Random walks and scenery reconstruction

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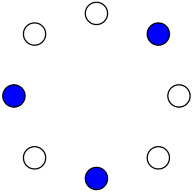
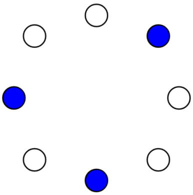
Setting and an initial question



- G is a graph
  - For example, G is a *cycle* with K vertices.
- $\xi : V(G) \to [c]$ a colouring of the vertices
  - For example, each vertex is coloured *White* or *Blue*.

Setting and an initial question



- G is a graph. ξ a colouring of the vertices
- $S_0, S_1, S_2, \dots$ is a random walk on G.
  - Each move has $\frac{1}{2}$ chance clockwise, $\frac{1}{2}$ chance anti-clockwise.
  - Initial position S_0 is unknown.
- Observe colours along the walk: eg $W, B, W, B, W, W, W, \dots$.

Central question

- Can you reconstruct ξ from these observations?

Reconstruct in what sense?

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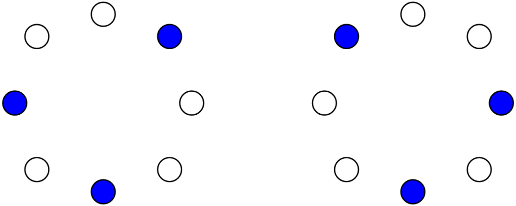

- Answer: Maybe, but only up to reflection / rotation.

Reconstruct in what sense?

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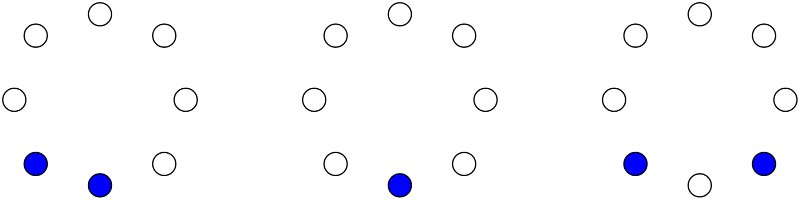
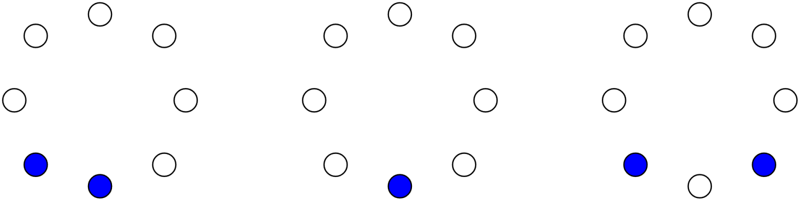


What does '*reconstruct*' actually mean?

- Possible that the walker doesn't explore the whole cycle.
  - Certainly can't reconstruct if that happens.
  - But this happens with $\mathbb{P} = 0$.

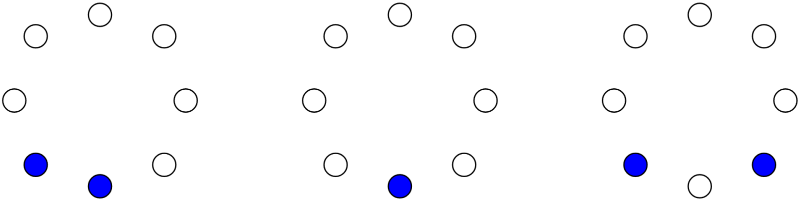
Reconstruct in what sense?



What does '*reconstruct*' actually mean?

- Possible that the walker doesn't explore the whole cycle.
  - Certainly can't reconstruct if that happens.
  - But this happens with $\mathbb{P} = 0$.
- So 'reconstruct' means
  - *Reconstruct with probability 1.*
  - Jargon for $\mathbb{P} = 1$: '*almost surely*'.

How many Whites and Blues?



In the long-run:

- Observed proportion of Bs $\longrightarrow$ Actual proportion of Bs.
- Meaning, with probability 1 / almost surely.
- So can distinguish various possibilities with probability 1
  - but this does require an *infinite* number of observations.

Technical aside (in the formal language of probabilistic convergence)

- n large: $S_n \stackrel{d}{\approx} \mathrm{Uniform}(G)$...
  - provided K is odd.
- But $S_n \longrightarrow X$ makes no sense. $\mathbb{P}(S_n \to \_) = 0$.
- However $S_n \xrightarrow{d} \mathrm{Uniform}(G)$ does make sense, as does

$$\frac{\#\{m \le n : S_m \text{ is Blue}\}}{n} \xrightarrow{\text{a.s.}} \frac{1}{K}.$$

General patterns amongst Whites and Blues

Question

- Can you reconstruct number of pairs $\underline{WW}$ and $\underline{WB}/\underline{BW}$ and $\underline{BB}$?
- Idea: compare

$$B\underline{B}B \quad \text{and} \quad \begin{cases} W\underline{B}B \\ B\underline{B}W \end{cases} \quad \text{and} \quad W\underline{B}W.$$

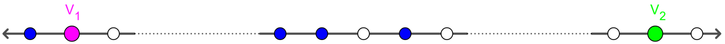
- We observe instances of "$\ldots, \underbrace{B, W}, \ldots$" in the sequence.
- In principle can induct on length of pattern to reconstruct the whole colouring with $\mathbb{P} = 1$.

Beyond finite settings - reconstruction on $\mathbb{Z}$

Challenges and possible questions:

- Can you reconstruct the colouring with $\mathbb{P} = 1$?
- What would 'proportion of Bs' mean here?
  - Proportion of time at any given vertex $\longrightarrow 0$,
  - 'Uniform distribution on $\mathbb{Z}$'?
- Can you distinguish any two non-equivalent colourings with $\mathbb{P} = 1$?
  - In fact, even this weaker result is false!
  - But *random colourings* can be reconstructed with $\mathbb{P} = 1$.
- What about higher dimensions $\mathbb{Z}^d$ for $d = 2$, $d = 3$, etc?
  - Lots of open questions!

Beyond finite settings - sketch argument



- Two *special* vertices v_1, v_2, distance L apart.
  - Have two special colours $\mathbf{S_1}$ and $\mathbf{S_2}$
- $\mathbb{P}(\text{RW goes } v_1 \xrightarrow{\text{directly}} v_2) = 2^{-L} > 0$.
- If you ever observe

$$\ldots \quad B, B, \mathbf{S_1}, \underbrace{W, B, W, W, B, \ldots, B, W}_{L-1}, \mathbf{S_2}, W, B, \quad \ldots,$$

  Then you have learned the scenery between v_1 and v_2.
- What if you don't know L? Observe

$$\ldots, B, \mathbf{S_1}, \underbrace{W, \ldots, W}_{L-1}, \mathbf{S_2}, W, \ldots \qquad \ldots, B, \mathbf{S_1}, \underbrace{W, \ldots, W}_{L-2}, \mathbf{S_2}, W, \ldots$$

Reconstruction - the other direction

Notation reminder:

$$\text{Walk on graph } G: \quad S_0, S_1, S_2, \ldots,$$

$$\text{Giving colours } \xi(S_0), \xi(S_1), \xi(S_2), \ldots.$$

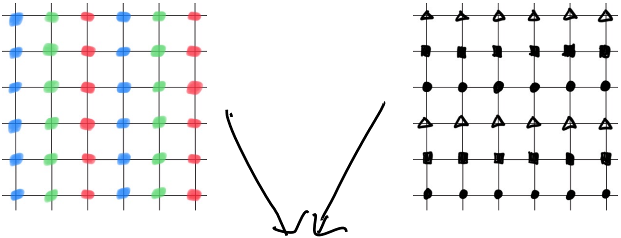
So far, we have discussed

- Reconstructing the scenery ξ from the colours on the walk $(\xi(S_0), \xi(S_1), \ldots)$.

What about

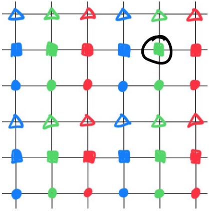
- Reconstructing the *walk* $(S_0, S_1, \ldots)$ from $(\xi(S_0), \xi(S_1), \ldots)$?

Two ideas



Idea 1

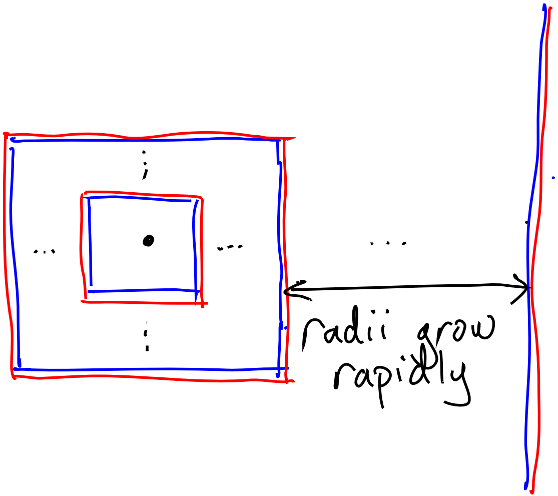
Product colourings



Idea 2

Colouring is an injection on neighbourhoods

Idea 3: finite reduction of x_0



Idea 4+: distinguishing finite collections of walks

Proposal: take a product of the two colourings discussed so far with an **IID uniform colouring** on sufficiently many colours.

Does this work?

- $\mathbf{x}$ is a **self-avoiding walk**.
  - Yes, works with $O(\Delta^2)$ colours.
- $\mathbf{x}$ is simple **random walk** on (transient) G.
  - Yes, works a.s. with two colours.
- $\mathbf{x}$ is a general deterministic transient walk
  - Conjecture: yes it also works in this case.