compute $\text{fac}(n)$ using the last program. Then

$$t_n = t_{n-1} + 1,$$

so $t_n = n$. Once again, the total time equals $\Theta(n)$, but we also require memory that grows linearly with n (unlike in the first case). Indeed, the equations stored expand and contract, like the stacking of trays. At some point of the process, we will have

$$\text{fac}(5) = 5 * (4 * (3 * (2 * (1 * \text{fac}(0))))),$$

and we are stuck if the process is interrupted.

3.2. Powers by squaring

Example. Consider computing x^n, where n is a positive integer and x is a number to a given precision. This could be carried out by iteration, mimicking what we first did for factorials:

```
def pow(x,n):
    y = 1
    for i in range(1,n+1):
        y = y*x
    return y
```

This method requires $n-1$ multiplications, but we can find a much more efficient way by squaring at intermediate stages. For example, the calculation

$$\begin{aligned} x^{19} &= (x^9)^2\,x \\ &= ((x^4)^2\,x)^2\,x \\ &= (((x^2)^2)^2\,x)^2\,x \end{aligned}$$

uses only 6 multiplications. Here, we have effectively written the exponent $19 = 10011_2$ in binary, adding x added on the right if and only if the remainder is 1. To convert the binary expansion of the exponent n into a systematic procedure, read it from the left with initial value 1 in the 'register'. Then

- square for the privilege of processing the digit,
- multiply by x for each digit '1' encountered.

Starting with 19, the first '1' on the left allows us to write $1^2 * x = x$. This is then squared three times, though in processing the next '1', we multiply by x. The final '1' causes us to square and multiply by x again, and we are finished. It would be more efficient to ignore the first squaring (which is always $1*1$) and to process x starting from the second binary digit, but the instructions are slightly neater to state using the full binary expansion. Here is the pseudocode:

```
pow(x,n):
    y = 1
    find the expansion n.binary()
    for each bit from left to right:
        y = y*y
        if bit == 1:
            y = y*x
    return y
```

To implement this properly in SAGE one would need to define 'bit' as a successive element in the string consisting of the binary digits of n, but this would obscure the instructions. As it stands, the process should be sufficiently clear by carrying it out by hand. The only values stored are x, n and each current value of y. The only operations are squaring and multiplying by x.

Example. If $x = 3$ and $n = 11 = 1011_2$, we display each loop vertically.

n	1	0	1	1
y in	1	3	9	243
y squared	1	9	81	59049
y out	3	9	243	177147

Thus $3^{11} = 177147$ was computed with three squarings (ignoring $1*1$) and three multiplications. Here is the same example modulo 16:

y in	1	3	9	3
y squared	1	9	1	9
y out	3	9	3	11

Therefore $3^{11} = 11 \bmod 16$.

Let's analyse the efficiency. Recall from §1 that the number of bits needed to encode n in binary is $\lfloor \lg n \rfloor + 1$. For each bit, we need to square (a multiplication). Ignoring the first $1 * 1$ gives $\lfloor \lg n \rfloor + 1 - 1 = \lg n$ operations. Each '1' after the first gives an additional multiplication, so there are at most $\lfloor \lg n \rfloor$ of these. So we have a total of $2\lfloor \lg n \rfloor$ operations, and the algorithm requires $O(\lg n)$ operations, which is much more efficient than the iteration program.

3.3. Euclid's algorithm

Let a, b be integers with $b > 0$. The aim is to compute their greatest common divisor $\gcd(a,b)$. Suppose that

$$a = qb + r, \qquad 0 \leqslant r < b,$$

which implies that

$$a/b = q + r/b. \qquad 0 \leqslant r/b < 1.$$

In this situation, we know that $r = a \bmod b$, but to emphasize that $r < b$ we can use the exact formula

$$r = a - \lfloor a/b \rfloor b.$$

This remainder is denoted $a\%b$ in C++ or SAGE.

Lemma. With this notation, $\gcd(a,b) = \gcd(b,r)$.

Proof. Suppose that $s = \gcd(b,r)$. Then $s|b$ and $s|r$. Thus $s|a$, and s is a common divisor of a and b. Suppose that t is *another* common divisor of a and b. Then $t|r$, so t is also a common divisor of b and r, and (since s is the *greatest* such) $t \leqslant s$. Therefore s is indeed the *greatest* common divisor of a and b. $\square$

Eulcid's algorithm now consists of starting from (a,b), and then repeatedly performing divison and applying the lemma. Since $r_{i+1} < r_i$, we must have $r_n = 0$ for some n:

$$\begin{aligned} a &= q_0 b + r_1 \\ b &= q_1 r_1 + r_2 \\ r_1 &= q_2 r_2 + r_3 \\ &\cdots \\ r_{n-2} &= q_{n-1} r_{n-1} + 0. \end{aligned}$$

Applying the lemma, we are led to $\gcd(r_{n-1}, 0) = r_{n-1}$. So this equals $\gcd(a,b)$. The code is therefore quite simple:

```
def euc(a,b):
    if b == 0:
        return a
    else:
        return euc(b,a%b)
```

If we keep track of r_i as a linear combination of r_{i-1} and r_{i-2}, simplified at each stage, this we will obtain integers x, y for which

$$\gcd(a,b) = xa + yb.$$

This is called *Euclid's Extended Algorithm*. There is a quick way of carrying this out by hand using matrices. We shall illustrate the method next. We first set up a matrix of the form

$$\begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$$

in which the second column keeps track of coefficients of a and the third column those of b: the first row is to be interpreted as telling us that $a = 1*a+0*b$ and the second that $b = 0*a+1*b$. One then subtracts the row with the smallest first entry (initially the second if $b < a$) from the other row. One repeats this step until one row row begins with a '0', in which case the entry above or below it equals $\gcd(a, b)$, and the remaining entries of that row give x and y.

Example. To compute $\gcd(33, 93)$, we have

$$\begin{array}{ccccc} \begin{pmatrix} 93 & 1 & 0 \\ 33 & 0 & 1 \end{pmatrix} & \rightarrow & \begin{pmatrix} 27 & 1 & -2 \\ 33 & 0 & 1 \end{pmatrix} & \rightarrow & \begin{pmatrix} 27 & 1 & -2 \\ 6 & -1 & 3 \end{pmatrix} \\ & & & & \downarrow \\ & & \begin{pmatrix} 3 & 5 & -14 \\ 0 & -11 & 31 \end{pmatrix} & \leftarrow & \begin{pmatrix} 3 & 5 & -14 \\ 6 & -1 & 3 \end{pmatrix} \end{array}$$

There is no need to swap the rows each time provided one is happy to subtract the first from the second. The final '0' tells us that $\gcd(93, 33) = 3$ (of course, this was obvious) and

$$3 = \gcd(33, 93) = 5*93 - 14*33$$

(the coefficients $x = 5$ and $y = -14$ were less obvious).

We shall not prove formally that this method *does* produce the correct result, but one can understand it as follows. The new numbers $27, 6, 3$ in the first columns are simply the remainders r_1, r_2, r_3, with $r_4 = 0$. And since we are performing elementary row operations, each row $(r_i\ x_i\ y_i)$ tells us that $r_i = x_i * 93 + y_i * 33$. The penultimate one gives us the desired expression for $r_3 = \gcd(93, 33)$.

3.4. Consolidation

The aim of this section is to draw together many of the topics we have seen so far, namely Induction (§1,1), the Fibonacci numbers (§2.2), and logarithms (§1.4 and §3.2), in order to analyse the efficiency of Euclid's algorithm (§3.3). We shall show that, like our method of exponentiation by repeated squaring, it executes in 'log time'.

Definition. Suppose that $a > b > 0$. Let $s(a, b)$ denote the number of steps needed in executing Euclid's algorithm so that the remainder becomes 0 in the final step.

It is reasonable to suppose that $s(a, b)$ estimates the time required to compute $\gcd(a, b)$. If $s(a, b) = n$ then we are stating that (in our previous notation) $r_n = 0$ and $r_{n-1} \neq 0$.

Examples. One has

$$\begin{aligned} s(93, 33) &= 4 \\ s(33, 93) &= 5. \end{aligned}$$

Computer software quickly reveals that

$$\begin{aligned} \gcd(100! + 1,\ 100^{100} - 1) &= 101 \\ s(100! + 1,\ 100^{100} - 1) &= 337. \end{aligned}$$

Fibonacci numbers are relevant to the implementation of Euclid's algorithm. The following scheme implements the algorithm to determine the greatest common divisor of two adjacent Fibonacci numbers:

$$\begin{array}{rcr} F_{n+2} & = & F_{n+1} + F_n \\ F_{n+1} & = & F_n + F_{n-1} \\ \dots & & \dots \\ F_5 & = & F_4 + F_3 \\ F_4 & = & F_3 + F_2 \\ F_3 & = & 2\,F_2 + 0 \end{array}$$

This is because each Fibonnaci number like F_{n+1} divides exactly once into the next higher one F_{n+2} with remainder F_n (because twice F_{n+1} into F_{n+2} won't go!). The conclusion is that

$$\gcd(F_{n+2}, F_{n+1}) = F_2 = 1,$$

so any two adjacent Fibonacci numbers are coprime. Here is an easy example:

$$\begin{array}{rcl} 13 & = & 1*8+5 \\ 8 & = & 1*5+3 \\ 5 & = & 1*3+2 \\ 3 & = & 1*2+1 \\ 2 & = & 2*1+0. \end{array}$$

Note that $s(13,8) = s(F_7, F_6) = 5$. More generally, one needs exactly n steps to compute $\gcd(F_{n+2}, F_{n+1})$, because there are n remainders (including 0) in the general scheme above. The next results records this fact and generalizes it:

Proposition. (i) We have $s(F_{n+2}, F_{n+1}) = n$ for all $n \geqslant 1$.
(ii) If $a > b > 0$ and $s(a,b) = n$ then

$$a \geqslant F_{n+2} \quad \text{and} \quad b \geqslant F_{n+1}.$$

Proof. It remains to prove the second statement, which we do by induction on n. It is obviously true for $n = 1$ since $F_3 = 2$ and $F_2 = 1$ and one only step is needed. Assume that the second statement is true when n is replaced by $n-1$. Let r be the first remainder:

$$a = qb + r = q_0 b + r_1.$$

Since $s(a,b) = s(b,r) + 1$, we have $s(b,r) = n-1$. By hypothesis,

$$b \geqslant F_{n+1} \quad \text{and} \quad r \geqslant F_n.$$

But then

$$a \geqslant b + r \geqslant F_{n+1} + F_n = F_{n+2}.$$

So the second statement is true for our fixed value of n. Therefore it is true for all n. □

There are other striking properties relating Fibonacci numbers to division and Euclid's algorithm. We quote without proof the

Theorem. Let $m, n \geqslant 3$. Then $m|n$ if and only $F_m|F_n$.

This means that the mapping $n \mapsto F_n$ 'respects' divisibility, and (as an exercise) it follows from the theorem that

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}.$$

A simple example is $\gcd(F_8, F_{12}) = \gcd(21, 144) = 3 = F_4,\ 4$ being $\gcd(8, 12)$.

Another problem a bit beyond the scope of the course is to find the most efficient way of computing the Fibonacci numbers themselves. For example, how many steps and how much memory is needed to compute F_{12}?

Returning to Euclid's algorithm, we shall use the formula

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$$

from §2.2, where (recall) φ and ψ are the roots of $x^2 - x - 1 = 0$ with $\psi < 0$. If n is odd then $\psi^n < 0$; we deduce that

$$F_n > \frac{1}{\sqrt{5}}\varphi^n \quad \text{if } n \text{ is odd.}$$

Now suppose that $a > b > 0$ and $s(a, b) = n$. By part (ii) of the last Proposition,

$$b \geqslant F_{n+1} > \frac{1}{\sqrt{5}}\varphi^n.$$

The second equality is true for all n, because if n is even we can actually replace φ^n by φ^{n+1}, whereas if n is odd we first use the fact that $F_{n+1} \geqslant F_n$. Therefore $\sqrt{5}\,b > \varphi^n$ and

$$\tfrac{1}{2}\lg 5 + \lg(b) > n\lg(\varphi).$$

Dividing by $\lg(b)$ shows and letting $b \to \infty$ shows that there exists a constant $c > 0$ such that

$$s(a, b) = n < c\lg b.$$

This can be expressed in 'big O' notation by the

Theorem. Let $a > b > 0$. Then

$$s(a, b) = O(\lg b) \quad \text{as} \quad b \to \infty.$$

We conclude on a more elementary note by constructing a counterpart of the greatest common divisor. Let $a, b \in \mathbb{N}$, and set $g = \gcd(a, b)$. In particular, g is a common divisor and we can write

$$a = ga', \qquad b = gb'.$$

Consider the positive integer

$$\ell = \frac{ab}{g} = ga'b'.$$

Proposition. ℓ is the lowest common multiple of a and b. That is,

1. $a \mid \ell$ and $b \mid \ell$;
2. if $a \mid m$ and $b \mid m$ then $\ell \leqslant m$.

Proof. Condition 1. is immediate.
For 2., write $m = am_1 = bm_2$, and recall that $g = xa + yb$ for some $x, y \in \mathbb{Z}$. Consider

$$gm = (xa + yb)m = xabm_2 + ybam_1 = ab(ym_1 + xm_2).$$

Since $ym_1 + xm_2 \in \mathbb{Z}$, we have $\ell \mid m$. In particular $\ell \leqslant m$. $\square$

4. Basic graph theory

4.1. Definitions

A *graph* consists of a finite set V of vertices and a finite family E of pairs of elements of V, the edges. (The edges are defined as a *family* rather than a *set* so as to allow for multiple edges between two vertices. Moreover, an edge could consist of a loop from a vertex to itself, so the pair should be an ordered pair even though the order will not matter until we discuss digraphs.)

Examples. A 'triangle' with three vertices:

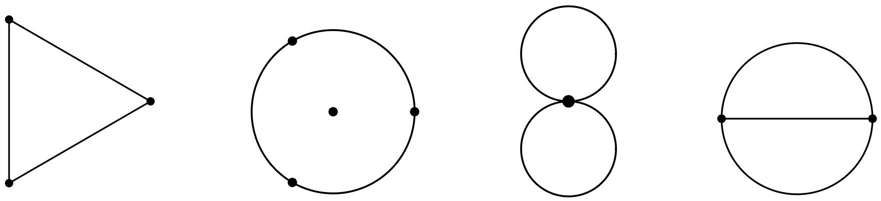
$$V = \{a, b, c\} \quad \text{or} \quad (a, b, c), \qquad E = (ab, bc, ca).$$

If we add an isolated vertex d, $V = \{a, b, c, d\}$ but E stays the same.
A 'figure eight' with one vertex:

$$V = \{o\}, \qquad E = (oo, oo).$$

A lower case 'theta' with 2 vertices and 3 edges:

$$S = \{a, b\}, \qquad E = (ab, ab, ab).$$



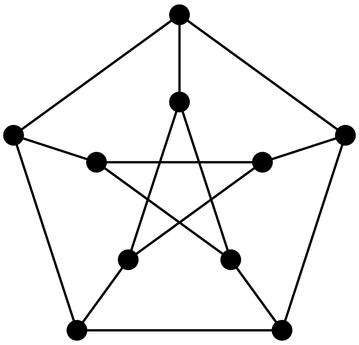
A graph is *simple* if there are no multiple edges and no loops. (In this case, E can be defined as a *set* of unordered pairs of vertices, but it is still easier to write ab or v_1v_2, or even 12, than $\{a, b\}$, $\{1, 2\}$ etc.)

The graph is *directed* or a *digraph* if each edge has an arrow, in which case each edge really is (in the logical sense) an ordered pair like (a, b). To emphasize that the order is now important, one can denote the edge by $a \to b$, notation that may be closer to its meaning (like a one-way flow).

Example. Quite simple sets give rise to interesting graphs. Let $S = \{1, 2, 3, 4, 5\}$ and let V be the set of all subsets of S of size 2. So

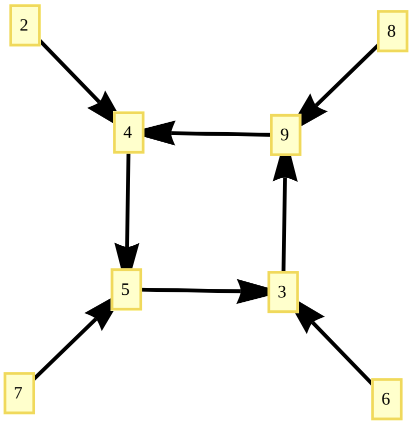
$$V = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$$

(where 12 is shorthand for $\{1, 2\}$ etc., as explained above) and $|V| = \binom{5}{2} = 10$. We shall join two vertices (elements of V) by an edge if and only if the two subsets are *disjoint*. The result is called the *Petersen graph*. It is an example of a regular graph: the degree if every vertex is the same:



There are only 15 vertices, though it is impossible to represent the graph in 2 dimensions without edge crossings, since it is not a *planar* graph, a topic to be explored later.

Example. We shall define a digraph with vertex set $V = \{2,3,4,5,6,7,8,9\}$ using modular arithmetic. Regard the elements of V as congruence (or residue) classes modulo 11 (we have excluded 0, 1 and $10 \equiv -1$). The set of directed edges consists of pairs (i,j) for which $j = i^2 \bmod 11$. The vertices $3,4,5,9$ of the 'square' are the so-called *quadratic residues* modulo 11; they are elements admitting a square root mod 11:



The *degree* of a vertex v, written $d(v)$, is the number of occurrences of v as an endpoint in the family of edges. Note that a loop will contribute 2 to the degree. If the graph G is simple then the degree is also the number of vertices joined to v by an edge. One often denotes the maximum degree of any vertex in G by $\Delta(G)$, and the minimum by $\delta(G)$.

Proposition. For *any* graph, the sum of the degrees of all vertices equals twice the number of edges: $\sum_{v\in V} d(v) = 2|E|$.

Proof. We can prove this by induction on $|E|$. Given a graph with n edges, remove any one. Either it joined two distinct vertices, or it was a loop at one vertex. In either case, we have reduced the sum of the degrees by 2. So assuming the result for $n-1$ edges (and it certainly holds for one edge), it remains true for n edges. □

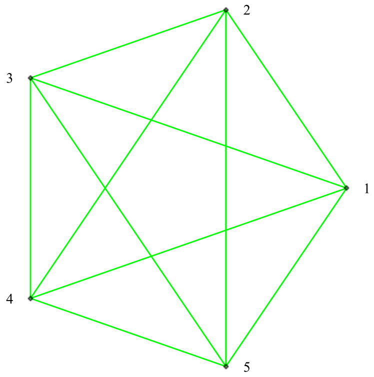
Example. There is no graph with vertex degrees $2, 3, 3, 5$.

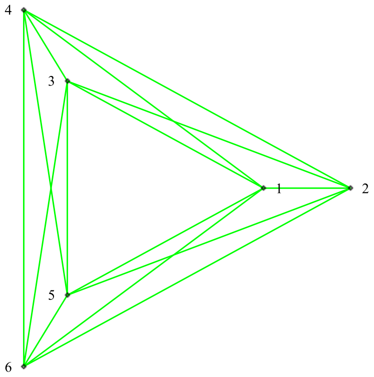
Definition. Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f\colon V_1 \to V_2$ between their vertex sets such that the number of edges between any two vertices $a, b \in V_1$ equals the number of edges between $f(a), f(b) \in V_2$. For simple graphs, this amounts to the assertion that

$$(a,b) \in E_1 \quad \Longleftrightarrow \quad (f(a), f(b)) \in E_2.$$

It is often easy to see that two graphs are *not* isomorphic, less easy to prove that they are. To show that two graphs are isomorphic, one must construct an isomorphism. To show that they are not isomorphic, one looks for some property which is different in the two graphs, such as the sequence of vertex degrees (if one is lucky), or the existence of cycles of a given length (see §4.2).

Examples. A complete graph with n vertices is a simple graph in which any two vertices are joined by an edge, so there are $\binom{n}{2}$ edges. Any two are isomorphic so we can speak of *the* complete graph with n vertices. It is denoted K_n. The figures shown are representations of K_5 and K_6:





4.2. Connectivity

Two vertices u, v of a graph G are *adjacent* if there is an edge $uv \in E$ whose endpoints are u and v. The edge is said to *join* u and v.

A *walk* from u to v is a sequence of edges

$$v_0v_1,\ v_1v_2, \ldots, v_{n-1}v_n$$

with $u = v_0$ and $v = v_n$. The length of the walk is the number n of edges (we also allow $u = v$ and $n = 0$).

A walk may be written $v_0v_1 \cdots v_n$. Best not to write $v_0 \to v_1 \to \cdots \to v_n$ if G is not a digraph.

A *trail* is a walk with no repeated edges.

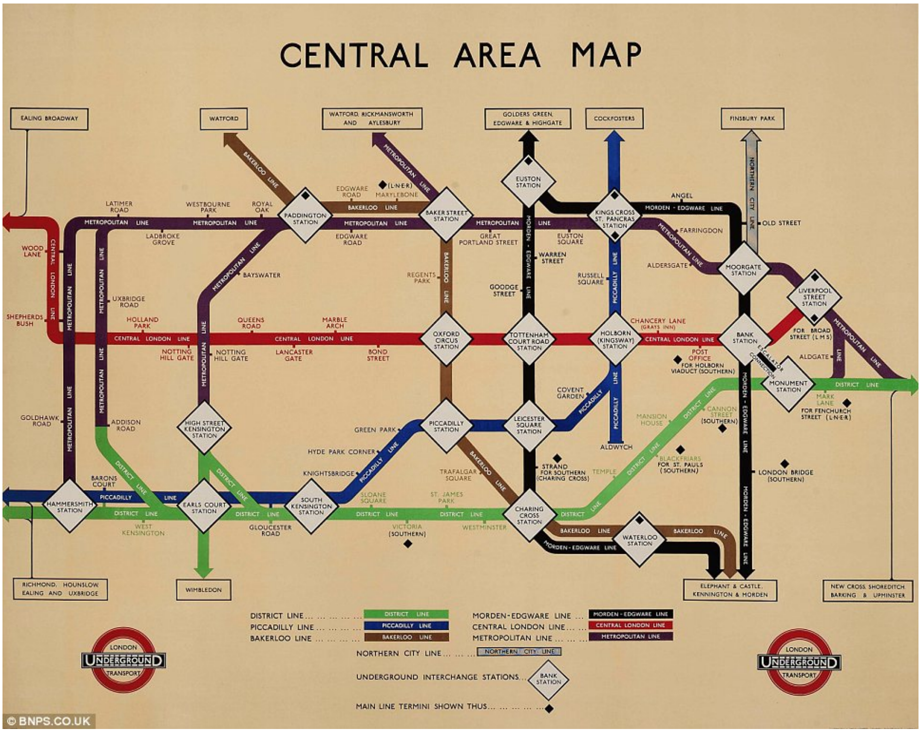
A *path* is a trail with no repeated vertices except possibly $v_0 = v_n$.

A walk/trail/path is *closed* if $v_0 = v_n$, i.e. it starts and finishes at the same vertex.

A *cycle* is a closed path with at least one edge. An n-cycle is a cycle of length n. Hence a loop is a 1-cycle and a 2-cycle only appears if there is a multiple edge.

Examples. Referring to the old underground map below, consider the stations

A Aldwych!	
C Charing Cross	H Holborn
O Oxford Circus	P Picadilly Circus
S South Kensington	T Tottenham Court Road.



Then

LTOPL	is a	4-cycle (and path)	
SPOTHA		path of length 5	
SPLTOPC		trail (and walk)	[repeated vertex]
SPLTOP		trail (and walk)	[P still repeated]
SPLTOPLH		walk	[repeated edge]

Definition. Two vertices u, v of a graph G are *connected* if one can walk from one to the other and we can write $u \sim v$. (This implies there is a *path* between any two vertices, why?) Then $\sim$ is an equivalence relation, and it partitions V into one or more subsets, the *components* of G. The graph G itself is *connected* if there is just one component, in which case any two of its vertices are joined by a path.

A *subgraph* of a graph $G = G(V, E)$ is any graph $G' = G'(V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Note that E' might not include all the edges in G that join vertices in V', though if it does one says that G' is *vertex-induced* (from V').

A component of a graph G is then a maximal connected subgraph G', i.e. G' is connected but if one more vertex or edge from G is added then the subgraph is no longer connected.

A *disconnecting set* of a connected graph G is a set of edges whose removal makes the new graph disconnected. When an edge is removed the vertices which are its endpoints are retained. Of particular importance is the special case:

Definition. A *cutset* of a connected graph G is a disconnecting set, no proper subset of which is a disconnecting set.

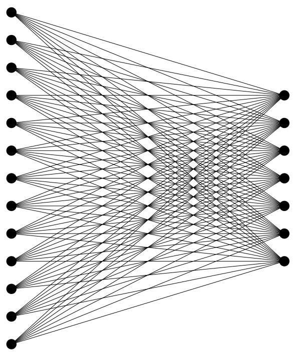
Thus, if any one of the edges in a cutset is retained then the graph stays connected. If a cutset consists of a single edge then this edge is called a *cut-edge* or a *bridge*.

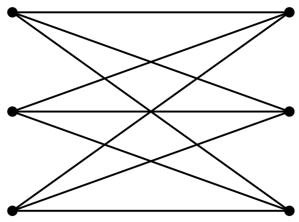
A *tree* is a connected graph with no cycles, though there are alternative definitions (to be seen later).

A *bipartite graph* is a graph in which V, the set of vertices, is the union of two disjoint non-empty sets V_1 and V_2 and all the edges have one endpoint in V_1 and the other endpoint in V_2. (Hence no two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent.) One can also prove the useful

Theorem. A graph is bipartite if and only if there are no cycles of odd length.

Example. $K_{n,m}$ is the complete bipartite graph where $|V_1| = n$, $|V_2| = m$ and every vertex in V_1 is adjacent to every vertex in V_2. Here we see $(m, n) = (13, 7)$ and $(3, 3)$:





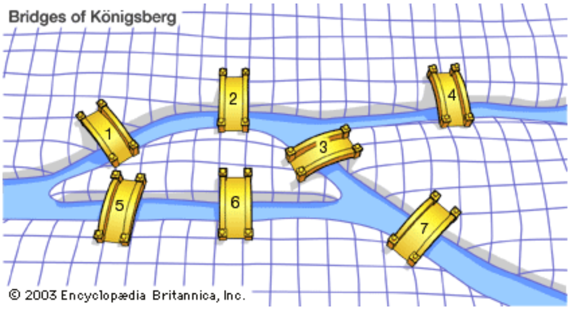
4.3. Eulerian graphs

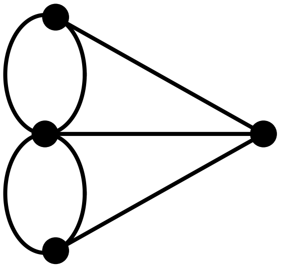
Recall that a cycle is a closed path. It is therefore a walk that passes through no edge twice and (apart from being joined up so that it finishes where it started) no vertex twice.

Lemma. If a graph G is connected and every vertex has degree at least 2 then G contains a cycle.

Proof. A loop defines a 1-cycle, and a multiple edge defines one or more 2-cycles, so we may assume that G is simple. Choose any vertex v_1. Let v_2 be an adjacent vertex (meaning there is an edge v_1v_2). Since v_2 has degree at least 2, it must have an adjacent vertex v_3 distinct from v_1. Continuing in this way, since V is a finite set, we must eventually find a vertex already in the list, say $v_j = v_i$ with $1 \leqslant i < j$, and $v_{i+1}, \ldots, v_j$ all distinct. Then $v_iv_{i+1} \cdots v_j$ is the desired cycle. $\square$

Example. Graph theory is said to have begun with Euler's paper solving the problem of the seven bridges of Könisberg near the Baltic Sea (now the Russian city of Kaliningrad sandwiched between Poland and Lithuania and *disconnected* from Moscow). The question was whether there exists a *trail* that crosses each bridge exactly once.





There can't be one because when one converts the map into a graph with each land mass a vertex and each bridge an edge, all the vertices have odd degree, so it is impossible to 'come and go' without repeating an edge. Next we'll make this precise.

Definition. A connected graph is *semi-Eulerian* if there is a trail which includes every edge of the graph. (Thus it passes along each edge exactly once and goes through every vertex, but some vertices can be visited more than once.) The graph is *Eulerian* if such a trail can be found that is closed.

Theorem. Suppose that G is a connected graph. Then G is Eulerian if and only if every vertex has even degree.

Corollary. A connected graph is semi-Eulerian if and only if it has at most two vertices of odd degree.

These results were known to Euler, who studied the bridges problem in 1736, though a formal proof was first given in 1871 by Hierholzer (who died that year aged 31).

Note. No graph can have an odd number of vertices of odd degree, because the 'total degree' must be even. We shall deduce the corollary from the theorem by a neat trick.

Proof of the theorem. If G has a closed 'Eulerian trail', then follow along it from a starting vertex. At each new vertex, we can mark off the arriving edge and the departing edge. Both are traversed once and only once, so the degree of each vertex (including the start=finish one) must be even. This justifies the 'only if' part of the theorem.

The harder 'if' part can be proved by induction on the number of edges of G, using the lemma to first remove a cycle and work on what is left of G. However we shall explain how one can effectively construct a closed Eulerian trail using (what is now called) Hierholzer's algorithm.

Suppose then that $G(V, E)$ is a connected graph, all of whose vertices are of even degree. Fix a vertex $v_0 \in V$. A first observation (which is a refinement of the lemma) is that one always find a trail (with at least one edge) in G that eventually returns to v_0. This is because, whichever edge one chooses to walk along and whichever vertex one arrives at, there will always be an edge available to leave that vertex. (This is guaranteed by the even degree property, even if the trail has already passed through the same vertex.) But the graph is finite, so we must eventually return to v_0.

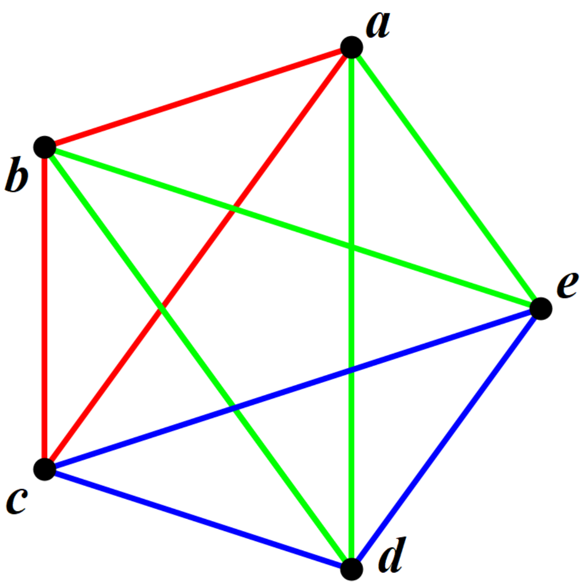
With this observation, we type out the algorithm in pseudocode. Each step can in fact by implemented by instructing a computer how to process data (lists of vertices and edges) representing the graph. Let T be a closed trail of any length (even zero) starting and ending at v_0.

```
# Hierholzer's algorithm
def aug(T):
    if T contains all the edges of G:
        return T
    else:
        walk to the first vertex v1 in T with a spare edge
        construct new trail T1 from v1 to v1 using spare edges
        insert T1 as a detour into T to form a longer trail T+
    return aug(T+)
```

A 'spare edge' means an edge of G that does not appear in G; there will always be one if T is not already Eulerian. The argument above shows that the new trail T_1 can be found, and inserting it as a detour into T yields the 'augmented' trail T^+. If T is empty, T_1 will be a trail with at least one edge (it could be a loop) based at v_0. We could have asked the programme to output T^+, but in that case it would be necessary to start afresh with T^+ in place of T. By calling aug(T^+), we have made the function recursive by requiring the program to keep applying the 'else' step until it finds a trail with all the edges of G.

Example. The complete graph K_n is Eulerian if and only if n is odd (for then all vertex degrees are even). Let us apply the algorithm to K_5 with vertices a, b, c, d, e. Take T to be the empty trail (denoted $\varnothing$) at $v_0 = a$. When it comes to spare edges, let's always move to a vertex of least alphabetical order. This causes the program to manufacture three separate detours, illustrated below with the colours red, green, blue, and to return the Eulerian path $abdaebcdeca$ with (as has to be) 10 edges.

$$\begin{array}{rlcll}
\text{input } \rightarrow & T & = & \varnothing & \\
& T_1 & = & \overbrace{abca} & \\
& T^+ & = & abca & \\
& T & = & a\underline{b}ca & \\
& T_1 & = & \overbrace{bdaeb} & \\
& T^+ & = & abdaebca & \\
& T & = & abdaeb\underline{c}a & \\
& T_1 & = & \overbrace{cdec} & \\
& T^+ & = & abdaebcdeca & \\
& T & = & abdaebcdeca & \leftarrow \text{ output}
\end{array}$$



Proof of the corollary assuming the theorem. Suppose that G has exactly two vertices u, v of odd degree. Add an *extra* edge from u to v. In this section, we are not assuming graphs are simple, so if there was already an edge from u to v (or more than one), we simply add another. The point is that the new graph has all its vertices of even degree so, by the theorem, it possesses a Eulerian trail, which must incorporate the extra edge. If the latter is removed, we end up with a Eulerian trail for G that starts at u and finishes at v. Conversely, if we had such a trail we can make it closed by again adding an extra edge, and conclude that the modified graph has all its other vertices of even degree. $\square$

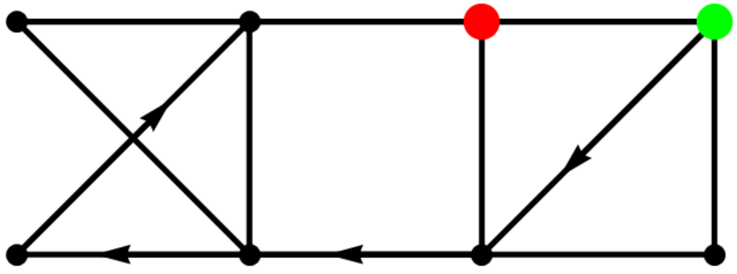
A more popular way of finding an Eulerian trail is *Fleury's algorithm*. It can be summed up by the slogan "do not burn bridges". Recall that a *bridge* in a connected graph is an edge that when removed will cause the new graph to be disconnected. We shall mimic our description of the previous algorithm to define Fleury's, but instead of proving that it always succeeds we make do with an example.

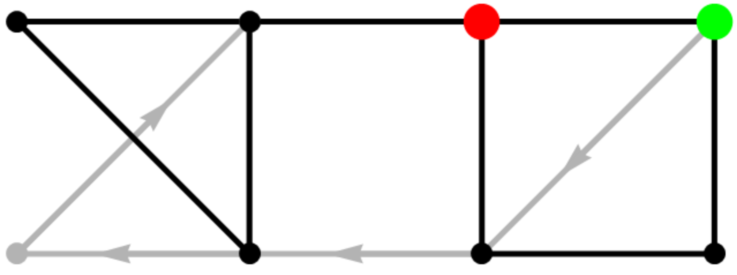
We shall implement the algorithm to construct a semi-Eulerian trail, so this time let G be a connected graph with at most two vertices of odd degree. Fix a vertex v_0 of odd degree if there is one, and a trail T starting at v_0, possibly empty.

```
# Fleury's algorithm
def fl(T):
    G' = G with edges and isolated vertices of T removed
    w = last vertex in T
    choose an edge of G' at w avoiding a bridge if possible
    T+ = T with the new edge added
    return fl(T+)
```

Then fl(T) will be a semi-Eulerian trail for G.

Example. In the graph illustrated immediately below, there are two 'odd' vertices, so we start top right (green), and aim to finish at the adjacent (red) vertex. After traversing 4 edges, we have constructed a trail T shown in grey. To obtain G', we discard its 4 grey edges and 1 isolated vertex. The point then is that we must complete the 'left wing' before crossing the top bridge. (This is obvious to the human eye, but programming a computer to recognize a bridge would be an unwanted complication.) Thus, T^+ will be formed from T by adding either the *left* horizontal edge or the vertical one.





4.4. Hamiltonian cycles

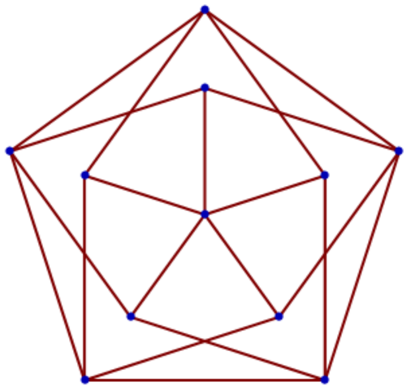
Definition. A connected graph is *Hamiltonian* if there is a closed path (and so, a cycle) that visits every vertex exactly once.

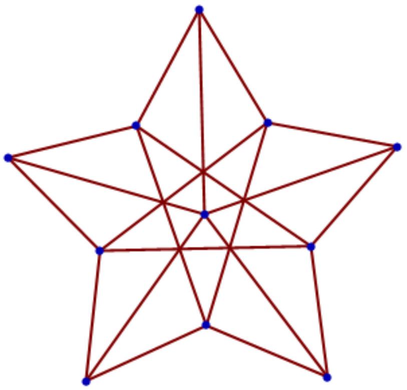
The closed path is called a *Hamiltonian cycle*. A Eulerian trail must, by its very nature visit every vertex, but it is allowed to do so more than once. By contrast, a Hamiltonian cycle must visit each vertex only once, and will not in general pass along every edge. Despite the analogy with Eulerian trail, deciding when a graph is Hamiltonian is much harder and remains the subject of current research.

There are no wide-ranging theorems that characterize Hamiltonian graphs, and most results that are known are rather restrictive and of limited value for the graphs encountered in this course. The following is one such result, which we state without proof.

Gabriel Dirac's Theorem. Let $G(V, E)$ be a simple graph with $|V| = n \geqslant 3$. If $d(v) \geqslant n/2$ for all $v \in V$ (equivalently, $n \leqslant 2\delta(G)$) then G is Hamiltonian.

Examples. Since $\delta(K_n) = n - 1$, the theorem does tells us that K_n is Hamiltonian for all $n \geqslant 2$.
A bipartite graph with an odd number of vertices cannot be Hamiltonian, because any Hamiltonian cycle would be odd.
The 12 edges and 6 vertices of an octahedron form a graph that (like those arising from the other platonic solids) is Hamiltonian.
The Petersen graph is not Hamiltonian, but it does have a (non-closed) path passing though every vertex. The *Grötzsch graph* $\ddot{G}$ has 11 vertices and 20 edges, and by contrast *is* Hamiltonian. Here are two representations of it:





Exercise. For which values of n does $\ddot{G}$ possess an n-cycle?

A celebrated example of a Hamiltonian graph is the *knight's graph* N with 64 vertices, defined as follows. Consider a knight moving freely on a standard (8×8) chess board, without other peices to get in the way. Assign a vertex to each square, and declare two vertices to be adjacent if a knight can move legitimately between them (that is, two squares one way and one sideways). If a knight starts on one of the 16 squares near the centre of the board, it has 8 squares it can move to, but this choice is reduced closer to an edge of the board. The edges of the central 6×6 square allow 6 moves, except its corners that allow only 4. Overall, N has

4	vertices of degree	2
8	”	3
20	”	4
16	”	6
16	”	8

It follows that the total number of edges is

$$e = \tfrac{1}{2}(4 * 2 + 8 * 3 + 20 * 4 + 16 * 6 + 16 * 8) = 168,$$

and a Hamiltonian cycle uses 64 of them. Not surprisngly, it is difficult to find such a knight's tour from scratch. The following instance was constructed (by the lecturer for Module Selection) by starting in a corner, and using Warnsdorff's (not infallible) rule: *move the knight to a square from which there is the least number of successive moves possible*. The knight therefore hugs the edges as long as it can; each number (from 1 to 64) labels the start of the corresponding edge of the cycle:

7	42	9	24	5	40	19	22
10	25	6	41	20	23	4	39
43	8	55	34	51	38	21	18
26	11	58	37	56	33	52	3
59	44	35	54	63	50	17	32
12	27	62	57	36	53	2	49
45	60	29	14	47	64	31	16
28	13	46	61	30	15	48	1

The knight's graph is bipartite because the vertices (squares) are divided into black and white, and a knight changes colour each move. From the remarks above, there cannot exist a knight's tour on (for example) a 7×7 chequer board.

5. Vertex colouring

5.1. Chromatic number

In this section, **all graphs will be simple**. The problem then is to assign a colour to each vertex of a graph so that no two adjacent vertices have the same colour, and to do this using the least number of colours.

In mathematical language, a vertex colouring of a graph $G(V,E)$ is a mapping $c\colon V \to \mathbb{N}$ with the property

$$uv \in E \quad\Rightarrow\quad c(u) \neq c(v).$$

This means that adjacent vertices have different colours (values of c), and to do this with as few colours as possible means reducing the image of c. We are using positive integers to label the colours, though in examples we shall use Greek letters in their alphabetical order

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \ldots$$

On screen, we can use actual colours, such as red, green, blue, and yellow.

A graph is *k-colourable* if we can find a vertex colouring with $|\operatorname{Im} c| = k$. In this case, one of k colours can be assigned to each vertex such that no two adjacent vertices have the same colour.

Definition. The *chromatic number* of a simple graph G, denoted $\chi(G)$, is the least value of k for which G is k-colourable.

Observation. It is impossible to find a vertex colouring if G has a loop uu. Multiple edges do not affect the colouring problem, but in any case, we restrict to simple graphs.
$\chi(G) = 1$ if and only if all the vertices of G are isolated, not an interesting scenario.
$\chi(G) = 2$ if and only if G is bipartite (see the end of §4.2). This is really the definition of bipartite: one can regard the colours as 'positive' and 'negative' or (in the chess example in §4.4) white and black.
If $|V| = n$ then obviously G is n-colourable and $\chi(G) \leqslant n$, but usually we can make do with *many fewer* colours because $\chi(G)$ is much more closely related to the degrees of vertices in G. In this sense, the next example is not typical.

Examples. Since every vertex of K_n is joined to every other, we have $\chi(K_n) = n$.
Let C_n denote the 'cycle graph' consisting of the n vertices and n edges of a polygon. Then

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

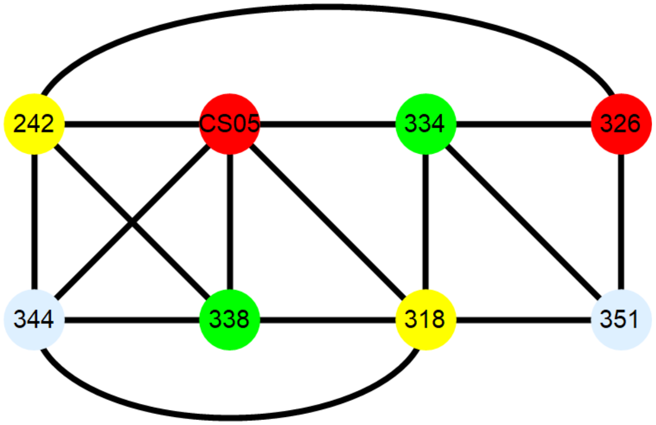
It follows that:

$$\begin{array}{rcl} G \text{ contains } K_n \text{ as a subgraph} & \Rightarrow & \chi(G) \geqslant n \\ G \text{ contains an odd cycle as a subgraph} & \Rightarrow & \chi(G) \geqslant 3. \end{array}$$

Application. Vertex colouring can be used to solve the *timetabling problem* with:

- a set of modules (the vertices);
- groups of students who have selected pairs of modules (the edges);
- a limited number of time slots (the colours);
- an unlimited number of lecture rooms (to simplify the problem).

If no adjacent vertices have the same colour all students can attend all the lectures for the modules they have chosen!



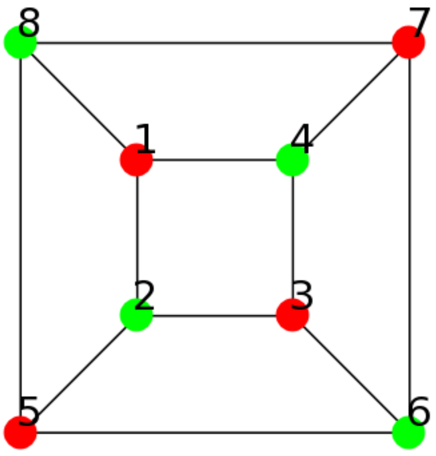
In the graph G above, a popular module has 'degree' 5. Four modules (left) form a complete subgraph K_4, so no less than 4 colours will suffice. Thus $\chi(G) = 4$.

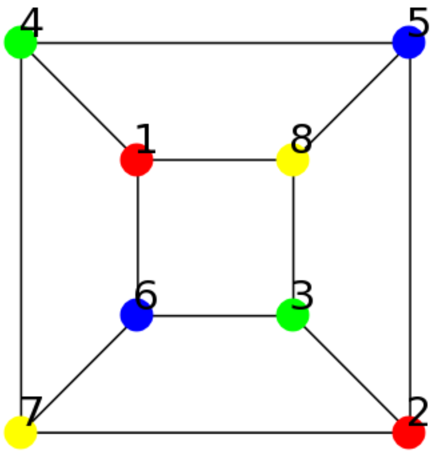
5.2. Colouring results

The whole theory of vertex colouring depends on the so-called *greedy algorithm*. This is a natural way of colouring the vertices when they are put in order, and (in very informal language) can be exporessed as follows:

```
label the vertices v_1,v_2,...,v_n
label the colours 1,2,...,n
assign to v_1 colour 1
for j in range(2,n+1):
    S = {colours assigned to vertices adjacent to v_j}
    assign to v_j the smallest colour not in S
return S and the assignments
```

Example. Different vertex orderings can give very different numbers of colours. The 'cube graph' is bipartite so $\chi = 2$, and this is illustrated on the left. But the greedy algorithm produces 4 colours (here $1 = R$, $2 = G$, $3 = B$, $4 = Y$) when applied to the ordering on the right:





For any G, it can be shown that there always exists some vertex ordering for which the greedy algorithm gives the minimum number (namely, $\chi(G)$) of colours.

Recall that

$$\Delta(G) = \max_{v \in V} d(v),$$

where $d(v)$ is the degree (valency) of the vertex v. For example, $\Delta(K_n) = n - 1$ and $\chi(K_n) = n$. Here are the main results on vertex colouring, in increasing difficulty.

Lemma. If G is simple then $\chi(G) \leqslant \Delta(G) + 1$.

Proposition. If G is simple, connected and not regular (not all vertices have degree Δ) then $\chi(G) \leqslant \Delta(G)$.

Brooks' theorem. If G is simple, connected and neither complete nor an odd cycle (so $G \not\cong K_n$ and $G \not\cong C_{2k+1}$) then again $\chi(G) \leqslant \Delta(G)$.

Question. Why must we add 'connected' in the last two statements?

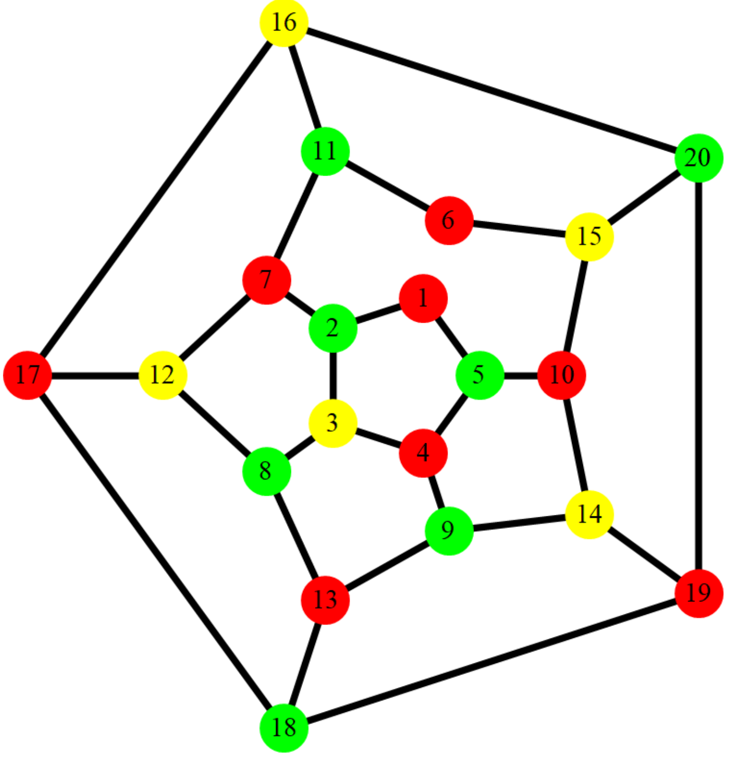
Proof of the lemma. Let $k = \Delta(G)$, and fix a set of $k + 1$ colours. Take the vertices in *any* order. Suppose we have managed to colour some (or $\varnothing$) of them. The next (or first) vertex is surrounded by at most n adjacent vertices, so we can colour it without a clash, and move on to the next vertex in the list. $\square$

Note that the proposition reduces the proof of the theorem to the case of regular graphs. We shall prove the former, but not the latter. Incidentally, if we assume that $\Delta(G) \geqslant 3$, there is no need to mention the odd cycle in the theorem.

5.3. Brooks' algorithm

The proof of the proposition is accomplished by implementing a 2-step process that is sometimes called *Brooks' algorithm*. This produces an ordering of the vertices (or a re-ordering if we have one already), relative to which (as we shall explain) the greedy algorithm will *always* succeed with at most Δ colours.

We shall illustrate it with a modification of the dodecahedral graph with 12 vertices, in which we have removed one edge (leaving 29) so that the graph is no longer regular. We have labelled the vertices with numbers $1, 2, \ldots, 20$ in no special way, and the missing edge is $(1, 6)$.



Rather than type out the instructions, we shall explain the process with a table.

In general, let G be a graph with $\Delta(G) = k$ but not regular. Choose a vertex with degree less than k to start, and place the vertex in a 'queue'. At each stage, the first element in the queue is moved to the left-hand list, and any adjacent vertices not previously queued are added at the back of the queue (in any order, but for definiteness one can add them in increasing order). In our example, $k = 3$, and we can start with vertex 6. Later on, vertex 5 is adjacent to $1, 4, 10$, but $1, 10$ have already appeared in the queue, so only 4 is added (see the boxes).

At the end of the process, *the list of vertices must be read in reverse order*. In our case, we obtain

$$v_1 = 13, \quad v_2 = 9, \quad \ldots \quad , v_{20} = 6.$$

Now apply the greedy algorithm to this (reversed) list. By construction, each vertex in the list can be adjacent to at most $k - 1$ previous elements in the list, because it is also adjacent to one of the vertices above it in the table. For example, there is no problem colouring v_{10} because it is only adjacent to v_4, and without checking we know it must be adjacent to some v_j with $j > 10$ (it first entered the queue when 10 entered the list).

This scheme shows that we can colour G with at most k colours. In our case, we recover the colouring shown. If we restore the missing edge, this is not a valid colouring, though Brooks' theorem implies that the dodecahedral graph is also 3-colourable and has $\chi = 3$.

	list	Q						
		6						
$v_{20} =$	6	11	15					
$v_{19} =$	11	15	7	16				
	15	7	16	$\boxed{10}$	20			
	7	16	10	20	2	12		
	16	10	20	2	12	17		
	10	20	2	12	17	5	14	
	20	2	12	17	5	14	19	
	2	12	17	5	14	19	$\boxed{1}$	3
	12	17	5	14	19	1	3	8
	17	5	14	19	1	3	8	18
$v_{10} =$	$\boxed{5}$	14	19	1	3	8	18	$\boxed{4}$
	14	19	1	3	8	18	4	9
	19	1	3	8	18	4	9	
	1	3	8	18	4	9		
	3	8	18	4	9			
	8	18	4	9	13			
	18	4	9	13				
$v_3 =$	$\boxed{4}$	9	13					
$v_2 =$	9	13						
$v_1 =$	13							

This table emphasizes the dynamic nature of the process, and how the data might be stored on a computer. Each vertex is processed separately and its 'new neighbours' put in the queue using the *First In First Out* (FIFO) principle, which contrasts with that of a stack (FILO) we saw in recursive relations.

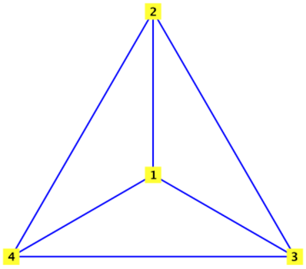
However, there is a lot of redundant information with the diagonals. In practice, one can form a long queue, ticking off the vertices as they are processed and crossing out vertices on the graph as soon as they enter the queue (as the initial one or neighbours of the current vertex).

Exercise. Retain the edge $(1, 6)$ and re-do the table starting with vertex 6 again. You will probably find that most vertex colours are the same but that at the final stage one requires a fourth colour. This is not a contradiction: even though the dodecahedral graph has $\chi = \Delta = 3$, it is important to remember that Brooks' *algorithm* is only guaranteed to use at most Δ colours when G is *not* regular. We have not proved Brooks' theorem!

6. Planarity

6.1. The Platonic graphs

We can represent K_4 the 'complete graph on four vertices' by the edges and vertices of a tetrahedron with transparent faces:



Unlike a square with its two diagonals added, this has the advantage that there are no 'false intersections' of its edges.

Definition. A graph is called *planar* if it can be drawn in the plane without crossings, so that edges intersect only in vertices. When it has been drawn that way we shall call it a *plane drawing*.

The phrase 'can be drawn' means 'is isomorphic to a graph', so 'planar' is a property of an *isomorphic class* — if it is true for one graph, it is true for any isomorphic graph. Our problem then is to understand how to decide whether such a class is planar.

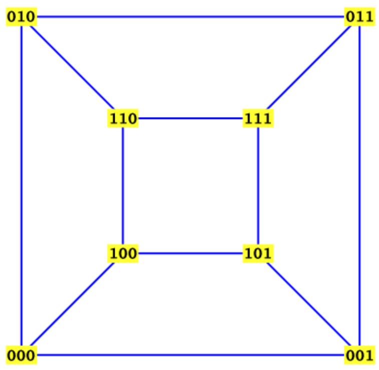
We shall begin with four more regular planar graphs, namely the ones determined by the vertices and edges of the remaining platonic solids. A *platonic solid* is a convex polyhedron (formed by intersecting a number of planes in space) with *congruent* faces each of which is a *regular polygon*. It is known that such a face must be a triangle, square or pentagon. Let

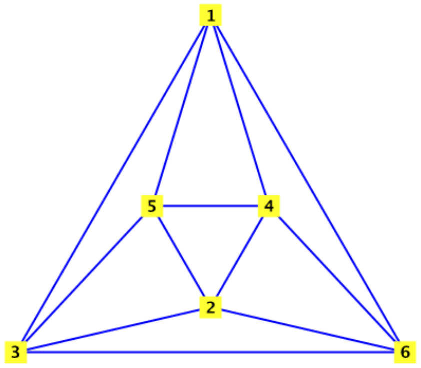
n	denote the number of	vertices
e		edges
f		faces
p		edges bounding each face
$\Delta = q$		edges joined at each vertex
χ	chromatic number	

Then the five Platonic solids and properties of the associated graphs are given by the table

name	n	e	f	p	q	χ	Eulerian?	Hamiltonian?
tetrahedron	4	6	4	3	3	4	no	yes
cube	8	12	6	4	3	2	no	yes
octahedron	6	12	8	3	4	3	yes	yes
dodecahedron	20	30	12	5	3	3	no	yes
icosahedron	12	30	20	3	5	4	no	yes

The Greek prefixes refer to the number of *faces*, for example *dodeca* means $2 + 10 = 12$. The last four come in pairs, in their data we swap $n \leftrightarrow f$ and $p \leftrightarrow q$. Here is the cube (or hexahedral) graph and the octahedral graph:

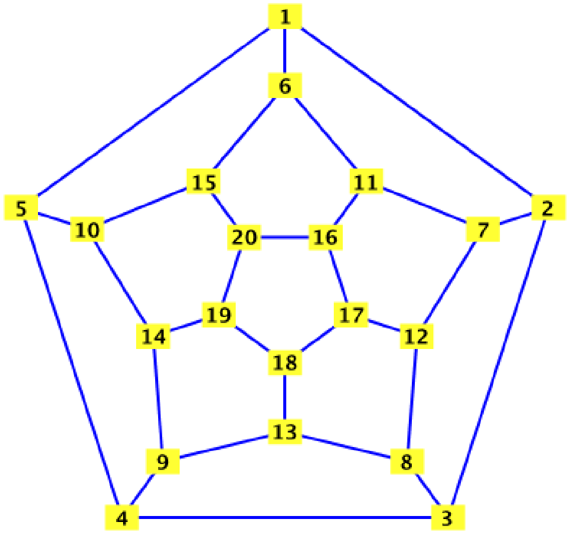


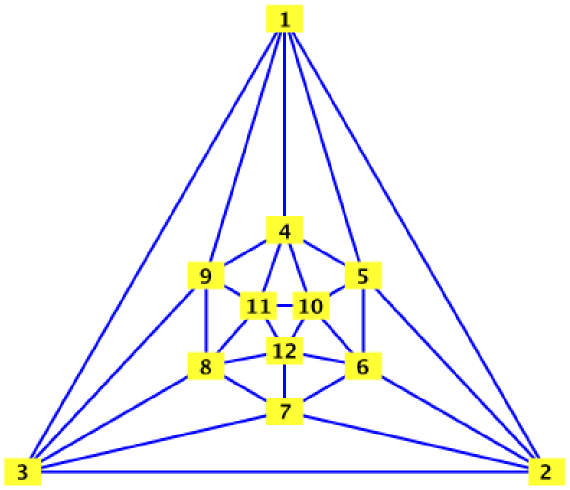


Recall that the cube graph is bipartite. Its 2^3 vertices can be labelled by their Cartesian coordinates, which in the image are listed without commas:

$$000, \quad 001, \quad 010, \quad 011, \quad 100, \quad 101, \quad 110, \quad 111.$$

The octahedral graph is the only one of the five whose vertices all have even degree. The dodecahedral and icosahedral graphs are more complicated:





To convert f into a number for a plane graph, we must count the outside as one face.

Theorem (Euler's formula). For any connected plane graph drawing, $n - e + f = 2$.

Proof. This is a remarkably universal formula. It is valid when there is just 1 isolated vertex (and so 1 outside face). We can proceed by induction on n. Each time we add an edge joined to the previous graph, either its other end is 'free' (the vertex has degree 1) or it joins up an existing vertex. In the former case, we have added 1 edge and 1 vertex, in the latter case 1 edge and 1 face. Either way $n - e + f$ does not change. □

6.2. Detecting non-planarity

Recall that:

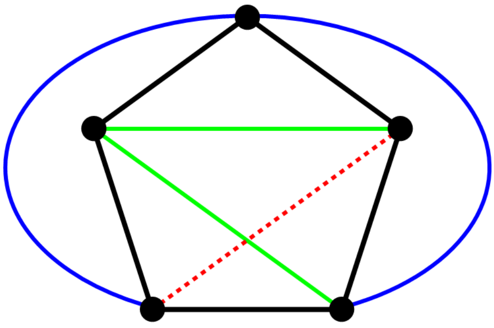
- K_n is the complete graph with n vertices and so $\binom{n}{2} = \frac{1}{2}n(n-1)$ edges;
- $K_{m,n}$ is the complete bipartite graph with $m + n$ vertices and mn edges.

In particular, K_5 is the 'starred pentagon' and $K_{3,3}$ is the 'utility' graph representing the distribution of broadband, electricity, water to three consumers. Note that the edges and vertices of a cube form a subgraph of $K_{4,4}$ obtained by removing four edges.

Proposition. Neither K_5 nor $K_{3,3}$ is planar.

It follows that no planar graph can contain K_5 or $K_{3,3}$ as a *subgraph*.

Proof. This can be done by first principles (so no theory). Consider K_5; suppose it has a plane drawing with no redundant crossings. Since the abstract graph has a 5-cycle, that (taken on its own) will determine a pentagon in the plane. The positions of the other five edges in the drawing remain to be specified, and each one must either lie inside or outside the pentagon. They cannot all lie outside without a crossing, so let us suppose one lies inside. There is no way to distinguish any of the 5 remaining edges, so we pick one and assume it lies inside. After one more choice, the inside/outside positions are forced upon us, and we are stuck at the final step.



A similar argument works for $K_{3,3}$, though this has a 6-cycle. (Recall that any cycle in a bipartite graph must be even.) □

To formulate two important results, we need two new concepts for modifying graphs, namely *homeomorphism* and *contraction*.

Definition. Two graphs G_1, G_2 are called *homeomorphic* if vertices of degree 2 can be added to one or both so that the resulting graphs are isomorphic.

Roughly speaking, this means 'adding blobs' on one or more egdes so that the two graphs correspond. Adding a single vertex of degree 2 will split one edge into two, so adding several will generate more edges.

To compare G_1 and G_2, one could also 'strip' them of vertices of degree 2 one by one provided the result is still a graph. However, this stripping operation can lead to graphs that are not simple, so the notion of isomorphism is a little more complicated. If G_1 and G_2 are homeomorphic then their degree sequences can only differ by their numbers of entries that are '2'. This is important when one is looking for subgraphs that are homeomorphic to a specific graph like K_5, which has vertex sequence $(4,4,4,4,4)$.

Example. Let G be a graph with degree sequence $(3,3,3,3,6,6)$. Draw one that is simple. Any graph homeomorphic to G must have degree sequence

$$(2,\ldots,2,3,3,3,3,6,6),$$

with zero or more '2's. If there are no '2's then the graph will be isomorphic to G, which is a special case of homeomorphism.

Homeomorphism defines an equivalence relation on graphs in which one disregards vertices of degree 2. In particular, all cycle graphs C_n with $n \geqslant 1$ are homeomorphic! But a 1-cycle (a single vertex with a loop) cannot have its vertex removed, because the result is not a graph!

Adding or removing vertices of degree 2 can be regarded as a 'trivial' operation that does not affect the essence of the graph. It is a topological notion. What we are really doing is concentrating on the set of points formed by the edges only (this set forms a 'topological space'), pretending they are made of stretchable wire. We are only interested in whether one set can be transformed into another by bending and stretching/compressing.

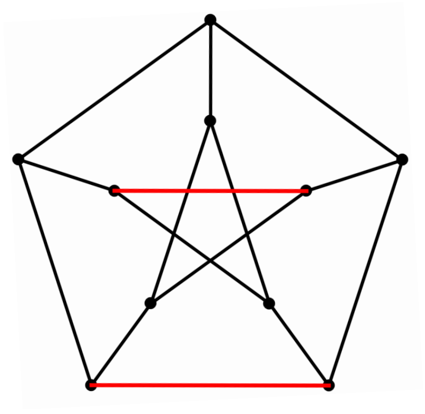
Theorem (Kuratowski, 1930). A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

Expressed another way,

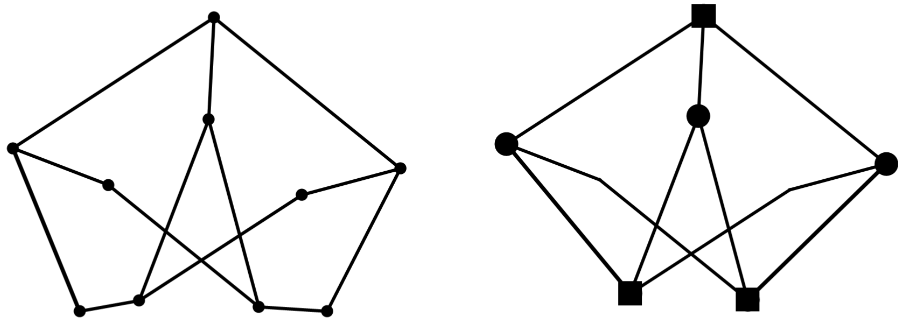
$$\text{non-planar} \iff \exists \text{ subgraph homeomorphic to } K_5 \text{ or } K_{3,3},$$

though (as remarked above) the implication $\Leftarrow$ is obvious. One can think of K_5 and $K_{3,3}$ as 'germs', one of which will *always* be present if the original graph is non-planar.

Example. Recall the Petersen graph P, which has 10 vertices and 15 edges. One guesses correctly that it is not planar. It is a regular graph with vertex degree 3, so there is no hope of finding a K_5 inside, but it does contain a subgraph homeomorphic to $K_{3,3}$. One needs to remove the two edges that are horizontal in the image below, leaving four vertices of degree 2. Note that the edges of a *subgraph* must be edges of P, so in order to talk of a subgraph we must leave the vertices in place, since they lie on other edges. But we can remove them and unite the edges so as to form a new graph homeomorphic to the subgraph:



The new graph has 6 vertices and 9 edges, since we removed 2 edges, and another four pairs of edges became four single ones. It is easy to see that the 6 vertices are partitioned into two groups of 3, with all possible edges going from one group to the other, so we are dealing with $K_{3,3}$.



This is all very well, but the operation we have performed is not very natural. Staring at P, it seems much closer to K_5 than $K_{3,3}$, and the next approach makes this precise.

Definition. If G is a graph, and uv is an edge joining vertices u and v, then the graph obtained from G by *contracting* uv, written G/uv, is formed by making u and v coalesce so that any edges that arrived at either of them now arrive at the new common vertex. In this process, any loops are eliminated and any multiple edge just becomes a single one.

A simple example would be a triangle with 3 vertices (i.e. a 3-cycle). If we contract any edge, we simply get a single edge with its two ends as vertices. Notice that the loop and extra edge are suppressed.

If we contract an edge in a plane diagram, it remains a plane diagram. One can contract a number of edges by doing one at a time. Contraction is a rough analogue of taking a *quotient* in other branches of mathematics.

Five contractions convert the Petersen graph P to the complete graph K_5. In its pentagonal representation above, we contract the 5 spokes by joining each outer vertex to its nearest neighbour inside. This is a painless operation that does not even produce multiple edges to combine.

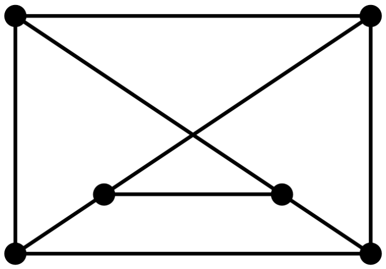
Theorem (Wagner, 1937). A graph is planar if and only if it does not contain a subgraph that can be contracted to (a graph isomorphic to) K_5 or $K_{3,3}$.

Expressed another way,

$$\text{non-planar} \iff \exists \text{ subgraph contractible to } K_5 \text{ or } K_{3,3}.$$

But this time, neither of the implications is elementary. The forward direction ($\Rightarrow$) follows immediately from Kuratowski's theorem, because any subgraph *homeomorphic* to K_5 (resp. $K_{3,3}$) can itself be contracted to K_5 (resp. $K_{3,3}$) by contracting edges so as to delete the superfluous vertices of degree 2. But there are complications the other way – if G is contractible to K_5 then it might not contain a subgraph homeomorphic to K_5, but one homeomorphic to $K_{3,3}$ instead.

Exercise. Identify three edges of P whose contraction yields $K_{3,3}$. It might help to use the following representation of $K_{3,3}$:



6.3. Further results

Recall Euler's formula for a plane graph drawing:

$$n - e + f = 2.$$

We gave an informal proof in §5.3 by showing that the left-hand side does not change when the graph is 'grown' by adding one edge at a time. A more rigorous proof can be given by induction on the number e of edges. We shall illustrate this for the special case of trees.

Recall that a *tree* is a *connected* graph with *no cycles*.

Proposition. If G is a tree then $e = n - 1$.

Proof. If $e = 1$ then $n = 2$, the result is valid. Assume the result is true for $e = N - 1$. Let G be a graph with N edges. Now any edge uv of a tree is a *bridge* – its removal disconnects the graph (because if not, there must be a path from u to v which becomes a cycle with uv). So if an edge is removed we obtain a graph with exactly two components (no more than two, because a single edge cannot connect three components). Both of the components must be trees, by definition. By assumption,

$$e = 1 + e_1 + e_2 = 1 + (n_1 - 1) + (n_2 - 1) = n - 1,$$

as stated. $\square$

Remarks. (i) We can use a similar induction argument to prove that any tree has a plane diagram (i.e. without crossings) with just an outside face. Thus $f = 1$, and the proposition is compatible with Euler's formula.
(ii) For a fixed number of vertices n, no connected graph can have less than $e - 1$ edges (exercise). Using this one can show that if G is connected then $e = n - 1$ if *and only if* G is a tree.

Proposition. If G is a simple planar graph with $e \geqslant 3$ then $e \leqslant 3n - 6$.

Proof. Represent G by a plane diagram. Each face (even if there is only one) borders at least 3 edges. Each such edge will be counted twice if it has different faces either side of it, otherwise counted once. Thus

$$2e \geqslant 3f = 3(2 + e - n),$$

which gives the result. $\square$

Lemma. Any simple planar graph has a vertex of degree at most 5, and is 6-colourable.

Proof. Recall that the sum of the vertex degrees equals $2e$. If all the degrees are at least 6, then

$$6n \leqslant \sum_{v \in V} d(v) = 2e \leqslant 6n - 12,$$

a contradiction. $\square$

We prove that $\chi \leqslant 6$ by induction on the number of vertices, n. Obviously $\chi \leqslant n$, so the result is true when $n \leqslant 6$. Suppose it is true for $n = N-1$. If G now has N vertices, remove one (call it v) of degree at most 5 and its associated edges (at most 5 of them). By assumption, the remaining graph is 6-colourable. Moreover, v only had 5 neighbouring vertices, so when we replace v and its edges we still have a 6th colour left for v. □

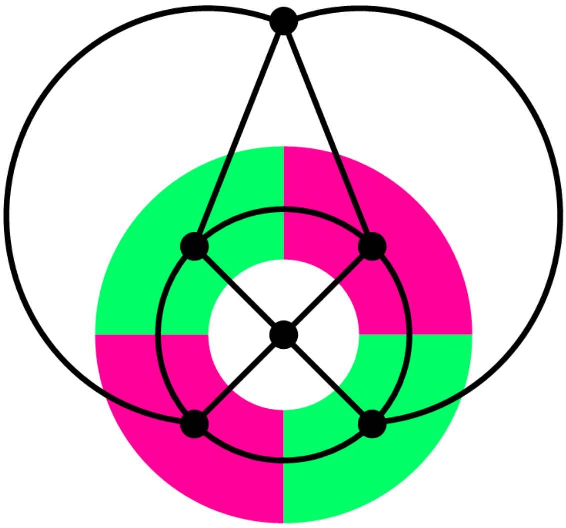
The lemma is analogous to saying that $\chi \leqslant \Delta + 1$, and it is not too hard to refine the proof to conclude that $\chi \leqslant 5$. This is effectively the five colour theorem, whose equivalent statement for maps was proved by Heawood in 1890. The four colour theorem was not resolved until almost a century later, but not without a 'proof' that involved substantial computer verification of special cases:

Theorem (Appel-Haken, 1976). Any simple *plane* graph is 4-colourable, i.e. $\chi \leqslant 4$.

*6.4. The four-colour theorem

The last theorem is more familiar as a statement about *maps* – only four colours are needed in such a way that contiguous countries are distinguished.

Here is the idea. A *map* can be defined as a plane graph drawing for which removing one or two edges will not disconnect the graph. (This excludes vertices of degree 2 and an outside face reaching both sides of an edge.) Given a map M, we can form its 'dual', which is a plane graph diagram denoted M^*, which has a vertex for each face of M and an edge joining two vertices whenever the corresponding faces are contiguous (and this edge crosses only that common border). Because of our assumptions, M^* will have no loops or multiple edges. Then a vertex colouring of M^* corresponds to a valid colouring of the map, so the theorem implies the map colouring result.



The concept of duality is well known in the context of polyhedra – as we remarked, the cube (6 faces, 8 vertices) and octahedron (8 faces, 6 vertices) are dual pairs, as are the dodecahedron (12 faces, 20 vertices) and icosahedron (20 faces, 12 vertices). The dual of a tetrehedron is another tetrahedron (the graph being K_4 with 4 faces, 4 vertices).

We conclude with some comments that link this topic to *Geometry of Surfaces*. Planar graph diagrams can be regarded as graphs on the surface of a sphere in which one point of the sphere (say the north pole p) corresponds to infinity in the plane. The outside face in the plane becomes a normal face containing p on the sphere, so this is more natural.

One can show that both K_5 and $K_{3,3}$ can be drawn on the torus without artificial crossings. One can also try to draw graphs on more complicated surfaces, in particular on a surface of *genus* g, which means a 'torus with g holes'. Provided there are no crossings, it is well known that if f now counts 'faces' on the surface, then

$$n - e + f = 2 - 2g,$$

this quantity being the so-called *Euler characteristic* of the surface. A graph that can be drawn on a surface of genus g but not on one of genus $g-1$ is called a *graph of genus* g. Thus K_4 is a graph of genus 0, and K_5 and $K_{3,3}$ are graphs of genus 1.

It is also known that a map drawn on a surface of genus g can always be coloured with a maximum of k colours, where

$$k = \lfloor \frac{7 + \sqrt{1 + 48g}}{2} \rfloor.$$

Taking $g = 0$ gives the four colour theorem as a special case. Taking $g = 1$ shows that 7 colours suffice to colour the vertices of a graph inscribed on a doughnut.

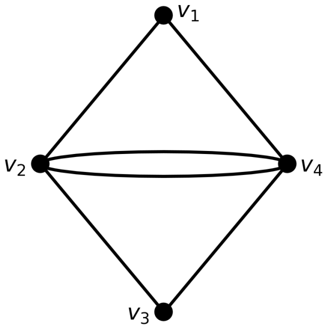
7. Navigation in graphs

7.1. Adjacency data

There are two common ways of representing graphs non-pictorially – by means of an *adjacency matrix* or an *adjacency table*. We consider these in turn.

Without assuming that it is simple, a graph G consists of a set V of vertices, and a family E of edges. Let us label the vertices $v_1 \ldots, v_n$. To specify E, we need to know the *number* a_{ij} of edges that join v_i to v_j. These numbers can be displayed as an $n \times n$ symmetric matrix, from which we can reconstruct G:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$



We can work out the degree of any vertex by taking the sum of the entries in the corresponding row (or column, since A is symmetric):

$$d(v_i) = \sum_{j=1}^{n} a_{ij}.$$

However, for this to work, a single loop at v_i should contribute 2 to a_{ii}. The graph is simple if and only if $a_{ii} = 0$ for all i, and every other entry is 0 or 1.

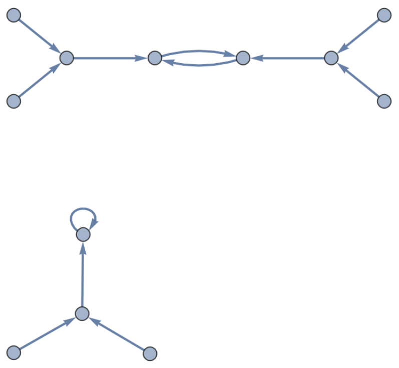
One can decide mechanically whether a graph is connected as follows. First replace any positive integer by 1, as multiple edges do not affect the answer.
Start from row $r_1 = 1$ and cross out the entire first column.
Scanning from left to right, note the first column with a 1, cross out that entire column and skip to the corresponding row, call it r_2.
Scan along r_2 to find the first 1, ignoring entries already crossed out, this defines r_3.
And so so. If at any stage, there are no more non-zero entries, return to re-scan (in any order) the rows already visited. If all their entries are eventually crossed out, the corresponding set of vertices form a component of G.

Two advantages of the matrix approach are that it can be generalized:

- to deal with digraphs by relaxing the condition that $a_{ij} = a_{ji}$, so $a_{ij} = 1$ means that $i \to j$ is a directed edge (with a loop now counting 1), see the next example;
- to deal with *simple* weighted graphs or digraphs, in which each edge is assigned a non-negative number. The lower triangular part of the matrix then resembles a table of distances between cities of the type that used to be common in motoring atlases, except that only *adjacent* nodes have non-zero entries.

Example. Below is the sparse adjacency matrix of the disconnected digraph that represents squaring modulo 13, with vertices representing the 12 non-zero residue classes:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$



The smaller component consists of the residue classes 1 (with the loop), $12 = -1$, 5 and $8 = -5$ (the last two square to $-1 \bmod 13$). If we re-label the vertices so that $1, 5, 21, 26$ come at the start then the matrix will have a block form, making it obvious that there are (at least) two components.

Another method of representing an ordinary graph is by its adjacency table. The one with adjacency matrix A above has table

1	2	3	4
2	1	2	1
4	3	4	2
	4		2
	4		3

The top row lists the four vertices, and each column below lists (in any order) all the vertices adjacent to the one on top. In this system, there is no need to label the vertices with numbers, in this example we could equally well use a, b, c, d. In the next section, we provide such a table by with the columns replaced by rows.

One can easily adapt the matrix methods, for example to detect the existence of cycles

$$1 \to 2 \rightrightarrows 4 \to 1, \qquad 1 \to 2 \to 3 \to 4 \to 1$$

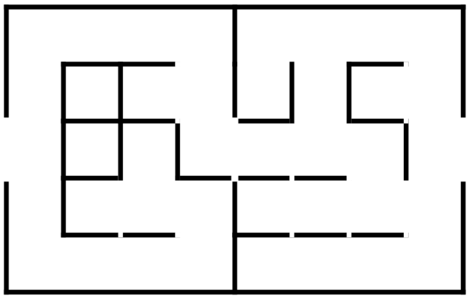
by passing from column to column.

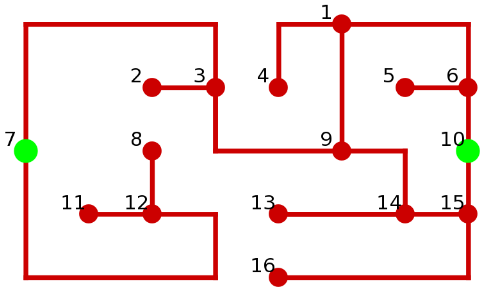
7.2. Search trees

We discuss two different methods of searching through the vertices of graphs. These are

- a *depth first search* (DFS) that uses a *stack*;
- a *breadth first seach* (BFS) that uses a *queue*.

Example. We can convert a maze (left) into a graph (right) by placing a vertex at each position where a choice is needed (including the start and end) and at a dead-end. A walk in this graph is a walk in the maze.





This graph can be input into a computer by typing its adjacency table. In SAGE, this is

```
adj   =   { 1:   [4,6],
            2:   [3],
            3:   [2,7,9],
            4:   [1],
            5:   [6],
            6:   [1,5,10],
            7:   [3,12],
            8:   [12],
            9:   [1,3,14],
           10:   [6,15],
           11:   [12],
           12:   [7,8,11],
           13:   [14],
           14:   [9,13,15],
           15:   [10,14,16],
           16:   [15] }
```

Depth first search. This is carried out by processing the data from the adjacency table in a **stack**, shown below (overleaf). At each stage, the vertex being processed is the one on the right. This represents the 'top' of the stack, which we are allowed to 'peek'. We seek to add or 'push' an *adjacent* vertex to the stack if there exists an adjacent vertex that has not already been processed.

In our particular implementation of DFS, we add at most §one adjacent vertex to the stack (for definiteness, the smallest in our list), and we keep a separate record of those vertices that have (at some point) entered the stack. If the vertex being processed has no new neighbours (either because it has degree one, or because its neighbours are in the stack), we remove or 'pop' it from the stack. Each vertex can appear at most once in our stack, and each vertex appears twice in our table, once added, once removed. This means that the table must contain $2n$ rows, where n is the number of vertices (here $n = 16$).

DFS stack $\leftrightarrows$	added	removed
7	7	
$7, 3$	3	
$7, 3, 2$	2	
$7, 3$		2
$7, 3, 9$	9	
$7, 3, 9, 1$	1	
$7, 3, 9, 1, 4$	4	
$7, 3, 9, 1$		4
$7, 3, 9, 1, 6$	6	
$7, 3, 9, 1, 6, 5$	5	
$7, 3, 9, 1, 6$		5
$7, 3, 9, 1, 6, 10$	10	
$7, 3, 9, 1, 6, 10, 15$	15	
$7, 3, 9, 1, 6, 10, 15, 14$	14	
$7, 3, 9, 1, 6, 10, 15, 14, 13$	13	
$7, 3, 9, 1, 6, 10, 15, 14$		13
$7, 3, 9, 1, 6, 10, 15$		14
$7, 3, 9, 1, 6, 10, 15, 16$	16	
$7, 3, 9, 1, 6, 10, 15$		16
$7, 3, 9, 1, 6, 10$		15
$7, 3, 9, 1, 6$		10
$7, 3, 9, 1$		6
$7, 3, 9$		1
$7, 3$		9
7		3
$7, 12$	12	
$7, 12, 8$	8	
$7, 12$		8
$7, 12, 11$	11	
$7, 12$		11
7		12
$\varnothing$		7

DFS abbreviated version. Use of a table is advised to gain confidence in the method, but for some purposes it suffices to list the vertices in the order in which they are encountered, and add ‘ties’ joining neighbouring vertices that are not already adjacent in the list:

$$\overbrace{7,\ \overbrace{3,\ 2,\ 9},\ \overbrace{1,\ 4,}\overbrace{6,\ 5,\ 10},\ \overbrace{15,\ 14,\ 13,\ 16},}\ \overbrace{12,\ 8,\ 11}.$$

It is a consequence of the methods that all such ties are ‘nested’, with no crossings.

We adopt the following conventions, reflecting the fact that we read/type/write from left to right.

- for a *stack*, items are added on the right, and removed from the right;
- for a *queue* (next example), items are added on the right, but removed from the left.

Breadth first search. This is carried out by inserting data into a **queue**. When processing a vertex in a BFS, it is important to add *all* its adjacent vertices before moving on. This is exactly what we did in the 2-part algorithm to re-order vertices prior to colouring them. Although our table should show each addition and removal step by step, we can save time by using one row for each vertex being processed (having just been removed). This way, we only have n rows, excluding the first:

removed	$\leftarrow$ BFS queue $\leftarrow$
	7
7	$3, 12$
3	$12, 2, 9$
12	$2, 9, 8, 11$
2	$9, 8, 11$
9	$8, 11, 1, 14$
8	$11, 1, 14$
11	$1, 14$
1	$14, 4, 6$
14	$4, 6, 13, 15$
4	$6, 13, 15$
6	$13, 15, 5, 10$
13	$15, 5, 10$
15	$5, 10, 16$
5	$10, 16$
10	16
16	$\varnothing$

BFS abbreviated version. As in the 2-part colouring algorithm, some of the information can be presented by a single long list:

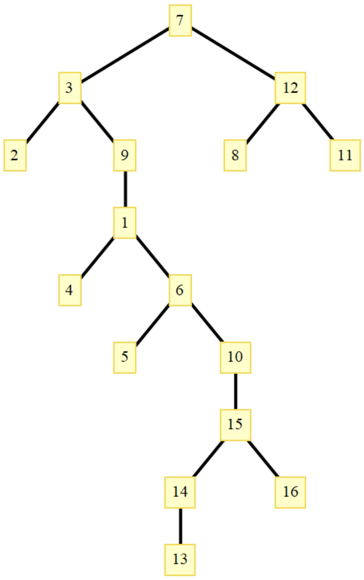
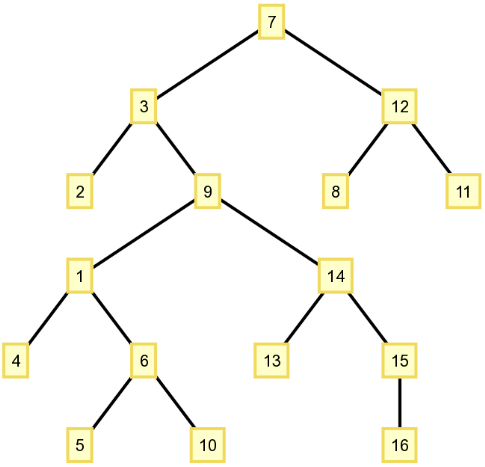
	✓	✓	✓	✓	✓	✓	✓	✓	✓							
vertices:	7	3	12	2	9	8	11	1	14	4	6	13	15	5	10	16
level:	0	1	1	2	2	2	2	3	3	4	4	4	4	5	5	5

Here we have processed up to and including vertex 14, in the latter case by adding $13, 15$. It is a consequence of the method that vertices are added level by level, and the last row indicates their 'distance' to the start.

Definition. Given a connected graph G, a *spanning tree* is a subgraph of G without cycles that includes all the vertices of G.

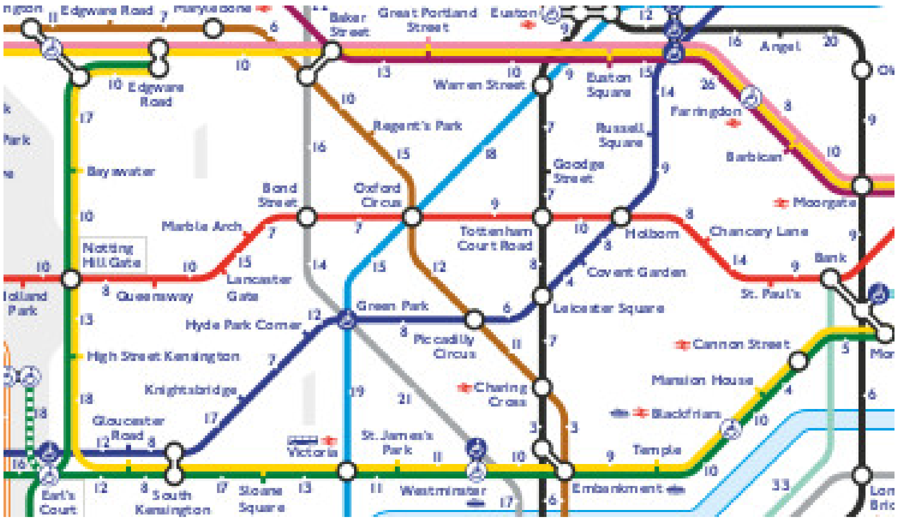
Both searches determine such a spanning tree, although the way they do this reflects their names. The trees are *rooted*, because we have distinguished a starting point – vertex 7 is the root. We can now re-draw the tree growing (by convention) downwards, level by level. The 'dead-end' vertices

$2, 4, 5, 8, 11, 13, 16$ all define *leaves* of the trees, but the BFS tree (right) happens to have an extra leaf at the finish.

In our example, G was not itself a tree, having a cycle $(9, 1, 6, 10, 15, 14, 9)$ and the two trees break this in different ways. The DFS tree omits the edge $(14, 9)$ whereas the BFS one omits $(10, 15)$. The DFS tree (left) has *height* 8, this being the maximum number of edges from root to leaf, achieved by arriving at the dead-end 13. By contrast, the BFS tree is more spread out and has height 5.

7.3. Shortest paths



A *weighted graph* consists of a graph $G = G(V, E)$ in which positive numbers (often, integers) have been assigned to each edge. We shall assume that G is simple. The weights define a function

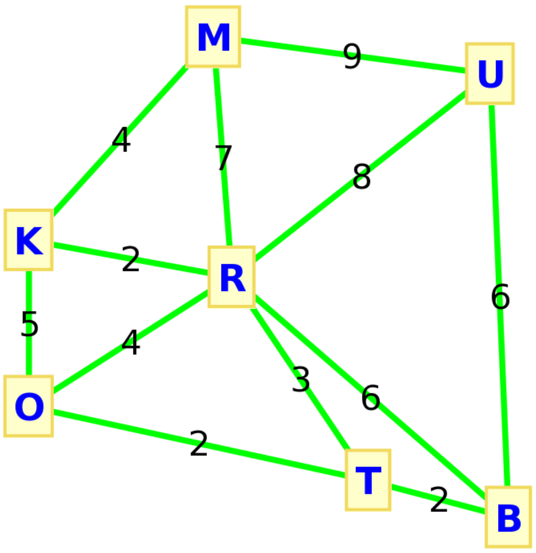
$$d\colon E \longrightarrow (0, \infty)$$

(we use d for 'distance' because w could be confused with a vertex). An example consists of points on a transport network with distances or travel times between nodes. The TfL map is arguably less practical – it displays walking times between stations – but is a good illustration.

The aim of this section is to present and justify Dijkstra's algorithm for finding shortest paths from some fixed root vertex to all the others in a weighted graph.

An easy way of doing this in which the weight of each edge equals 1 is based on BFS. Dijkstra's algorithm generalizes this procedure by introducing a priority in the queue. It also produces a tree incorporating the shortest paths but the nature (breadth/depth) of this tree depends on the weights.

Simple Example. The vertices of this graph represent cities taken from a former Soviet road atlas (R is Rostov on the river Don):



The problem is to find the *shortest path* from B to M, in these notes written $\mathrm{SP}(B, M)$. But first we find the 'length' of this path, meaning the sum of the weights of its edges, this is the *shortest distance* from B to M, written $\mathrm{SD}(B, M)$.

In this example, we might (or might not) be able to spot the SP: it is

$$B \to T \to R \to K \to M,$$

and $\mathrm{SD}(B, M) = 11$. We illustrate the method with a table. At the start, the shortest distance from $B = v_0$ to itself is obviously 0 and this it its permanent label. The other distances are provisionally set to ∞:

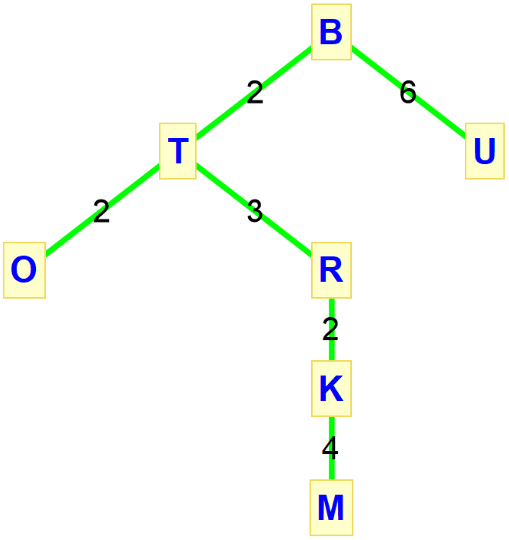
	B	K	M	O	R	T	U
	$\boxed{0}$	∞	∞	∞	∞	∞	∞
$B \rightarrow$		∞	∞	∞	6	$\boxed{2}$	6
$T \rightarrow$		∞	∞	$\boxed{4}$	5		6
$O \rightarrow$		9	∞		$\boxed{5}$		6
$R \rightarrow$		7	12				$\boxed{6}$
$U \rightarrow$		$\boxed{7}$	12				
$K \rightarrow$			$\boxed{11}$				
$M \rightarrow$							

Then B is selected as a 'permanent' vertex, meaning that its label correctly indicates its shortest distance from B (obviously so, since this is 0). The row against B then updates the distances of those vertices adjacent to B. The least of these arises from the edge BT and so T is moved over and assigned a permanent label of 2 [also boxed] that does not change thereafter. Other distances are updated if we can reach a vertex in less distance from T (e.g. the distance from B to R reduces to 5).

We conclude that $\mathrm{SD}(B,M) = 11$. The method actually determines a tree that encodes the shortest path from B to every other vertex. To find the edges of the tree, one inspects each column to find where the boxed distances *first* appeared:

$$BT,\ TO,\ TR,\ BU,\ RK,\ KM.$$

A tree spanning 7 vertices must have $7-1=6$ edges.



Let's write down the general procedure.

Dijkstra's algorithm. This takes as input: a weighted graph $G(V,E)$ and a distinguished vertex v_0. Let $V = \{v_0, v_1, \dots, v_n\}$. Its output consists of a list of the shortest distances $L_i = \mathrm{SD}(v_0, v_i)$ for each vertex v_i. Obviously $L_0 = 0$.

```
$V_{\text{perm}} = \varnothing$
$V_{\text{temp}} = V$
$L_0 = 0$
for $j > 0$:
    $L_j = \infty$

while $V_{\text{perm}} \neq V$:
    choose $v_i \in V_{\text{temp}}$ with $L_i$ minimal
    $V_{\text{perm}} = V_{\text{perm}} \sqcup \{v_i\}$
    $V_{\text{temp}} = V_{\text{temp}} \setminus \{v_i\}$
    for $v_j \in V_{\text{temp}}$ adjacent to $v_i$:
        $L_j = \min(L_i + d(v_i v_j),\ L_j)$

return $(L_0, L_1, \dots, L_n)$
```

Before applying each 'while' loop, the vertex set is partitioned into a disjoint union

$$V = V_{\text{perm}} \sqcup V_{\text{temp}}.$$

(Initially, V_{perm} is empty.) We then select the temporary vertex v_i with the least label L_i and make it permanent. Its label L_i is not modified again, and (we shall show) represents the length of a shortest path $v_0 \rightsquigarrow v_i$. (In the first loop, this is obvious, because we are obliged to choose v_0 with $L_0 = 0$.) We then use v_i to scan each remaining temporary vertex v_j adjacent to it, and update the label L_j if we obtain to shorter path to v_j via v_i. (In the second loop, this simply amounts to providing finite labels $L_j = d(v_0, v_j)$ to all vertices adjacent to v_0, but in the third loop we may discover a vertex v_j for which

$$d(v_0, v_i) + d(v_i, v_j) < d(v_0, v_j)$$

in which case L_j decreases.)

Sometimes, V_{perm} is called the *closed set*, and the subset of V_{temp} with finite labels (consisting of those temporary vertices that have been visited) is called the *open set*.

Special case. An ordinary graph can be treated as a weighted one in which any two adjacent vertices u, v have $d(u,v) = 1$. In this case, we can still apply the boxed algorithm, but it will just be matter of moving vertices one by one from V_{temp} to V_{perm}. The labels will be incremented by 1 once we have moved all those with equal labels into V_{perm}. Once a vertex acquires a finite label, no re-labelling takes place because an inequality like the one above is impossible. This process then provides a very effective method for finding the shortest path between any two vertices in a complicated graph.

8. Optimality

8.1 Shortest paths

The aim of this section is to prove that Dijkstra's algorithm is 'correct'. The next result is phrased using the notation of §7.3.

Lemma. Suppose that a shortest path $v_0 \rightsquigarrow u \rightsquigarrow v_i$ between two vertices v_0, v_i in a weighted graph passes via an intermediate vertex u. Then both subpaths $v_0 \rightsquigarrow u$ and $u \rightsquigarrow v_i$ are shortest paths between their respective vertices, and in particular

$$\mathrm{SD}(v_0, v_i) = \mathrm{SD}(v_0, u) + \mathrm{SD}(u, v_i).$$

Proof. If one of the 'subpaths' is not shortest, then we could substitute it with a shorter one, giving a shorter path $v_0 \rightsquigarrow v_i$. It's that simple! □

For the lemma, we are assuming that the path between v_0 and v_i is a shortest one. If this were not the case, we only have the triangle inequality

$$\mathrm{SD}(v_0, v_i) \leqslant \mathrm{SD}(v_0, u) + \mathrm{SD}(u, v_i).$$

The lemma is an instance of an important argument called 'Bellman's optimality principle' that crops up in different guises in many problems in optimization theroy.

Let us return to Dijkstra's algorithm. We wish to prove that all the permanent labels of vertices in V_{perm} are correct shortest distances:

Theorem. The label L_p that Dijkstra's algorithm assigns to each vertex $v_p \in V_{\text{perm}}$ does indeed equal its shortest distance from v_0.

Proof. We shall prove this by induction on $|V_{\text{perm}}|$.
The statement is certainly true when $|V_{\text{perm}}| = 1$ because then $V_{\text{perm}} = \{v_0\}$ and $L_0 = 0$.
Now let $v_i \in V_{\text{temp}}$ denote the temporary vertex chosen at a later stage because its label L_i is minimal. Since the algorithm will then move v_i into V_{perm} and make L_i permanent, it suffices to show that

$$L_i = \mathrm{SD}(v_0, v_i).$$

Suppose that

$$v_0 \rightsquigarrow v_p \to v_q \rightsquigarrow v_i$$

is a shortest path from v_0 to v_i. Here we have chosen intermediate and *adjacent* vertices $v_p \in V_{\text{perm}}$ and $v_q \in V_{\text{temp}}$; this is clearly possible and it may be that $q = i$. Our inductive hypothesis is that every vertex in V_{perm} is labelled by its correct shortest distance, so in particular $L_p =$

$\mathrm{SD}(v_0, v_p)$. Then

$$\begin{array}{rlll}
L_i & \leqslant & L_q & \text{minimality} \\
& \leqslant & L_p + d(v_p, v_q) & \text{definition of } L_q \text{ when } v_p \text{ scanned } v_q \\
& \leqslant & L_p + \mathrm{SD}(v_p, v_i) & \text{edge is part of the SP} \\
& = & \mathrm{SD}(v_0, v_p) + \mathrm{SD}(v_p, v_i) & \text{by hypothesis} \\
& = & \mathrm{SD}(v_0, v_i) & \text{by the previous lemma.}
\end{array}$$

But L_i is the distance to v_i via *some* path, so it must *equal* $\mathrm{SD}(v_0, v_i)$, and (incidentally) all the inequalities are equalities.
This completes the induction. □

Our previous example. The proof helps to explain why after this step

B	T	O	R	U	‖	K	M
0	2	4	5	6	‖	$\boxed{7}$	12

we can be certain that $\mathrm{SD}(B, K) = 7$. Since $R \in V_{\text{perm}}$ we know that $\mathrm{SD}(B, R) = 5$. Since $7 < 12$, we can be certain that the SP from B to K must pass through R, and we can take

$$v_0 = B, \quad v_p = R, \quad v_q = v_i = K.$$

Therefore $\mathrm{SD}(B, K) = 7$.

The crucial technique in Dijkstra's algorithm is the act of relabelling: once v_i becomes permanent we scan its adjacent vertices and reduce their labels if passing through v_i gives a shorter path:

$$L_j = \min(L_i + d(v_i, v_j),\ L_j).$$

The act of relabelling is called *relaxation* of the edge $v_i v_j$, and its repeated use allows one to decrease the estimated shortest distances until they become correct. Dijkstra's algorithm has the characteristic that it grows a tree of shortest paths from the root. There are other shortest path algorithms that apply relaxations in a more brute force (and thus, simpler) way, whilst still eventually achieving a shortest path.

8.2. Kruskal's algorithm

In this section, G is always a *simple connected* weighted graph. Let n denote the number of its vertices.

We have seen a number of algorithms that, when applied to G, construct a *spanning tree*. This is a subgraph with the same vertex set as G, and since it is a tree, it will have $n - 1$ edges. In particular, Dijkstra's algorithm does this, provided we give it a starting vertex to act as root.

A different connectivity problem concerns the construction of a *minimal spanning tree* or MST. This means a spanning tree whose *total weight*

$$\sum_{uv \in E} d(u, v)$$

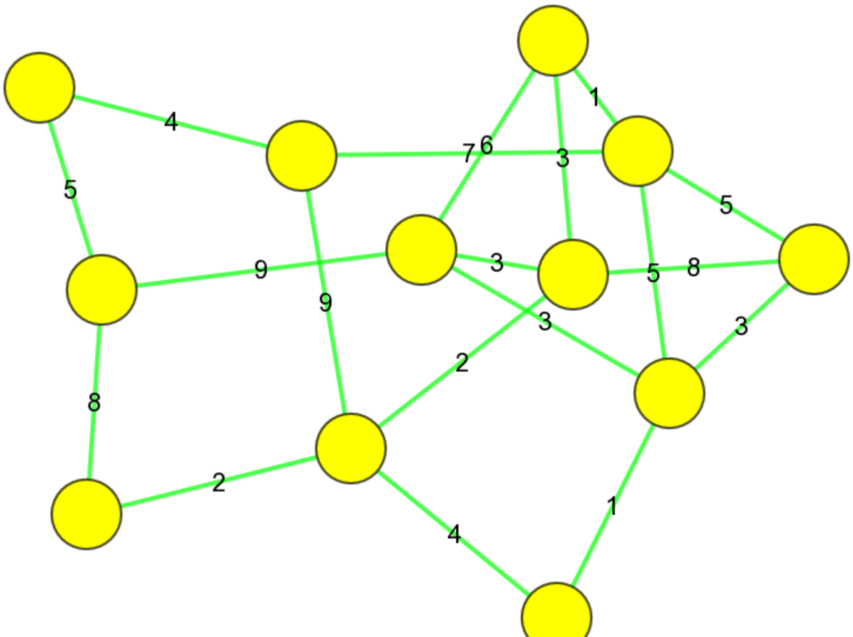
is least. Any connected graph has a spanning tree, because whenever there is a cycle one can remove any edge in that cycle, leaving all the vertices connected, and continue until there are no more cycles. Therefore a *minimal* spanning tree must exist, and (if there is more than one) the total weight of any two are equal by definition.

Kruskal's algorithm. Its input is a simple connected graph $G(V, E)$ with $|V| = n$. Its output is a subset $T_{\mathrm{K}} \subseteq E$ of size $n - 1$ that forms a minimal spanning tree.

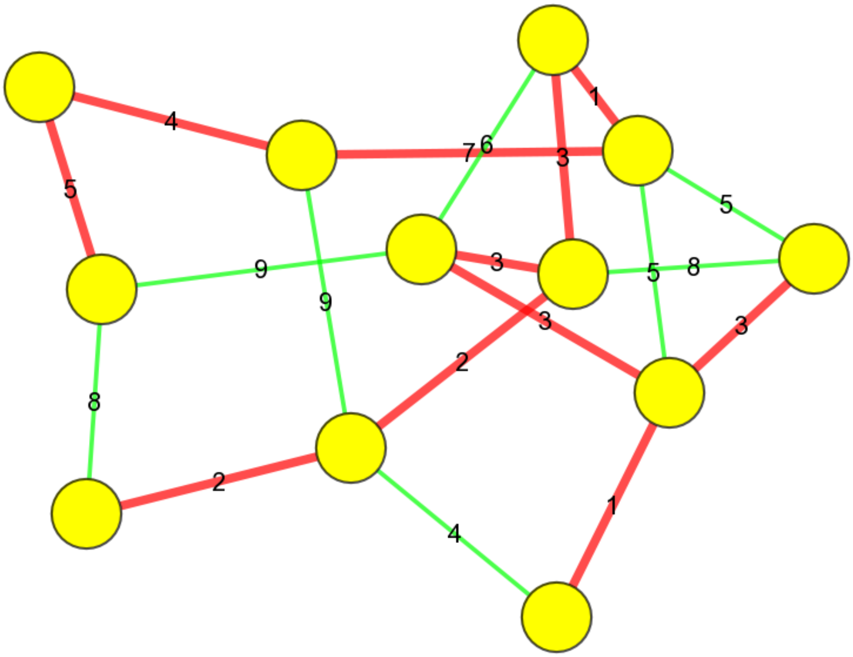
It is run by means of the following instructions:

```
<$F = \varnothing$
while $|F| < n - 1$:
    choose an edge $uv \in E \setminus F$ of minimal weight:
        if adding $uv$ to $F$ gives a cycle:
            discard $uv$
        else:
            $F = F \sqcup \{uv\}$
$T_{\mathrm{K}} = F$
return $T_{\mathrm{K}}$
```

Example. Applying Dijkstra's algorithm with root bottom left in the following graph gives a tree of weight 41. Applying Kruskal's algorithm gives a MST with weight 34.



The vertices are drawn with large circles to allow one to record the labels (temporary and permanent) in applying Dijkstra's algorithm.



Kruskal's is a prototype 'greedy algorithm' since it executes what seems to be the optimal choice at each step. One could imagine (for example) that at each stage one should only add edges that are connected to ones already chosen, so that the MST is 'grown' branch by branch. (It turns out that this procedure is also valid – it is called Prim's algorithm and works well when the graph is defined by an adjacency matrix.) In any case, it is far from obvious that Kruskal's procedure works, but this is what the next result assures us:

Theorem. T_{K} is a minimal spanning tree.

Proof. Note that at each intermediate stage, F is 'forest' consisting of one or more trees, and $|F| < n$. Since $|T_{\mathrm{K}}| = n-1$ it can have only one connected component, and must include all n vertices. We need to prove that its total weight is *minimal* amongst all spanning trees.

Let T_{J} be a *minimal* spanning tree (one certainly exists!) such that $T_{\mathrm{J}} \cap T_{\mathrm{K}}$ is as large as possible. Think of elements of T_{J} as red, of T_{K} as blue.

Choose an element of $T_{\mathrm{K}} \setminus T_{\mathrm{J}}$ of least weight, call it $\boldsymbol{uv}$ and set $d = d(u, v)$. Think of it as DEEP BLUE!

In a tree, any two vertices are joined by a *unique* path. (For if the path were not unique, we would have a cycle, which is impossible.) This observation is used twice next:

- The path $u \rightsquigarrow v$ in T_{J} must have an edge $\boldsymbol{u'v'} \in T_{\mathrm{J}} \setminus T_{\mathrm{K}}$, otherwise T_{K} would have a cycle. Think of $\boldsymbol{u'v'}$ as DEEP RED, and call its weight d'.
- The path $u' \rightsquigarrow v'$ in T_{K} must similarly have an edge $\boldsymbol{u''v''} \in T_{\mathrm{K}} \setminus T_{\mathrm{J}}$. It is again DEEP BLUE; call its weight d''.

We now claim that

(i) $d \leqslant d''$,

(ii) $d \leqslant d'$.

(i) is true because we chose uv to have *least* weight in $T_{\mathrm{K}} \setminus T_{\mathrm{J}}$.

(ii) is true because if not d' would have been added to $F \subseteq T_{\mathrm{K}}$ before uv:

> To see this, suppose for a moment that $\ell' < d$. At the stage the algorithm is applied to any edges of weight d', the edge $\boldsymbol{u''v''}$ would not have been part of F since $d' < d < d''$. nor could adding $\boldsymbol{u'v'}$ have created a cycle in F, because in that case there would already be a path $u' \to v'$ in T_{K} preventing the later addition of d''. So $\boldsymbol{u'v'}$ would have been added to $F \subseteq T_{\mathrm{K}}$. But this is a contradiction.

Finally, consider

$$(T_{\mathrm{J}} \setminus \{\boldsymbol{u'v'}\}) \sqcup \{\boldsymbol{uv}\};$$

this is an MST with a *larger* intersection with T_{K}, which is a contradiction. $\square$

8.3. Back to matrices

The aim of this section is to link our study of spanning trees to some matrix algebra. We'll be using the prefix 'ADJ' for both the 'ADJacency' matrix and the 'ADJugate' (sometimes called 'ADJoint') matrix.

Revision on matrix algebra. Let $A = (a_{ij})$ be a square $n \times n$ matrix, and recall the notion of *cofactor*, used in the computation of inverses. Namely, define

$$c_{ij} = (-1)^{i+j}(\text{sub-determinant formed from } A \text{ by deleting row } i \text{ and col } j).$$

The transpose $C^\top$ is the so-called *adjugate* matrix $\widetilde{A} = \operatorname{adj} A$:

$$\widetilde{A}_{ij} = c_{ji},$$

and

$$A\widetilde{A} = (\det A)I = \widetilde{A}A.$$

Of course,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \operatorname{adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If A is invertible then

$$A^{-1} = \frac{1}{\det A}\widetilde{A}.$$

However, $\widetilde{A}$ can still be useful when A is not intertible; here's an enticing example:

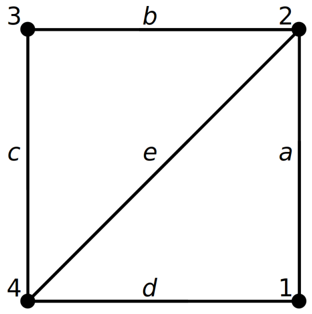
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad \widetilde{A} = -3 \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

Let G be a simple graph with n vertices. Consider the following $n \times n$ matrices:

$$\begin{aligned} A &= \text{the adjacency matrix of } G \\ D &= \text{the diagonal matrix of vertex degrees} \\ L &= D - A. \end{aligned}$$

L is called the *Laplacian matrix* of th graph G. It is obviously symmetric, and all its rows (or columns) add up to zero. In fact, *the rank of* L *equals* n *minus the number of components of* G.

Example. Consider this graph with 4 vertices and 5 edges:



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Exercise. Compute A^2 and check that its (i,j)th entry is the number of walks of length 2 from vertex i to vertex j. Explain why, more generally, $(A^n)_{ij}$ is the number of walks of length n from i to j.

*8.4. The graph Laplacian

Now we turn attention to the matrix L. Consider the characteristic polynomial

$$\det(L - xI) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_4 - x).$$

Since L is not invertible, at least one eigenvalue must vanish, say $\lambda_4 = 0$. Then

$$\det(L - xI) = x^4 - 10x^3 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)x^2 - \lambda_1\lambda_2\lambda_3 x.$$

One the other hand, this equals

$$\begin{aligned}
\det\begin{pmatrix} 2-x & -1 & 0 & -1 \\ -1 & 3-x & -1 & -1 \\ 0 & -1 & 2-x & -1 \\ -1 & -1 & -1 & 3-x \end{pmatrix} &= \det\begin{pmatrix} 2-x & -1 & 0 & -1 \\ -1 & 3-x & -1 & -1 \\ 0 & -1 & 2-x & -1 \\ -x & -x & -x & -x \end{pmatrix} \\
&= \det\begin{pmatrix} 2-x & -1 & 0 & -x \\ -1 & 3-x & -1 & -x \\ 0 & -1 & 2-x & -x \\ -x & -x & -x & -4x \end{pmatrix} \\
&= -4x\, c_{44} + O(x^2),
\end{aligned}$$

where $O(x^2)$ gathers all the terms in x^2, x^3, x^4. (The first step above was to add the first three rows to the last one to give a row of $-x$'s, the second was to add the first three columns to the last one.) Therefore,

$$\lambda_1\lambda_2\lambda_3 = 4c_{44} = 32.$$

More to the point, by crossing out other rows/columns, we can see that *all* the cofactors of L are are equal! The same argument gives

Lemma. For a simple connected graph, all the cofactors of L are equal (to $1/n$ times the product of its non-zero eigenvalues).

Kirchoff's matrix tree theorem. This number equals the number of spanning trees in the simple connected graph G.

Idea of proof. This is based on another matrix associated to a graph, its incidence matrix. Or rather, the incidence matrix M associated to a digraph. First, we need to 'orient' the edges of G arbitrarily, as in the picture on the previous page. Then rows of M represent vertices, columns edges, and a 1 (resp. -1) in a column means that the edge leaves (resp. enters) the vertex associated to that row. With vertices labelled $1, 2, 3, 4$ and edges labelled $a, b, c, d, e,$ this gives

$$M = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

It is easy to understand that

$$L = MM^\top.$$

The sum of the rows of M is also zero. In general, for a connected graph, the rank of M equals $n-1$, and (this is the key point) one can show that *a subset of $n-1$ edges forms a tree if and only if the determinant of that submatrix is non-zero.* We do not lose information by deleting any row of M, say the last, to define the *reduced incidence matrix* R.

The proof of Kirchoff's theorem is now a matter of computing

$$c_{nn}(L) = \det(RR^\top)$$

as a sum of products of sub-determinants of R. A generalization of the usual rule for $\det(AB)$ says how to do that. $\square$

Example. The complete graph K_3 obviously has 3 spanning trees, and K_4 has $\binom{6}{3} - 4 = 16$. Using Kirchoff's theorem (and the trick of adding rows to simplify the cofactor calculation), one quickly obtains

Corollary. The complete graph K_n has n^{n-2} spanning trees.

Actually, we can forget about K_n, and n^{n-2} counts *labelled trees with n vertices.* This fact was known to Cayley, and Prüfer explained how such trees can be described by sequences of numbers $(a_1, \ldots, a_{n-2})$ with $a_i \in \{1, \ldots, n\}$.

9. Networks and flows

Assumptions. For the purpose of this course, a *network* is a weighted digraph, so each edge has both a *positive* number and an arrow associated to it, and the underlying graph is simple and connected.

A network therefore possesses a set V of vertices, and a set E of directed edges that we can regard as a subset of $V \times V$. We shall often write uv or $u \to v$ to indicate that $(u,v) \in E$, and $u \rightleftharpoons v$ to mean that $\{u,v\}$ is an edge of the underlying graph. Thus, $u \rightleftharpoons v$ means that either $(u,v) \in E$ or $(v,u) \in E$, but not both because we are assuming that G is simple. If $(u,v) \in E$, we shall denote by $d(u,v)$, or (from §9.2 onwards) $c(u,v)$, the positive weight assigned to it.

A *path* will mean a path in the underlying graph without reference to the arrows, so if it has k edges we can indicate it by

$$v_0 \rightleftharpoons v_1 \rightleftharpoons v_2 \rightleftharpoons \cdots \rightleftharpoons v_k.$$

We shall use the expression *aligned path* from v_0 to v_k to imply that all the arrows are forward pointing:

$$v_0 \to v_1 \to v_2 \to \cdots \to v_k.$$

All the networks in this section will possess two special vertices, a 'start' or 'source' s from which all adjacent edges point away, and an 'end' or sink' t to which all adjacent edges point towards. We shall impose one extra technical connectivity condition, namely that every vertex v lies on an aligned path from s to t.

In a network, the vertices are often called 'nodes' and the directed edges 'arcs'.

*9.1. Activity networks

In this set-up, one is given a list of tasks or 'activities', each of which takes a given time to perform. Some activities cannot be started until others have been completed – these are the so-called dependencies. One must determine (i) the minimum total time for all the activities to be completed with the dependencies being preserved, and (ii) which activities are critical, meaning ones that delay the whole project if they overrun.

The problem is represented by a network in which:

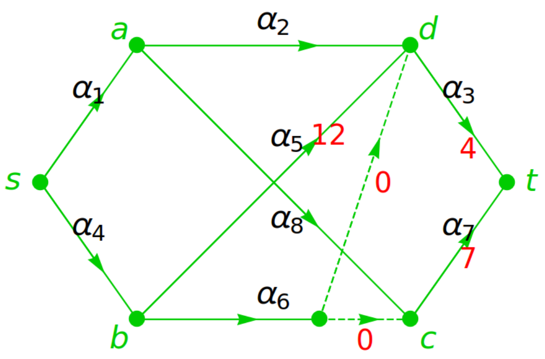
- the vertices are events, which are points in time;
- the edge represent activities, weighted by duration;
- there is a start vertex s and a finish vertex t.

At the event u, each activity with u as starting point depends on all the activities that have u as a finishing point. In particular:

- activities that do not depend on others will have s as initial point;
- activities that have no others depending on them have t as endpoint.

Example. The next table generates the network shown below it:

activity:	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
duration:	6	7	4	9	12	7	7	8
dependent on:	$-$	α_1	α_2, α_5	$-$	α_4	α_4	α_6, α_8	α_1



It is easy to unintentionally create extra dependencies when drawing the directed graph, so always check your graph against the original table.

One associates to each event u two numbers:

- $E(u)$ is the *earliest start time* for all successive activities. It is the length of a *longest* aligned path from s to u, and (by the 'optimality lemma' from §8.1) satisfies

$$E(u) = \max_{x \to u} \Big\{ E(x) + d(x,u) \Big\}$$

(the maximum is taken over all arcs from x to u), with $E(s) = 0$. We set

$$E(t) = \tau,$$

this is the least time required to complete the entire project.

- $L(u)$ is the *latest finish time* for all preceding activities in order to finish the project without overrunning. It satisfies

$$L(u) = \min_{u \to y} \Big\{ L(y) - d(u,y) \Big\},$$

with $L(t) = \tau$. Then $\tau - L(u)$ is the length of a *longest* aligned path from u to t.

Starting from $E(s) = 0$, one finds all the E values by working from start to finish (the 'forward pass'). At each stage, pick a vertex u for which the E values of all its predecessors have been calculated and use the formula above to calculate $E(u)$. Stop when all the E values have been found, and t has been reached.

Then set $E(t) = L(t)$, and work backwards to find the L values. At each stage, pick a vertex u for which the L values of all its successors have been calculated and use the formula above to calculate $L(u)$. Stop when all the L values have been found.

Given a vertex u, there exist longest aligned paths

$$s \rightsquigarrow u \text{ with length } E(u), \qquad u \rightsquigarrow t \text{ with length } \tau - L(u).$$

The combined path is not necessarily longest, so its length is at most τ, thus

$$E(u) \leqslant L(u).$$

But we have $E(t) = \tau - L(s)$, i.e.

$$L(s) = 0.$$

If these relations do not hold there is a mistake!

Back to the example. We find the event table

	$E(u)$	$L(u)$
s	0	0
a	6	10
b	9	9
c	16	18
d	21	21
t	25	25

How much time is allowed for each activity $u \to v$? This is the difference between the latest time it can finish and the earliest time it can start, i.e. $L(v) - E(u)$, and can be compared with the actual duration. The difference is the *float* or 'slack', the extra time that the activity can take without holding up the project:

$$F(u,v) = L(v) - E(u) - d(u,v).$$

Since $L(v) - d(u,v) \geqslant L(u)$, we have

$$F(u,v) \geqslant L(u) - E(u).$$

A similar argument allows us to replace u by v on the right-hand side. In any case, we can be certain that $F(u,v) \geqslant 0$.

Definition. (i) An event u is called critical if $E(u) = L(u)$.
(ii) An activity uv is called critical if $F(u,v) = 0$.

The start s and finish t are automatically critical – it's the other critical events that interest us. From above, the two ends of a critical activity will be critical events, but more it true:

Lemma. If an activity is critical, it forms part of an aligned path from s to t all of whose edges are critical.

Such a path is called a *critical path* and is merely a aligned path from s to t of maximal length τ. The lemma can then be proved by noting that

$$F(u,v) = 0 \quad \Rightarrow \quad L(v) = E(u) + d(u,v) \ \leqslant \ E(v),$$

which impies that $E(v) = L(v)$. This means that there is an aligned path from start to finish via v of length $E(v) + (\tau - L(v)) = \tau$. □

Sometimes the critical activities can be easily identified from a critical path. But if the critical events give rise to several possible paths from start to finish, one must check each to see which paths are critical.

Our example has activity table

activity	$u \to v$	$E(u)$	$L(v)$	$L(v) - E(u)$	$F(u,v)$
α_1	$s \to a$	0	10	10	4
α_2	$a \to d$	6	21	15	8
α_3	$d \to t$	21	25	4	0
α_4	$s \to b$	0	9	9	0
α_5	$b \to d$	9	21	12	0
α_6	$b \to c$	9	18	9	2
α_7	$c \to t$	16	25	9	2
α_8	$a \to c$	6	18	12	4

Therefore

- s, b, d, t are the critical events;
- there is a unique critical path $s \to b \to d \to t$;
- the critical activities are $\alpha_4, \alpha_5, \alpha_3$.

There are two techniques that are often needed to prevent superfluous dependencies. Namely:

(i) Create a dummy activity between two existing events, treat it like an ordinary activity but with a duration of 0, and denote it by a dotted line.

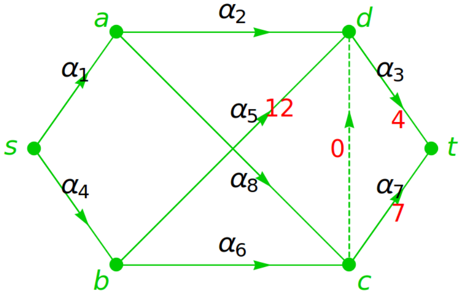
(ii) Take two copies of an event and add in one or more dummy activities.

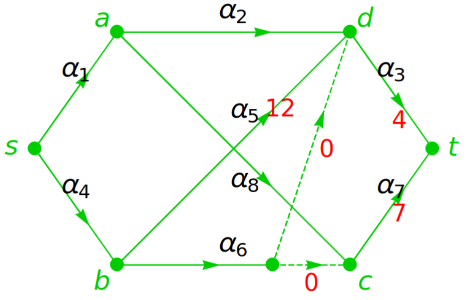
Two variations to our original problem will illustrate these respective techniques:

(i) α_3 now depends on α_6 and α_8 as well as α_2 and α_5, no other dependencies are affected.

(ii) α_3 now depends on α_6 as well as as α_2 and α_5, nothing else affected.

The modified networks are illustrated:





9.2. Network flow

We now wish to consider the situation in which a weighted directed graph represents a network of one-way roads (like motorway carriageways and sliproads) carrying traffic, pipes with carrying fluid or gas at pressure, or a national electricity grid.

Let E denote the set of directed edges, which are ordered pairs of adjacent vertices u, v. As before, we assume as before that $(u,v) \in E$ (also written $u \to v$) or $(v,u) \in E$ (also written $v \to u$ or $u \leftarrow v$), but not both. If we are not sure which, we shall occasionally type $u \rightleftharpoons v$.

The weight of each arc $u \to v$ will now represent *capacity*, the maximum permitted flow from u to v, and will accordingly be denoted $c(u,v)$ rather than $d(u,v)$. For example, in the case of a gas network, the capacities will be determined by the diameter of the pipes, along which the gas may travel at something like 20m/s.

As explained at the start of §9, we also assume that our network has a *source* s and a *sink* t, and that every other vertex v belongs to an aligned path $s \to \cdots \to v \to \cdots \to t$ from source to sink. Although t is like the final vertex in an activity network, the set-up is very different; we are no longer concerned with longest paths in which some arcs are irrelevant, but in attempting to use all the arcs in collaboration to maximize capacity.

Definitions. A *flow* is a function $f\colon E \to [0,\infty)$ that assigns a non-negative number $f(u,v)$ to each directed arc $u \to v$ with the following properties:

1. $0 \leqslant f(u,v) \leqslant c(u,v)$. If $f(u,v) = c(u,v)$ then the arc is called *saturated*.
2. At any vertex $v \in V \setminus \{s,t\}$, flow is conserved, so total 'inflow' equals the total 'outflow':

$$\sum_{x \to v} f(x,v) = \sum_{v \to y} f(v,y).$$

In the electrical setting, this is Kirchoff's law.

Remark. We have only defined $f(u,v)$ for an edge $u \to v$. The left-hand side of the last equation means that we only sum over those edges *entering* v. One can extend the definition of f by defining $f(v,u) = -f(u,v)$. Conservation of flow could then be expressed more succintly by the equation

$$\sum_{x \rightleftharpoons v} f(x,v) = 0,$$

in which we sum over *all* vertices adjacent to v. (One could even ensure that flow is conserved at s and t by artifically adding a directed edge $t \to s$ to the network.) However, we shall always assume that $f(u,v) \geqslant 0$, so in writing $f(u,v)$ we are declaring that $u \to v$; similarly for $c(u,v)$. We use this approach in the argument directly below.

Consider the linear combination

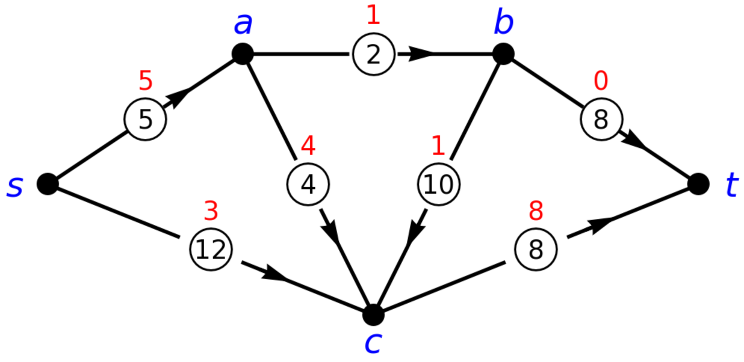
$$\sum_{u \to v} \big(f(u,v)u - f(u,v)v\big)$$

taken over *all directed eges*, in which we treat vertices as a basis for a vector space. In this sum, the terms arising from each individual vertex other than s, t cancel out by condition 2. Thus

$$\sum_{s\to x} f(s,x) = \sum_{y\to t} f(y,t),$$

i.e. the flow out from the source equals the flow in to the sink. This number is called the *value* of the flow.

Example. The diagram illustrates a flow of value 8 on a network whose capacities are indicated by the ringed numbers. There are three saturated arcs: $s \to a$, $a \to c$, $c \to t$.



Problem. Given a network with source and sink, find a flow with the maximum possible value, a so-called *maximum flow*.

We shall solve problem this using the so-called *Labelling Algorithm* for augmenting flow, which when iterated constructs a maximum flow. Each iteration uses a type of BFS with a queue and (if successful) produces an increment ε and a path along which each flow number can be modified by $\pm\varepsilon$.

Here's how it works for the example. Starting with s as the current vertex under consideration, we form a queue by adding adjacent vertices which

(i) have not already been labelled in the current iteration,
(ii) have *spare capacity* if the arc points *forward* away from the current vertex,
(iii) have *non-zero flow* if arc points *backwards* towards the current vertex.

We'll apply the labels in a table to avoid further complicating the diagram

	s	a	b	c	t	
1st iteration:	∞	$4c^-$	$1c^-$	$9s^+$	$1b^+$	Queue is $scabt$

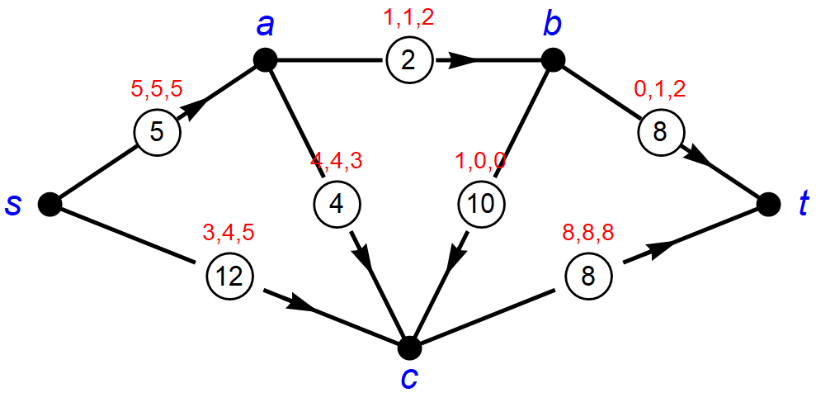
Below each vertex is a number indicating the amount of flow that can be transferred towards that vertex, from which vertex it was transferred, and in which direction. The recipe for giving this information is given in the next section, and one assigns ∞ to s to make the formula consistent.

We can now update the flow by $\varepsilon = 1$ (the amount reaching t) along the 'winning path'

$$s \to c \leftarrow b \to t,$$

which is remembered with the aid of the symbols in the last row. Forward arcs have the flow increased by ε, backwards ones have it reduced by ε. We can now remove all the labels and apply the same procedure to the updated flow to perform a second iteration:

	s	a	b	c	t	
2nd iteration:	∞	$4c^-$	$1a^+$	$8s^+$	$1b^+$	Q is $scabt$
3rd iteration:	∞	$3c^-$		$7s^+$		Q is sca



In the third iteration, we cannot augment any arcs beyond a or c because forward ones are saturated and backward ones have flow 0. This means that the second iteration produced a *maximum flow*. The second diagram shows the all the flow numbers after $n = 0, 1, 2$ iterations, and the maximum flow has value 10.

With hindsight, it was obvious that our network admits a flow with value 10 – we can send 2 units along the path $sabt$ and 8 units along the path sct. However, the Labelling Algorithm has the advantage that it can be applied to any *any* flow, including the one with all numbers 0, which would have produced the more obvious maximum flow.

9.3. Max flow, min cut

In our example, the third iteration defined a partition

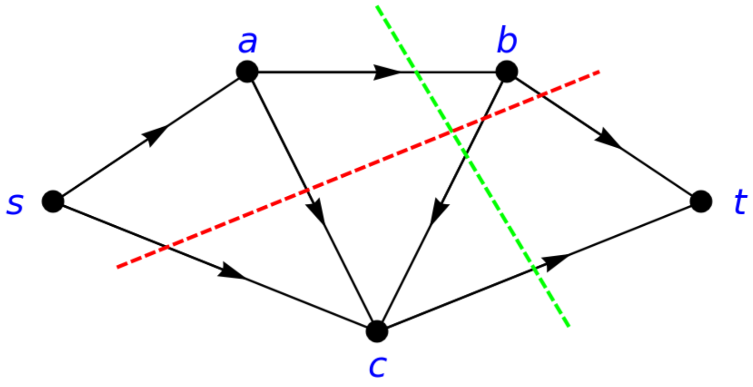
$$V = S \sqcup T = \{s, a, c\} \sqcup \{b, t\},$$

in which the first subset consists of all vertices to which we can increase the flow.

Definition. Given a network with source and sink, a *cut* is a partition of the set V of vertices of into two connected subsets, one of which contains s and the other t:

$$V = S \sqcup T, \qquad s \in S,\ t \in T.$$

A cut is completely specified by S since $T = V \setminus S$. It is also specified by the arcs that need to be removed to separate S from T. If we remove these arcs, we are assuming that the resulting subgraphs are both connected. A cut can be visualized by means of a line or curve cutting through the edges joining S to T. An obvious special case is always $S=\{s\}$, whilst our third iteration (attempted in vain) led to the green cut:



Each separating arc is classified as *forward* or *backward*, according as whether it runs from S to T or viceversa. Note that the description 'forward' and 'backward' for an arc only makes sense relative to a fixed vertex, cut, path, or similar.

For the red cut, sc, ac, bc, bt are removed and all are forward;
For the green cut, ab, bc, ct are removed and only bc is backward.

Definition. The *capacity* of the cut S is the sum of the capacities of only forward arcs, and therefore represents the maximum flow possible across the cut.

In the example, referring to the original diagram, red has capacity 34, and green only 10.

Lemma. Given any flow f and any cut S (S being the subset of V containing s),

$$\text{the value of } f \leqslant \text{the capacity of } S.$$

Proof. Define the *net flow* across S to be the sum of the forward flows minus the sum of the backward flows. A similar argument to defining the value of f in §9.2 shows that this value must equal the value of the net flow across any cut. This is particularly obvious when the cut is defined by setting $S=\{s\}$ or $S=V\setminus\{t\}$. □

Theorem ('max flow, min cut'). One can always find a flow and a cut for which there is equality in the lemma. Therefore, the maximum value of all possible flows equals the minimum capacity of all the cuts.

Definition. Given a network and a flow g, an *augmenting path* is a path

$$(s=u_0,\ u_1,\ u_2,\ \cdots,\ u_k=m)$$

from s to some vertex m (not necessaarily t) such that

(i) any forward edge is unsaturated, i.e.

$$(u_i, u_{i+1}) \in E \quad \Rightarrow \quad \boxed{c(u_i, u_{i+1}) - g(u_i, u_{i+1})} > 0$$

(ii) any backward edge has non-zero flow, i.e.

$$(u_{i+1}, u_i) \in E \quad \Rightarrow \quad \boxed{g(u_{i+1}, u_i)} > 0$$

Given such a path, let ε denote the minimum of the boxed quantities (one for each of the k edges defined by $i = 0, \ldots, k-1$). The flow from s to m can now by increased by

(i) adding ε to each forward arc;
(ii) subtracting ε from each backward arc.

The Labelling Algorithm is based on these observations. In our example, we found two augmenting paths, each with $\varepsilon = 1$, allowing us to increase the value from 8 to 9 to 10.

Proof of the theorem. Let g be a flow of maximum value. Let M denote the set of vertices for which there exists a flow-augmenting path $s \rightsquigarrow m$. We include the empty path, so $s \in M$. Then M cannot contain the sink, because we are assuming that the flow from s to t *cannot* be increased.

Let E' be the set of all arcs separating M from $V \setminus M$. If $(u, v) \in E'$ and $u \in M$ and $v \in V \setminus M$, then the arc must be saturated or else we could increase the flow to v, implying that $v \in M$. Similarly, if $v \in M$ and $u \in V \setminus M$ then the flow must be zero or else we could decrease it, giving an augmenting path to u.

So every forward arc in E' is saturated and every backward arc has zero flow. Hence the capacity of the cut $M \sqcup (V \setminus M)$ equals the value of the flow across the cut, which coincides with the value of g. $\square$

9.4. Labelling Algorithm

In this section, we describe more carefully the algorithm that provides an infallible method for increasing the flow through a network, if such an increase is possible.

Starting with an initial flow, the strategy is to try to get some extra flow from source to sink. The initial flow could be one with all numbers set to 0, but it helps to choose one that is non-zero. (Keep the choice simple, but try to saturate at least one arc, and if you use more than one path from source to sink make sure they are disjoint.)

Each iteration is performed using a stand-alone algorithm. If the iteration succeeds, one can start from scratch by applying a new iteration to the updated flow, and perform further iterations until it is no longer possible to reach the sink.

One uses a label for each vertex to indicate how much extra flow can come from the source to that vertex on the current iteration. In practice, the labels may be best written in a table (as we did in the previous section) rather than on the diagram.

It is implicit in the following description that we have a network with a set of vertices ordered (alphabetically or numerically) and including the source s and the sink t. Each directed arc $(u, v) \in E$ is weighted by a capacity $c(u, v)$.

One iteration. In addition to the network, the input consists of a given flow of value $\sigma \geqslant 0$. The instructions in the box were used to compile the tables shown in the preceding section. They perform a BFS by adding *certain* adjacent vertices to a queue Q that initially contains only s:

```
$\varepsilon = 0$
$Q = (s)$
$L(s) = \infty$
add $(s, L(s))$ to the table
while $Q$ is non-empty and has first element $x$:
    while there exists $y$ adjacent to $x$ not already in $Q$:
        if $(x,y) \in E$ and $f(x,y) < c(x,y)$:
            add $y$ to $Q$
            $L(y) = \min\{L(x),\ c(x,y) - f(x,y)\}$
            add $(y, L(y), x, +)$ to the table
            if $y = t$:
                $\varepsilon = L(t)$
                stop
        elif $(y,x) \in E$ and $f(y,x) > 0$
            add $y$ to $Q$
            $L(y) = \min\{L(x),\ f(y,x)\}$
            add $(y, L(y), x, -)$ to the table
    remove $x$ from $Q$
return $\varepsilon$ and the table
```

There are two possible outcomes:

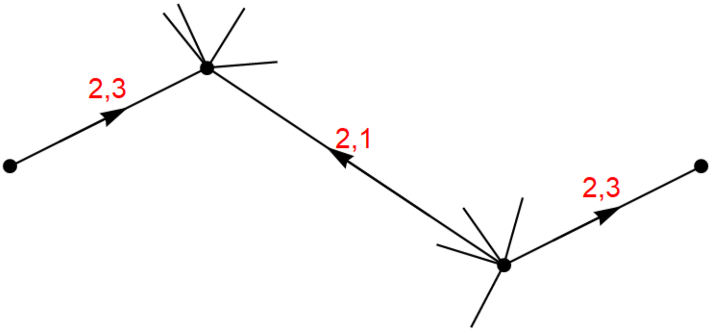
(i) ε is output as positive. In this case,the table can be used to construct an augmented flow with value $\sigma + \varepsilon$ (as in the first and second iterations in §9.3). One uses the table to trace a 'winning path' back from t to s, and one updates the flow on each arc on this path according to its recorded orientation:

$f(x,y) = f(x,y) + \varepsilon$ given the tag $(y, L(y), x, +)$,
$f(y,x) = f(y,x) - \varepsilon$ given the tag $(y, L(y), x, -)$.

(ii) ε is output as zero, which signals that the BFS will have stopped before reaching t. The table will be incomplete, and the set of vertices processed forms a proper subset S of V with S and $T = V \setminus S$ connected. In this case, one can show that the inputted flow was maximum and the cut $V = S \sqcup T$ minimum.

Why does the algorithm work? Consider the two output scenarios again:

(i) Since $\varepsilon > 0$, the various minima are strictly positive and we have increased the flow to t, provided we understand that the operations of updating the flow on the winning path are legitimate. No arc has been assigned a flow above its capacity. Conservation of flow has been preserved at each vertex, with no changes away from the path in question. If both alternatives $\pm$ have a vertex in common, the flow has merely been *diverted* to allow more through. The following sketch (in which the initial flow was 2 units on each edge) is designed to make the last point clear:



(ii) This case relates to the proof at the end of §9.3. We are assuming that for any arc crossing from S to T has flow to capacity, and every arc from T to S has zero flow. Thus the value σ equals the capacity of the cut, and the flow must be maximal.

*9.5. Dynamic programming

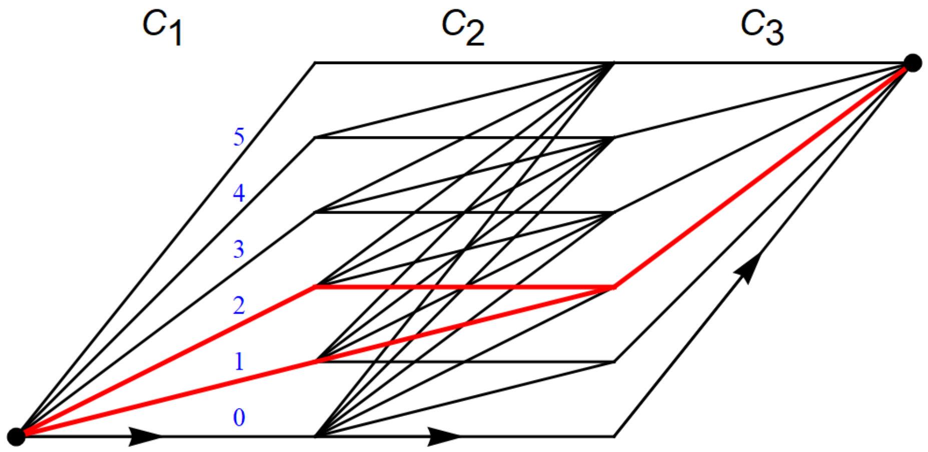
In this section, we shall once again encounter the optimality principle that underlies Dijkstra's algorithm for finding shortest paths in a weighted graph, and the labelling of an activity network to find critical paths.

Example. Deborah has £50K to invest in multiples of £10K in three companies C_1, C_2, C_3, and wants to maximise the return. All the money is to be used, but she is not allowed to invest more than once in any company. The following table shows the expected returns on investment, with the amount invested along the top row. All numbers are in units of £10K.

	0	1	2	3	4	5
C_1	0	3	5	6	7	8
C_2	0	2	4	8	10	11
C_3	0	0	2	10	11	11

The problem amounts to finding an aligned path in the following network that runs from bottom left to top right with *maximum* total weight. Although only three arrows are shown, all the arcs are oriented from left to right, and this is akin to the activity network, in which duration is replaced by financial return.

The first stage consists of deciding how to invest in C_1, the second how much to invest in C_2. The difference has to be invested in C_3: if x_1 units are invested in C_1 and x_2 units in C_2 then $0 \leqslant x_2 \leqslant 5 - x_1$, and $x_3 = 5 - x_1 - x_2$ units are invested in C_3:



With reference to the graph, each event has coordinates (i, y_i) where $0 \leqslant i \leqslant 3$ and $0 \leqslant y_i \leqslant 5$. At this event, one has concluded a total investment of $y = y_i$ in companies up to and including C_i. There are

$$6+5+4+3+2+1 = \tfrac{1}{2} * 6 * (6+1) = 21$$

aligned paths from start $(0,0)$ to finish $(3,5)$, but the problem can easily be generalized to more companies and investment choices.

Our solution below reveals that there are in fact two longest paths (shown red with a final arc in common) realizing a total return of £150K.

The special feature of this network is that although there are $6+21+6=33$ arcs, there are only 18 different weights. These are determined by the functions

$$r_i(x) = \text{return on investment of } x \text{ units in } C_i$$

from the table, with $0 \leqslant x \leqslant 5$. From these, we shall construct functions

$$f_i(y) = \text{ best return at the } i\text{th stage for a total investment } y\,,$$

where 'best' is taken over all possible investment strategies $x_1, x_2, \ldots, x_i$ up to the ith stage, and y denotes the sum $x_1 + \cdots + x_i$. Our aim is to find $f_3(5)$.

Fortunately we do not need to consider all choices. The optimality principle implies that the best investment (which is a *longest* path) at each stage will *necessarily* arise from a best investment (longest path) at all previous stages. Hence the

Corollary. The 'best return' function satisfies

$$f_i(y) = \max_{0 \leqslant x \leqslant y} \Big\{ f_{i-1}(y-x) + r_i(x) \Big\} \quad \text{or} \quad f_i(y_i) = \max_{x_i} \Big\{ f_{i-1}(y_{i-1}) + r_i(x_i) \Big\},$$

with $f_0 = 0$. In practice, it may help to use the second equation in which the values of $x = x_i$ and $y = x_1 + \cdots + x_i$ at each stage make the discrete nature of y more explicit.

The underlying logic of this formula is identical to that of the expression

$$E(u) = \max_{x \to u} \left\{ E(x) + d(x,u) \right\}$$

in §9.1 for finding latest start times. This is in turn a version of relabelling formula

$$L_j = \min(L_i + d(v_i, v_j),\ L_j)$$

in §8.1 to relax the temporary labels in Dijkstra's algorithm, 'min' here because we were seeking a *shortest* path.

In dynamic programming, the graph overcomplicates the situation, so the work is best organized into a table, which is headed by the values of y in (traditionally) reverse order. There is really a separate table for each stage, which applies the corollary to determine $f_i(y_i)$, though the tables can be joined together. Strictly speaking, *every* stage needs a triangular table to cater for all the combinations of $x = x_i$ and $y_i = x_1 + \cdots + x_i$. However, the first and last are simpler, and in our example only the second one illustrates the general technique (this is apparent from the structure of the graph!).

		5	4	3	2	1	0	$\leftarrow y$
		8	7	6	5	3	0	$\leftarrow f_1(y_1)$
$x_2 \downarrow$	$r_2(x_2) \downarrow$							
5	11						11	
4	10					13	10	
3	8				13	11	8	$\leftarrow f_1(y_1) + r_2(x_2)$
2	4			10	9	7	4	
1	2		9	8	7	**5**	2	
0	0	8	7	6	**5**	3	0	
		13	11	8	5	3	0	$\leftarrow f_2(y_2) = \max \text{ in } \Delta$
		13	11	10	**15**	14	11	$\leftarrow f_2(y_2) + r_3(x_3)$
		15						$\leftarrow f_3(y_3) = \max$

The first stage is straightforward: we set $x = x_1 = y$ and $f_1(y_1) = r_1(x_1)$.

At the second stage, for each value of y, we are allowed to choose any value of $x = x_2$ with $y + x_2 \leqslant 5$, and for each such value we add $r_2(x_2)$ to our return. The total investment $y + x_2$ is constant along each diagonal Δ, which we scan in order to find the maximum return. For example, the diagonal returns $7, 8, 9, 11, 10$ is associated to a total inestment of 4 units, and its best return is

$$f_2(4) = \max_{0 \leqslant x \leqslant 4} \left\{ f_1(4 - x) + r_2(x) \right\} = 11.$$

There are no entries above the main diagonal since those would represent current total investments of over £50K.

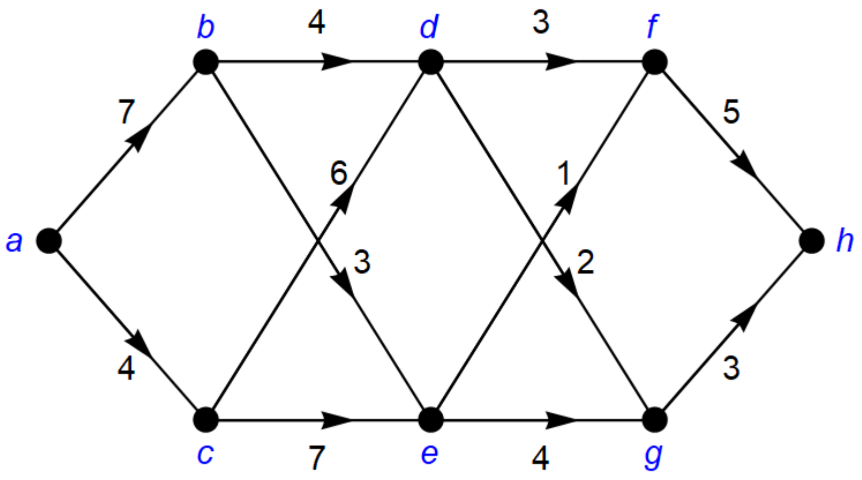
As we pass to the next stage, we associate the maximum return to the new value $y_i = y_{i-1}+x_i$, and place it under the bottom-left entry of the diagonal. In theory, we are ready for another triangular table, but in our example, there is *no choice left* since x_3 is determined by $x_1 + x_2$. Thus, there would only be one useful diagonal, which we have shown horizontally to save space.

The best return has value 15 coming from $x_3 = 3$, which in turn comes from an earlier best return of 5 with *either* $x_2 = 2$ *or* $x_2 = 1$. (These entries have been shown in bold.)

Conclusion. Wise investment produces a best possible return of £150K, arising in two ways: (i) £20K in C_1 plus £30K in C_3, or (ii) £10K in C_1 and C_2 plus £30K in C_3. These two solutions are the red paths, but the return is not good enough for our Dragon.

Dynamic programming is really a BFS with 'pruning' – one can forget results from previous stages. Here is an example in which one can compare the technique to Dijkstra's algorithm, though in this case the latter is probably quicker.

Example. Find the shortest path from a to h in the weighted digraph below:



Dijkstra's algorithm will work provided one takes account of the fact that the edges now directed; for example, d is adjacent to b but not vice versa, so one only scans from left to right. This problem is therefore layered like the investment one – for each vertex all paths from the start have the same length. The solution can be again be obtained in 'vertical' stages. The key point in problems like these is that *the best solution at the* $(i+1)$*th stage must arise from the best solutions at the* i*th stage*.

The shortest path can be obtained from the following table:

start at:	a	
	dist$=0$	
to get to:	b	c
	dist $=7$ from a	dist $=4$ from a
to get to:	d	e
	best dist $=10$ from c	best dist $=10$ from b
to get to:	f	g
	best dist $=11$ from e	best dist $=12$ from d
to get to:	h	
	dist$=15$ from g	

Exercise. Suppose that the weights in the above network now represent capacities, a is the source and h is the sink. Show that the value of a maximum flow is 7 and identify a minimum cut.

10. Codes

When sending a message, one might wish to:

- transmit it efficiently – achieved with error-correcting codes;
- keep the message private and authenticate the sender – the role of cryptography.

Here is the set-up to keep in mind:

$$\begin{tikzcd}
\text{ALICE} & & & \text{BOB} \\
\texttt{message} \arrow[d, "\text{encode/encrypt}"] & & & \texttt{message} \\
\texttt{codewords} & \arrow[r, squiggly] & \| \arrow[r, squiggly] & \texttt{received words} \arrow[u, "\text{decode/decrypt}"']
\end{tikzcd}$$

We shall consider codes first. Bob might receive

$$\texttt{10472\underline{7} IS A MEMORABLE \underline{Q}RIME}$$

but the underlined characters are not what Alice wrote. The second error needs both a dictionary (or spell-checker) and a realization that the number is not salty, not criminal, not dirty, nor is it 3 cents. The first needs an analysis of primes that differ 'minimally' from 104727 (which is itself divisible by 3).

10.1. Check digits

The idea here is to assign a check digit or digits to each block of numerical data:

$$\boxed{}\,\boxed{}$$

The check should involve the whole block, but ideally be small compared to it and easy to compute. It should be able to detect when a common error has occurred, even if it will not correct the error. Examples of some common systems follow.

Parity bit. Each block might consist of 7 bits, to which one check bit is added. For example

$$\boxed{1010100}\,\boxed{?}$$

where '?' is chosen so that the overall number of 1's is even. Here it is 1. More generally

$$\boxed{x_1x_2x_3x_4x_5x_6x_7}\,\boxed{x_8}$$

must satisfy $\sum_{i=1}^{8} x_i = 0 \bmod 2$. This system defines a set of $2^7 = 128$ code words requiring 8 bits for transmission. It will succeed in detecting an error if exactly one digit $x_1, \ldots, x_7$ is transcribed wrongly, but it does not 'see' the transposition of two bits.

ISBN 10 (International Standard Book Number, pre 2007). This is a 10-digit number, which we can type as $x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}$, in which the last digit x_{10} is a check. For a number to be valid

$$x_1 + 2x_2 + 3x_3 + \cdots + 9x_9 + 10x_{10} = 0 \bmod 11.$$

The check digit is easily determined by the formula

$$x_{10} = -10x_1 = x_1 + 2x_2 + 3x_3 + \cdots + 9x_9 \bmod 11.$$

If $x_{10} = 10 \bmod 11$, the character 'X' is used. For example, the ISBN 10 number of Iris Murdoch's novel "The Sea, the Sea" (the 1978 Booker prizewinner) is

`014118616X.`

ISBN 10 detects the two commonest errors:

(i) a single wrong digit, like 3491234287 instead of 3491242287.

(ii) a single adjacent transposition, like 3491224287 instead 3491242287.

Let's verify (i). Suppose that the original 10-digit 'word' is $\mathbf{x} = x_1x_2x_3 \cdots x_{10}$, but that this is received as $\mathbf{y}$ with a error in (say) the second position: $\mathbf{y} = x_1y_2x_3 \cdots x_{10}$. Set

$$f(\mathbf{x}) = \sum_{i=1}^{10} i\, x_i,$$

so that $f(\mathbf{x}) = 0 \bmod 11$. Then

$$f(\mathbf{y}) = f(\mathbf{x}) + 2(y_2 - x_2) = 2(y_2 - x_2) \bmod 11$$

can't be zero because 11 is prime.

IBAN (International Bank Account Number). The IBANs of a given country have the same number of digits: for example, the UK and Germany have 22, whilst France and Italy have 27. Here is a UK example:

`GB27 LOYD 3011 2700 1268 86`

The two digits (here, 27) after the country code form the check. In general, these two digits represent an integer c satisfying $2 \leqslant c \leqslant 98$ (with $c = 2$ giving 02 etc). The next four characters obviously indicates the bank (here Lloyds). What follows is the sort-code (30-11-27) and account number (00126886).

Digression. The sort-code and account number must pass a separate validation called 'modulus checking' that varies from bank to bank (and whether the account it a sterling or euro one!). Lloyds uses a system akin to ISBN 10 with (in our case) the weights shown in the first row:

0	0	3	2	9	8	5	2	6	5	4	3	2	1
3	0	1	1	2	7	0	0	1	2	6	8	8	6
0	0	3	2	18	56	0	0	6	10	24	24	16	6

The sum of the numbers in the last row is 165, which is a mltiple of 11, so the account number is valid.

Returning to the IBAN, move the first four characters to the back and remove spaces:

```
LOYD30112700126886GB76
```

Now replace any alphabetic letters by its position in the alphabet plus 9 (so A$\rightarrow$10,..., L$\rightarrow$11,..., Y$\rightarrow$34). This gives the 26-digit number

```
21243413301127001268861611 76
```

which (as it stands, expressed to base 10) must equal 1 modulo 97, which it does! This condition uniquely specifies the check digits, with the assumption $2 \leqslant c \leqslant 98$. Because 97 is prime, any one of these 97 possibilities can occur. The drawback is that validation requires a computer.

ISBN 13 (post 2007). First, some general theory. Suppose that we want to add a single check digit x_n to a string $x_1x_2\cdots x_{n-1}$, using the rule

$$c_0 + c_1x_1 + \cdots + c_nx_n = 0 \bmod N.$$

In order that x_n is determined, we need c_n to be coprime to N. For single errors to be detected we also need c_i to be coprime to N for all for all $i < n$. For transpositions to be detected, we need $c_i - c_j$ coprime to N for all $i \neq j$.

ISBN 13 is validated by the 'check function'

$$f(\mathbf{x}) = x_1 + 3x_2 + x_3 + 3x_4 + \cdots + 3x_{12} + x_{13} \bmod 10,$$

but this fails to detect transpositions in which the digits differ by 5, like $27 \leftrightarrow 72$. For

$$3x_i + x_{i+1}, \quad 3x_{i+1} + x_i, \quad x_i + 3x_{i+1}, \quad x_{i+1} + 3x_i$$

are all equal modulo 10, and the check digit will be the same.

Luhn algorithm. This is used to determine the final digit of credit card numbers. First define

$$\widehat{2x} = \begin{cases} 2x & \text{if } x \in \{0,1,2,3,4\}, \\ 2x - 9 & \text{if } x \in \{5,6,7,8,9\}. \end{cases}$$

The map $x \mapsto \widehat{2x}$ is the permuation

$$(0,1,2,3,4,5,6,7,8,9) \longmapsto (0,2,4,6,8,1,3,5,7,9)$$

with fixed points 0 and 9 (so $\widehat{2x}$ is not quite the obvious residue of $2x$ modulo 9). For a 16-digit number $\mathbf{x} = x_1 \cdots x_{16}$ we define

$$f(\mathbf{x}) = \widehat{2x_1} + x_2 + \widehat{2x_3} + x_4 + \cdots + \widehat{2x_{15}} + x_{16}.$$

We then require that $f(\mathbf{x}) = 0 \bmod 10$. Without the 'hats', the function f would not detect transcriptions differing by 5. But with the hats its detects all single transcriptions, and all adjacent transpositions except for $09 \leftrightarrow 90$ (thought to be less of a problem since 0 and 9 are far apart on the numerical keypad). It also corrects most twin errors $ii \leftrightarrow jj$, but not $22 \leftrightarrow 55$, $33 \leftrightarrow 66$ or $44 \leftrightarrow 77$, since (e.g.) $2 + \widehat{2} = 5 + \widehat{5}$.

10.2. Binary codes

These are based on the alphabet $\mathbb{B} = \{0, 1\}$. Later, it will be important to realize that (equipped with addition modulo 2 and multiplication) this set becomes a field. It is commonly denoted $\mathbb{Z}_2$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/(2)$. The problem with the first notation is that $\mathbb{Z}_2$ also stands for the (infinite) set of p-adic integers with prime $p = 2$. The other notations are clumsy, so we shall use $\mathbb{B}$ or (maybe later) $\mathbb{F}_2$.

The Cartesian product

$$\mathbb{B}^n = \mathbb{B} \times \cdots \times \mathbb{B}$$

is a vector space over the field $\mathbb{B}$ of dimension n with an obvious basis. We abbreviate $(x_1, \ldots, x_n)$ to $x_1 x_2 \cdots x_n$.

Definition. A *binary code* C is a set of strings of 0's and 1's of length n, i.e. it is a subset of $\mathbb{B}^n$.

We shall call an element of $\mathbb{B}^n$ a *string* or *word*, and each element of C a *codeword*. We regard $\mathbf{x} \in \mathbb{B}^n$ as 'valid' if $\mathbf{x}$ belongs to C, which we can think of (for the moment) as a set of valid account numbers expressed in binary.

Example 1. Take $n = 4$ and define $C = \{0000,\ 0101,\ 1010,\ 1111\}$. You receive 0111. This is not in <C, so there must be an error. You can compare it to each element of C:

received	codeword	# erroneous digits
0111	0000	3
0111	0101	1
0111	1010	3
0111	1111	1

The original message was likely to have been 0101 or 1111. But that is still two choices – we want to design codes so that there is only one choice.

Definition. The *Hamming distance* between two words $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$ is the number of bits by which they differ. It is denoted $\partial(\mathbf{x}, \mathbf{y})$.

This function satisfies the properties of a *metric* in the sense of metric space, including the triangle inequality (for a proof of the latter, see §10.3):

$$\begin{cases} \partial(\mathbf{x}, \mathbf{y}) = 0 \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{y} \\ \partial(\mathbf{x}, \mathbf{y}) = \partial(\mathbf{y}, \mathbf{x}) \\ \partial(\mathbf{x}, \mathbf{y}) \leqslant \partial(\mathbf{x}, \mathbf{z}) + \partial(\mathbf{z}, \mathbf{y}). \end{cases}$$

We shall always adopt the

Minimum distance (MD) or nearest neighbour principle. If an invalid word $\mathbf{x}$ is received, assume that the codeword $\mathbf{y}$ transmitted was one for which $\partial(\mathbf{x}, \mathbf{y})$ is as small as possible.

Example 2. Let $C = \{\mathbf{a} = 01101,\ \mathbf{b} = 10110,\ \mathbf{c} = 00011\}$. Then

$$\partial(\mathbf{b},\mathbf{c}) = 3, \quad \partial(\mathbf{c},\mathbf{a}) = 3, \quad \partial(\mathbf{a},\mathbf{b}) = 4.$$

If we receive $\mathbf{x} = 01011$, we test

$$\partial(\mathbf{x},\mathbf{a}) = 2, \quad \partial(\mathbf{x},\mathbf{b}) = 4, \quad \partial(\mathbf{x},\mathbf{c}) = 1,$$

so the MD principle tells us to assume that $\mathbf{c}$ was transmitted.

We want to design C so that each codeword has a unique nearest neighbour (as measured with ∂). One might expect to achieve this if the codewords are well dispersed, which amounts to requiring that the distance between any two is sufficiently large:

Definition. Let C be a binary code. Its *minimum distance* is given by

$$\delta = \min\{\partial(\mathbf{x},\mathbf{y}) : \mathbf{x},\mathbf{y} \in C,\ \mathbf{x} \neq \mathbf{y}\}.$$

Now suppose that $\delta \geqslant 2e+1$. If $\mathbf{x} \in \mathbb{B}^n$ and $\mathbf{y},\mathbf{y}' \in C$ then

$$\partial(\mathbf{x},\mathbf{y}) \leqslant e,\ \ \partial(\mathbf{x},\mathbf{y}') \leqslant e \quad \Rightarrow \quad \mathbf{y} = \mathbf{y}'.$$

This is an immediate consequence of the triangle inequality:

$$\partial(\mathbf{y},\mathbf{y}') \leqslant \partial(\mathbf{y},\mathbf{x}) + \partial(\mathbf{x},\mathbf{y}') \leqslant 2e < \delta,$$

so the definition of δ implies that $\mathbf{y} = \mathbf{y}'$. As a consequence, we obtain the well-known

Lemma 1. A binary code with $\delta \geqslant 2e+1$ will correct e errors using the MD principle.

Examples. In Example 1, $\delta = 2$ and this is not enough to correct any errors. In Example 2, $\delta = 3$ so one can detect and correct single errors.

Lemma 2. Let $C \subset \mathbb{B}^n$ be a binary code with $\delta \geqslant 2e+1$. Then

$$|C|\left(1 + n + \tbinom{n}{2} + \cdots + \tbinom{n}{e}\right) \leqslant 2^n.$$

Proof. The expression in parentheses on the left-hand side equals the number of elements in $\mathbb{B}^n$ that are within distance e of a given codeword $\mathbf{y}$. For example, there are n words that differ from $\mathbf{y}$ by exactly one digit, and $\binom{n}{2}$ that differ by exactly two digits. If we surround each codeword $\mathbf{y}$ by the 'ball' or neighbourhood

$$N_e(\mathbf{y}) = \{\mathbf{y} \in \mathbb{B}^n : \partial(\mathbf{x},\mathbf{y}) \leqslant e\},$$

no two balls can intersect, for Lemma 1 tells us that $N_e(\mathbf{y}) \cap N_e(\mathbf{y}') = \varnothing$. □

In the next section, we shall study the case $e = 1$ of '1-error correcting codes' in more detail, for which we need to assume that $\delta \geqslant 3$. Lemma 2 implies that $|C|(1+n) \leqslant 2^n$, and equality here would imply that both $|C|$ and $n+1$ are powers of 2. We shall show (in §10.4) that such codes do in fact exist.

10.3. Binary linear codes

We now specialize the set-up of the previous section to the case in which C is a sub*space* of $\mathbb{B}^n$. This condition makes sense because $\mathbb{B}^n$ is a vector space over the finite field $\mathbb{B}=\{0,1\}=\mathbb{F}_2$, with coordinate-wise addition.

Remember that a word like 010101 really stands for the vector $(0,1,0,1,0,1)$. We need not worry about scalar multiplication since $2=0$ and $-1=1$ in $\mathbb{B}$! So we just need to verify that

$$\mathbf{x},\mathbf{y}\in C\quad\Rightarrow\quad\mathbf{x}+\mathbf{y}\in C.$$

Any linear code must contain the zero vector $\mathbf{0}=00\cdots0$. Moreover, it has a dimension k with $k\leqslant n$, and a basis $\{\mathbf{x}_1,\cdots,\mathbf{x}_k\}$ consisting of k elements. It then follows that

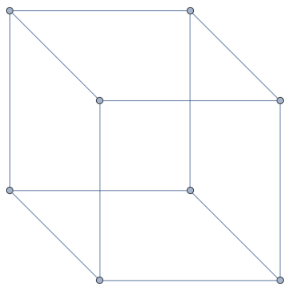
$$C=\Big\{\sum_{i=1}^{k}a_k\mathbf{x}_k : a_k\in\mathbb{B}\Big\}$$

has 2^k elements. The space $\mathbb{B}^n$ itself has dimension n and a basis consisting of the vectors $\{\mathbf{e}_i\}$ where $\mathbf{e}_i$ is the vector or word with a 1 in the ith position and zeros elsewhere.

Examples. Let $C=\{000,111\}\subset\mathbb{B}^3$. The two codewords can be visualized as the opposite vertices of a cube. Notice that $\delta=3$ and

$$\mathbb{B}^3=N_1(000)\sqcup N_1(111)$$

is partitioned into two subsets of size 4. This is an example of a *repeat code* in which each of two messages (0 and 1) is repeated twice to enable correction of 1 error.



Words in $\mathbb{B}^n$ can be thought of as vertices of an n-dimensional hypercube, but this is hard to visualize (at least for $n>4$!). Here is a linear code with $(n,k,\delta)=(5,2,3)$ that we shall return to:

$$C=\{00000,\ 10110,\ 01011,\ 11101\}.$$

Any two of the nonzero elements form a basis of C.

Definition. The *weight* of a word $\mathbf{x}\in\mathbb{B}^n$ equals the number of 1's it has:

$$\mathbf{x}=x_1\cdots x_n\quad\Rightarrow\quad w(\mathbf{x})=\sum_{i=1}^{n}x_i.$$

Recall the previous definition (of δ).

Lemma 3. Given a *linear* code C,

$$\delta = \min\{w(\mathbf{x}) : \mathbf{x} \in C,\ \mathbf{x} \neq \mathbf{0}\}.$$

So the minimum distance is also minimum *nonzero* weight in C.

Proof. Denote (temporarily) the right-handf side by δ'. The point is that

$$\partial(\mathbf{x}, \mathbf{y}) = \partial(\mathbf{x} - \mathbf{y}, \mathbf{0}) = \mathbf{w}(\mathbf{x} - \mathbf{y}),$$

which holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$ (we could equally well write $+$ in place of $-$). If the latter belong to C then so does $\mathbf{x} - \mathbf{y}$, so $\delta \geqslant \delta'$. But $w(\mathbf{z})$ is itself the distance of $\mathbf{z}$ from the zero vector $\mathbf{0} \in \mathbf{C}$, so $\delta' \geqslant \delta$. $\square$

We can also use w to prove the triangle inequality for ∂. If $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$ then

$$w(\mathbf{x} + \mathbf{y}) = \sum_{i=1}^{n} \widehat{x_i + y_i} \ \leqslant \ \sum_{i=1}^{n} (x_i + y_i) = w(\mathbf{x}) + w(\mathbf{y}),$$

where $\widehat{x_i + y_i} \in \{0, 1\}$ stands for the reduction of $x_i + y_i$ modulo 2. Thus w behaves like a *norm* on a real vector space, and

$$\begin{aligned} \partial(\mathbf{x}, \mathbf{y}) \ = \ w(\mathbf{x} - \mathbf{y}) \ &= \ w(\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}) \\ &\leqslant \ w(\mathbf{x} - \mathbf{z}) + w(\mathbf{z} - \mathbf{y}) \\ &= \ \partial(\mathbf{x}, \mathbf{z}) + \partial(\mathbf{z}, \mathbf{y}). \end{aligned}$$

Key example to illustrate the theory. The first two nonzero elements of the previous example $C \subset \mathbb{B}^5$ correspond to the columns of the matrix

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 \\ \\ A \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix. Our convention is that matrices always act on the left on column vectors, so this defines a linear transformation

$$E\colon \quad \mathbb{B}^2 \longrightarrow \mathbb{B}^5.$$

It follows that $C = \operatorname{Im} E$, and each element of C has the form

$$E\mathbf{v} = \begin{pmatrix} \mathbf{v} \\ \\ A\mathbf{v} \end{pmatrix},$$

where $\mathbf{v}$ is one of

$$\text{west} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{north} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{south} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{east} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We shall freely transpose from rows to columns, using the latter when we need to act on them by matrices. With this confusion,

$$E\mathbf{v} = \boxed{\quad\mathbf{v}\quad}\boxed{\quad A\mathbf{v}\quad}.$$

Seen this way, $A\mathbf{v}$ plays the role of a check block for each of the four possibilities for $\mathbf{v}$, which might be commands for a robot to move. Observe that here the check is longer than the original message. This is to enable error *correction* rather than mere *detection*: since $\delta = 3$, the block 'protects' the direction in the event it is corrupted by 90 degrees.

We can describe C in an equivalent way, using the matrix

$$H = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(A \;\middle|\; I_3 \right).$$

For let

$$\mathbf{x} = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} \in \mathbb{B}^5,$$

with $\mathbf{b} \in \mathbb{B}^2$ and $\mathbf{c} \in \mathbb{B}^3$. Then

$$\begin{aligned} H\mathbf{x} = \mathbf{0} \quad &\Leftrightarrow \quad A\mathbf{b} + \mathbf{c} = \mathbf{0} \\ &\Leftrightarrow \quad A\mathbf{b} = \mathbf{c} \\ &\Leftrightarrow \quad \mathbf{x} \in \operatorname{Im} E \;=\; C. \end{aligned}$$

The matrix H is called the *parity-check* or *check* matrix of the linear code C.

Definition. Suppose that $r < n$. Let H be a matrix of size $r \times n$ and entries in $\mathbb{B}$ (one can express this by writing $H \in \mathbb{B}^{r,n}$ and r is the number of rows). Then the subspace

$$\ker H = \{\mathbf{x} \in \mathbb{B}^n : H\mathbf{x} = \mathbf{0}\}$$

of $\mathbb{B}^n$ is called the linear code *with check matrix* H. We shall always assume that $r = \operatorname{rank} H$, since if not we can delete one or more rows without affecting the kernel.

Example. Take $r = 1$ and H to be the single row with all 1's. Then C consists of all the elements of $\mathbb{B}^n$ of even weight. We could regard the first $n-1$ bits of $\mathbf{x} \in B^n$ as the 'message', and the final bit x_n as a parity check digit, as in §10.1.

10.4. Correcting one error

We have already used three parameters to help describe a linear code:

$$\begin{aligned} n \quad &= \quad \text{number of bits in transmission} \\ k = n - r \quad &= \quad \text{dimension, so } |C| = 2^k \\ \delta \quad &= \quad \text{minimum distance between codewords.} \end{aligned}$$

Suppose that we need to correct one error in a transmitted message block. This requires a code (linear or not) with $\delta \geqslant 2e + 1 = 3$, and by Lemma 2 from §10.2,

$$s(1 + n) \leqslant 2^n,$$

where $s = |C|$ (s for *size*). A big question is

> *Given s, n satisfying this inequality, does there exist $C \subset \mathbb{B}^n$ with $\delta = 3$ and $|C| = s$?*

Easy exercise. There is no code (linear or not) with $|C| = 3,\ n = 4$ and $\delta = 3$.

At the risk of repetition, let's summarize the definition of linear codes using matrices.

Such a code is often defined by a check matrix H of size $r \times n$ with $r < n$. Then the set of codewords is

$$C = \{\mathbf{x}^\top : H\mathbf{x} = \mathbf{0}\} \subset \mathbb{B}^{\mathbf{n}}.$$

We naturally regard a *word* as a string written as a row, but it is always transposed to a column vector for the check matrix to test it. We shall usually omit the transpose symbol $^\top$, since context makes it clear whether one is dealing with a row or a column. So $C = \ker H$, i.e. C is the kernel of the linear transformation

$$\begin{array}{ccc} \mathbb{B}^n & \longrightarrow & \mathbb{B}^r \\ \mathbf{x} & \longmapsto & H\mathbf{x}. \end{array}$$

We assume that $\operatorname{rank} H = r$, so that $\dim C = n - r$. We call this dimension k, so that there are 2^k codewords.

To make clearer the analogy with check digits, one often takes

$$H = \left(A \;\middle|\; I_r \right)$$

so that the last block is the identity matrix. In this case, H is said to be in *standard form*, but this is not always convenient. Note that A has $n - r = k$ columns. We can define an 'encoding matrix'

$$E = \begin{pmatrix} I_k \\ A \end{pmatrix}.$$

By multiplying the blocks, we see that

$$HE = AI_k + I_rA = A + A = \mathbf{0}$$

is the zero matrix (of size $r \times k$). This means that H annihilates the k columns of E, which must therefore lie in C. But these k columns are linearly independent because they include the columns of I_k, and they span the image of $E\colon \mathbb{B}^k \to \mathbb{B}^n$.

Conclusion. $C = \ker H = \operatorname{Im} E$, so as a row any codeword can be written

$$\boxed{\ \mathbf{v}\ }\boxed{\quad A\mathbf{v}\quad}.$$

Some authors would (correctly) express this as $(\mathbf{r},\ \mathbf{r}A^\top)$ having preferred to make explicit the row vector $\mathbf{r} = \mathbf{v}^\top$ and having choden to use $E^\top$ instead of E.

Exercise. Suppose that $C = \ker H$, where

$$H = \left(\begin{array}{cc|c} 1 & 0 & \\ 1 & 0 & \\ 1 & 0 & \\ 1 & 1 & I_7 \\ 0 & 1 & \\ 0 & 1 & \\ 0 & 1 & \end{array}\right).$$

What is the size of C? How many errors does it correct?.

Lemma 4. Let H be the check matrix of a linear code C. Then $\delta \geqslant 3$ (so C corrects at least one error) provided no column of H is zero and no two columns are equal. Moreover, if $\mathbf{x}$ differs from a codeword $\mathbf{y}$ by just one bit in the ith position (i.e. $\mathbf{x} = \mathbf{y} + \mathbf{e}_i$), then $H\mathbf{x}$ is the ith column of H.

Proof. We need to ensure that C has no words of *weight* 1 or 2. A word of weight one means it is $\mathbf{e}_i$ for some i, and $H_i = H\mathbf{e}_i$ is the ith column of H. So this must be nonzero. Similarly, a word $\mathbf{x}$ of weight 2 must equal $\mathbf{e}_i + \mathbf{e}_j$ with $i \neq j$, and so

$$H\mathbf{x} = H\mathbf{e}_i + H\mathbf{e}_j = H_i + H_j = H_i - H_j$$

must be nonzero. Finally, if $\mathbf{x} = \mathbf{y} + \mathbf{e}_i$ with $\mathbf{y} \in C$ then $H\mathbf{x} = H_i$. □

Example. The matrix

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

obviously has rank 3, so defines a linear code of dimension $7 - 3 = 4$. Its parameters are $(n, k, \delta) = (7, 4, 3)$. If $\mathbf{x}$ differs from a codeword only in the ith position then $H\mathbf{x}$ (transposed to a row) is conveniently the binary expansion of i! If $H\mathbf{x}$ is nozero, it is called the *syndrome* of the word $\mathbf{x}$.

The best way to modify H so that the identity matrix appears on the right is to perform row operations (as for echelon form) because this will not change the kernel of the matrix. We take

$$\left\{ \begin{array}{rcl} \mathbf{r}_1' & = & \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{r}_2' & = & \mathbf{r}_1 + \mathbf{r}_3 \\ \mathbf{r}_3' & = & \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \end{array} \right.$$

to form

$$H' = \left(\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right).$$

The encoding matrix associated to H' is

$$E = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array} \right),$$

and any codeword then has the form $\boxed{\quad \mathbf{v} \quad}\boxed{\ A\mathbf{v}\ } \in \mathbb{B}^{4+3}$.

This time, the check block is smaller than the original message $\mathbf{v}$. To quantify this fact, one defines the *information rate* of the code as

$$\rho = \frac{k}{n} \qquad \left(= \frac{\lg |C|}{n} \text{ if } C \text{ is not linear} \right).$$

Here we have $\rho = 4/7 \sim 0.57$.

Exercise. Explain in what sense the code in the previous example can *detect* up to two errors, if it does not have to *correct* one!

Definition. Let H be a matrix whose columns are all $2^r - 1$ nonzero words formed from k bits. The linear code $C = \ker H$ is called a *Hamming code*; it has parameters $(2^r - 1, 2^r - r - 1, 3)$.

Any two Hamming codes of the same size ($|C| = 2^{2^r - r - 1}$) are essentially equivalent, because permuting the columns will merely permute all the bits in C. When $r = 2$, we can take

$$H = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right), \qquad E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

so as to recover the repeat code $C = \{000, 111\} \subset \mathbb{B}^3$.

Hamming codes are *perfect*, meaning that we have equality in Lemma 2 in §10.2. Another way of saying this is that the balls

$$N_e(\mathbf{y}) = \{\mathbf{x} \in \mathbb{B}^n : \partial(\mathbf{x}, \mathbf{y}) \leqslant e\}$$

partition C as $\mathbf{y}$ ranges over C. For the Hamming code in $\mathbb{B}^n$, we have 2^k balls each of size $1 + n = 2^r$. This was alluded to at the end of §10.2.

The information rate of a Hamming code is

$$\rho = \frac{k}{n} = \frac{2^r - r - 1}{2^r - 1} = \frac{1 - (r+1)2^{-1}}{1 - 2^{-r}} \to 1 \quad \text{as} \quad r \to \infty.$$

Already for $r = 6$ ($n = 63$) we have $\rho > 0.9$.

Apart from the Hamming codes, codes of size 1 and repeat codes

$$\{0 \cdots 0,\ 1 \cdots 1\} \subset \mathbb{B}^n$$

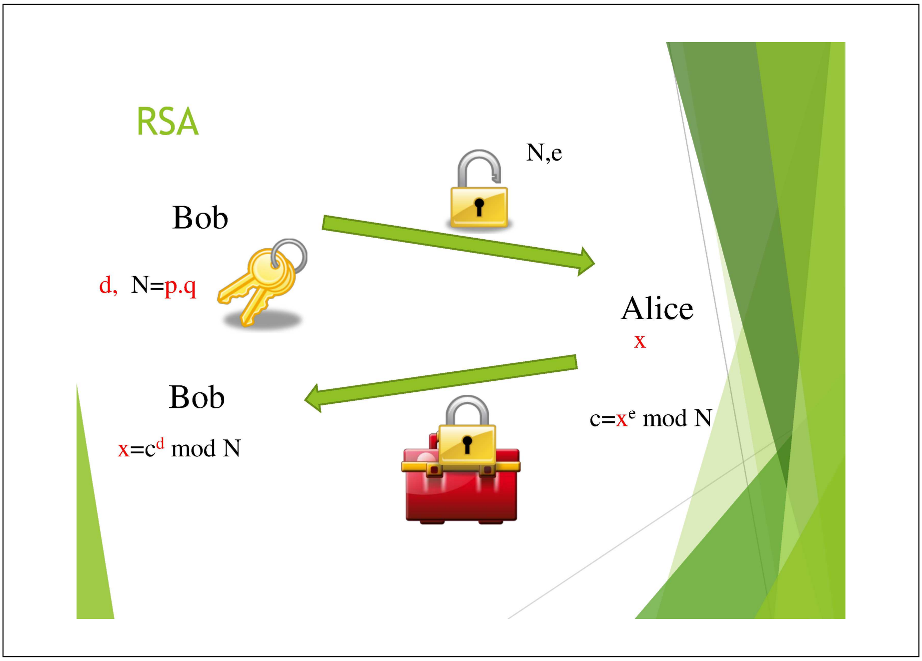
of size 2 with n odd, there is just one other perfect code. This is the mysterious *Golay code* G_{23}, a binary linear code with parameters $(23, 12, 7)$.

11. Cryptography

Until the 1970's encrypting messages required both sender and receiver to use the same key ('code-book') to encrypt and decrypt. This use of such symmetric keys is not practical for interchange of secret data on the internet. The concept that led to the introduction of all modern forms of cryptography is that of an asymmetrical system of keys based on a *trapdoor*, in the terminology of a famous paper by Diffie & Hellman (1976). The trapdoor is a mathematical function that can only be inverted using extra information. This idea was successively implemented in the celebrated RSA algorithm, named after Rivest, Shamir & Adleman, who discovered it in April 1977 and patented it that year.

11.1 The RSA algorithm

Years afterwards, it was revealed that the same algorithm had been described by Clifford Cocks in a GCHQ memo in 1973, working with James Ellis, who had already conceived of the trapdoor mechanism. Its essence was described by Professor Cocks at Cumberland Lodge in 2018:



Alice (now on the right) wants to send Bob a secret message x, in the form of a number in ordinary decimal notation. In preparation for this:

- [Key generation] Bob chooses two large prime numbers p, q (nowadays they will typically each have up to 2048 bits) and computes $n = pq$. He also chooses a number e (that need not be so large) that is coprime to $\phi = (p-1)(q-1)$, if not actually prime itself. He also uses Euclid's extended algorithm to find the inverse d to e modulo ϕ (with $0 < d < \phi$);

this is his private key that should stored with password protection.

- [Key distribution] Bob then makes available to Alice (and indeed, the world) the *public key* consisting of the pair (e, n) (in this order on past exams!). These are represented by the 'open padlock'. The number n can be thought of as the body of the lock, and e a safety catch needed to snap the lock shut.

All Alice has to do is:

- [Encryption] Make sure her plaintext message x is shorter than n, so we'll assume $0 < x < n$ (if not it must be split it into blocks). She then computes the ciphertext

$$c = E(x) = x^e \bmod n$$

using modular exponentiation by repeated squaring as taught in the module CCM251 (§3.2). This is sent to Bob, and forms the 'padlocked case'.

To undo the padlock Bob must:

- [Decryption] Take the ciphertext c and compute

$$D(c) = c^d \bmod n.$$

The next result shows that $D(c) = x$ is the original plaintext.

Theorem. With our notation, the operations D and E are inverse to each other, i.e.

$$\boxed{x^{ed} = x \bmod n}$$

Before embarking on a proof, we give some number theoretical background. For an arbitrary integer n, one denotes by $\varphi(n)$ the number of integers in the range $1, 2, \ldots, n$ that are coprime to n, i.e. positive integers $k \leqslant n$ such that $\gcd(k, n) = 1$. The integer mapping φ is called *Euler's totient function*. The notation is due to Gauss around 1800, though Euler had established the product formula

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

in the 1760's. If n is prime (so greater than 1), then $\varphi(n) = n - 1$.

Now take $n = pq$, and set $\phi = \varphi(n)$. The only numbers in the list $1, 2, \ldots, n$ that are *not* coprime to n are multiples of p or q. These are

$$\begin{array}{ccccc} p, & 2p, & 3p, & \ldots, & qp \\ q, & 2q, & 3q, & \ldots, & pq, \end{array}$$

and there are $q + p - 1$ of them (including the zero class, here represented by n). So

$$\phi = n - (q + p - 1) = pq - p - q - 1 = (p-1)(q-1),$$

consistently with Euler's formula. More generally one can show that $\varphi(mn) = \varphi(m)\varphi(n)$.

Proof of the theorem. By assumption, there is an integer y such that $1 = de + y\phi$. It suffices to show that

$$x = x^{ed} \bmod p \quad \text{and} \quad x = x^{ed} \bmod q,$$

since then pq must also divide $x - x^{ed}$. Consider the first assertion. If $p|x$ then both x and x^{ed} are congruent to 0 modulo p. If not, we can use Fermat's little theorem to deduce that

$$x = x^{ed+y\phi} = x^{ed}(x^{p-1})^{(q-1)y} = x^{ed} \mod p,$$

since $x^{p-1} = 1 \bmod p$. The same applies modulo q. $\square$

The secrecy part of the algorithm derives from the apparent impossibility of inverting the operation E and factoring large numbers into prime factors. In practice, p and q should have a similar length but differ but a few powers of 10. Prime numbers can be found using primality tests, like a probabilistic version of one we shall consider briefly in §10.6. The effectiveness of the algorithm in this respect cannot be *proved* mathematically, and there is a serious concern from experts that within a couple of decades quantum computers could crack the current public keys.

11.2. Examples and comments

We begin with a toy example to understand the procedure. Years provide a repertoire of 'small' memorable primes, such as $1999, 2003, 2011, 2017, 2027, 2029, 2039$. Bob chooses

$$p = 1999, \quad q = 2029,$$

so the 'key length' equals

$$n = pq = 4\,055\,971,$$

and

$$\phi = \varphi(n) = (p-1)(q-1) = 4\,051\,944.$$

Bob takes

$$e = 5$$

so as to make it easy for Alice for work out x^e with her primitive calculator. It is obviously coprime to ϕ; indeed choosing e to be a prime obviates the need to check that $\gcd(e, \phi) = 1$. In addition $\phi + 1$ is a multiple of 5, and in fact

$$\phi + 1 = 5 * 810389,$$

so Bob's 'PIN' is $d = 810389$. He considers swapping d with e but decides against it.

Alice's message x is in fact only 3 digits long, nonetheless x^e is about $996 * 10^9$, just less than one trillion. But she did the modular calculation almost by hand:

$$\begin{aligned} c = 251^5 &= 251 * (251^2)^2 \mod n \\ &= 251 * (63001)^2 \mod n \\ &= 251 * (3\,969\,126\,001) \bmod n \\ &= 251 * (2\,386\,363) \mod n \\ &= 2\,749\,376 \bmod n. \end{aligned}$$

Bob now uses a computer to discover that

$$c^d \bmod n = 251,$$

so Alice had encrypted her module code.

Further comments on the theorem:

- When x is coprime to n, which in practice it will almost always be, the boxed result is also a corollary of Euler's theorem:

  $$\gcd(x,n) = 1 \quad \Rightarrow \quad x^{\varphi(n)} = 1 \bmod n.$$

  This is proved in the same way as Fermat's little theorem: the congruence classes of those numbers that are coprime to n form a group (of size ϕ) under multiplication modulo n. By Lagrange's theorem, the order of any element divides the size of the group, so $x^{\varphi(n)}$ is the identity.
- Let $g = \gcd(p-1,q-1)$ and $\ell = \operatorname{lcm}(p-1,q-1)$. Recall that

  $$(p-1)(q-1) = g\ell,$$

  so that ℓ divides $(p-1)(q-1)$. The theorem above remains valid if $de = 1 \bmod \ell$. In our example, $g = 6$ and we can take d to be the smaller number $135\,065$.

Further comments on the algorithm:

- A secret memo dating from 1973 shows that Clifford Cocks actually took $e = n$ but uses the same operation E. In theory one can use quite a small value of e to make encryption easy, although this makes the process more vulnerable to attack (especially if x^e is already less than n).
- A popular, but more serious, choice of e is the Fermat prime $2^{16} + 1 = 65537$, since its binary form

  $$10000000000000001_2$$

  has small Hamming weight, which assists in computing x^e.
- In practice, the numbers p, q, e, d will be converted to base 2, and then divided into 64-bit blocks. These are then displayed using the 64 characters `A`...`Z`, `a`...`z`, `0`...`9`, `+` `/` as well as `=` and a series of check digits.

The RSA algorithm can be also be used to *authenticate* the sender of ciphertext by providing a digital signature linked to the message, and to enable *non-repudiation* – Alice can't deny she was the author of a command send to Bob.

A list of public keys in a typical `known _hosts` folder reveals a mix of 'ssh-rsa' and 'ecdsa-sha2-nistp256' algorithms. The latter are all based on the elliptic curve
$y^2 = x^3 - 3x + 41058363725152142129326129780047268409114441015993725554835256314039467401291$
whose study belongs to the realm of number theory and geometry.

11.3. Miller's test

This subsection presents a powerful method for detecting numbers that are not prime, generalizing Fermat's little theorem.

Let $n = 217$. It is patently obvious that n is divisible by 7. Indeed, $n = 7 * 31$.

Observe that $n-1$ is divisible by 2^3, so set $b = 2^j * 27$ with $0 \leqslant j \leqslant 3$. The following table displays the values of $a^b \bmod n$, for $1 \leqslant a \leqslant 10$ in the range $[-108, 108]$:

	$a{=}1$	2	3	4	5	6	7	8	9	10
$b{=}27$	1	−27	−8	78	−92	−1	−77	64	64	97
$b{=}54$	1	78	64	8	1	1	70	−27	−27	78
$b{=}108$	1	8	−27	64	1	1	−91	78	78	8
$b{=}216$	1	64	78	−27	1	1	35	8	8	64

If n were prime, by Fermat's little theorem we would have $a^{n-1} = 1 \bmod n$ if $0 < a < n$ and so the last row would be all 1's. Moreover, if n is prime and $n-1 = 2m$ then $x = a^m$ satisfies $x^2 = 1 \bmod n$, i.e. $(x-1)(x+1) = 0 \bmod n$, and n must divide $x-1$ or $x+1$ so either $a^m = 1 \bmod n$ or $a^m = -1 \bmod n$. Continuing in this way establishes the

Proposition. Let n be prime and set $n-1 = 2^j * k$ where k is odd. If $0 < a < n$ we have

either (i) $a^k = 1 \bmod n$,

or (ii) $a^{2^i * k} = -1 \bmod n$ for some i with $0 \leqslant i < j$.

Proof. Observe that a^{n-1} is found by successively squaring a^k. But if n is prime, 1 has no 'non-trivial' square roots modulo n, only ± 1. (As a contrasting example from the table, ± 92 is a square root of 1 modulo 217.) □

Definition. Fix $n > 2$ and then a.

- If either (i) or (ii) holds, we say that n *passes Miller's test to base* a. If n is not prime, then a is called a *liar* since the test would appear to indicate that n is prime; one also says that n *is a strong pseudoprime to base* a.
- If both (i) and (ii) fail, we say that n *fails Miller's test to base* a, and that a is a *witness* to the compositeness of n.

In the example above, the columns in the last row show that a is a witness for n for all $a \in \{2,3,4,5,7,8,9,10\}$. But 6 is a liar, and 217 is a strong pseudoprime to base 6. (The word 'base' is an unfortunate one, this has nothing to do with arithmetic to base 6.)

It is known that if n is an odd composite number then at least $3/4$ of the possible a with $0 < a < n$ are witnesses. If $n < 1\,000\,000$ it even suffices to take $a = 2$ or 3 to test the primality of n. Miller's test becomes a *very* reliable way of detecting whether n is prime when it is applied repeatedly for random values of a. This is the Miller-Rabin test, whose performance can also be related to the so-called generalized Riemann hypothesis.

Exercise. Determine whether 353 passes Miller's test to base 5. [Answer: Yes, since $352 = 2 * 176 = 2^5 * 11$ and one finds that $5^{176} = -1 \bmod 353$. In fact, 353 is prime so it must pass to any base!]