Water waves and transcendental numbers

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C.J. Amick and L.E. Fraenkel 1987
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Stokes waves

A Stokes wave is a steady periodic wave, propagating under gravity with constant speed on the surface of an infinitely deep irrotational flow. Its free surface is determined by Laplace's equation, kinematic and periodic boundary conditions and by a dynamic boundary condition given by the requirement that pressure in the flow at the surface should be constant (Bernoulli's theorem).

On one of the final passes in front of the airshow crowd, Gene Soucy and Teresa Stokes wave to the crowd.

Stokes waves

Free surface:

$$S := \{(u(s), v(s)) | \ s \in \mathbb{R}\},$$

where

- (u, v) is injective,
- $u'(s)^2 + v'(s)^2 > 0$,
- $s \mapsto (u(s) - s, v(s))$ is 2π–periodic.

Let Ω denote the open region of $\mathbb{R}^2$ below S.

The boundary value problem:

Find S for which there exists ψ such that

- $\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = 0$ in Ω,
- ψ is 2π–periodic in X,
- $\nabla \psi := \left(\frac{\partial \psi}{\partial X}, \frac{\partial \psi}{\partial Y}\right)$ is bounded in Ω and $\nabla \psi(X, Y) \to (0, 1)$ as $Y \to -\infty$,
- $\psi \equiv 0$ on S,
- $|\nabla \psi(X, Y)|^2 + 2\mu Y = 1$ on S (the Bernoulli boundary condition).

$\mu^{-1/2}$ is the Froude number, a dimensionless combination of speed, wavelength and gravitational acceleration.

If $\psi \equiv 0$ on S, then the Bernoulli condition

$$|\nabla\psi(X, Y)|^2 = 1 - 2\mu Y \text{ on } S$$

is equivalent to the Neumann condition

$$\frac{\partial\psi}{\partial\mathbf{n}}(X, Y) = h(Y) \text{ on } S,$$

where $1 - 2\mu Y = h^2$ and $\mathbf{n}$ is the outward unit normal to S.

Stokes (1847): nonlinear waves with small amplitudes.

Stokes Conjectures (1880)

First conjecture: There exists a large amplitude wave with a stagnation point and a corner containing an angle of 120° at its highest point (Stokes wave of extreme form).

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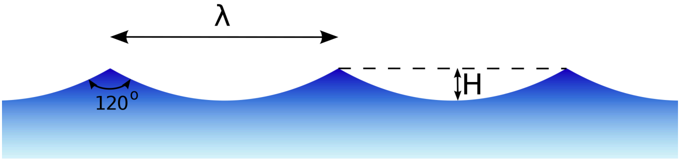

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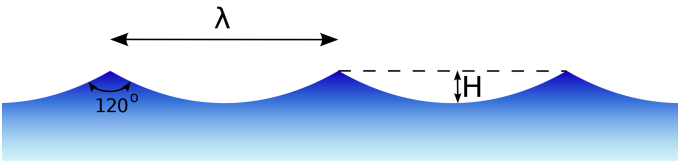

Second conjecture: The Stokes wave of extreme form is convex between successive crests.

Stokes first conjecture: C.J. Amick, L.E. Fraenkel, J.F. Toland (1982) and independently P.I. Plotnikov (1982)

Stokes first conjecture: C.J. Amick, L.E. Fraenkel, J.F. Toland (1982) and independently P.I. Plotnikov (1982)

Stokes second conjecture: P.I. Plotnikov and J.F. Toland (2004)

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C.J. Amick and L.E. Fraenkel, Trans. Am. Math. Soc. 299, 273-298 (1987):

$$\phi(s) = \frac{\pi}{6} + \sum_{n=1}^{k} a_n s^{\mu_n} + o\left(s^{\mu_k}\right) \quad \text{as } s \searrow 0$$

to arbitrary order;

J.B. McLeod, Trans. Am. Math. Soc. 299, 299-302 (1987):

$$a_1 < 0$$

Stokes second conjecture: P.I. Plotnikov and J.F. Toland (2004)

L.E. Fraenkel (2007): a new constructive proof of the existence of the Stokes wave of extreme form.

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Ludwig Edward Fraenkel
28 May 1927 – 27 April 2019

Amick & Fraenkel (1987)

$$\sqrt{3}\,(1+z) = \tan(z\pi/2) \tag{1}$$

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All roots of (1) with $\mathrm{Re}z > -1$ are simple and real,

$z_j \in (2j-2, 2j-1)$ for all $j \in \mathbb{N}$,

$z_j = 2j - 1 + O(j^{-1})$ as $j \to \infty$ (the O-term is negative).

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"It was shown by Grant (1973), albeit somewhat tersely, that z is irrational."

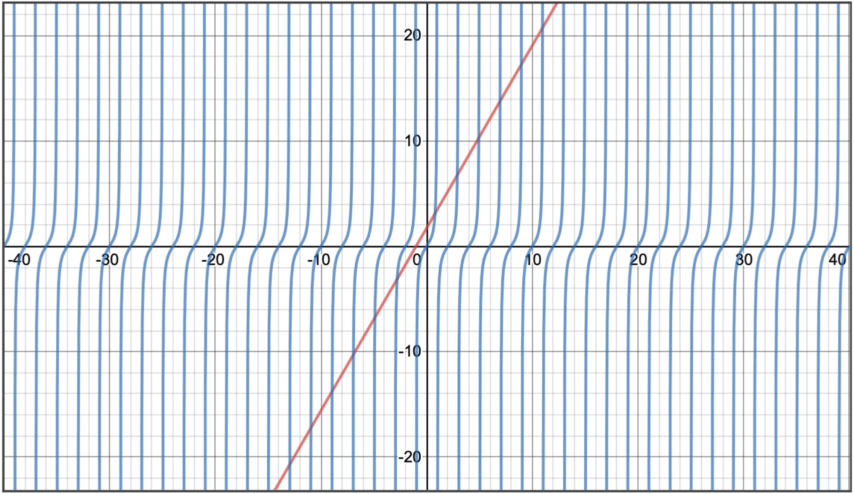

Figure: Graphs of $y = \sqrt{3}\,(1 + z)$ and $y = \tan(z\pi/2)$.

Amick & Fraenkel (1987)

$$\sqrt{3}\,(1+z) = \tan(z\pi/2)$$

ASSUMPTION: The sets

$$\Lambda := \left\{ \sum_{j=1}^{r} m_j z_j + m_{r+1} 2 \,\middle|\, \sum_{j=1}^{r+1} m_j \geq 2,\ m_j \in \mathbb{N}_0,\ r \in \mathbb{N} \right\}$$

and

$$B := \left\{ z_j \,\middle|\, j \in \mathbb{N} \right\}$$

are disjoint.

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and

$$B := \{ z_j \mid j \in \mathbb{N} \}$$

are disjoint.

Numerical calculation: None of the first 100 terms of Λ belongs to B.

Amick & Fraenkel (1987)

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CONJECTURE: The set $\{1, z_1, z_2, z_3, \ldots\}$ is linearly independent over the rationals.

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CONJECTURE: The set $\{1, z_1, z_2, z_3, \ldots\}$ is linearly independent over the rationals.

Theorem.

(i) Each number z_j is transcendental.

(ii) If $j \neq k$, then the set $\{1, z_j, z_k\}$ is linearly independent over the rationals.

Transcendental numbers

A number α is called algebraic if it is a solution of an equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are integer numbers.

Transcendental = not algebraic.

Transcendental numbers

J. Liouville (1844): transcendental numbers exist, e.g.

$$\sum_{n=1}^{\infty} 10^{-n!}$$

Transcendental numbers

G. Cantor (1874): the set of polynomials with integer coefficients is countable $\Longrightarrow$
the set of algebraic numbers is countable $\Longrightarrow$
the set of transcendental numbers is uncountable

Transcendental numbers (prehistory)

L. Euler (1737): e and e^2 are irrational.

J.H. Lambert (1768): proved that π is irrational,
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Transcendental numbers (prehistory)

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J.H. Lambert (1768): proved that π is irrational, conjectured that e and π are transcendental.

L. Euler (1744): claimed but did not prove that if a and b are rational numbers and b is not a rational power of a, then $\log_a b$ is transcendental.

$\pi = 3.14 \qquad \ldots$

$\pi = 3.14159265358979\ldots$

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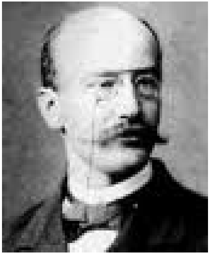
How I need a drink, alcoholic of course, after the heavy lectures involving quantum mechanics ...

Transcendental numbers

C. Hermite (1873): e is transcendental.



C.L.F. von Lindemann (1882): π is transcendental. If $\alpha \neq 0$ is algebraic, then e^{α} is transcendental.

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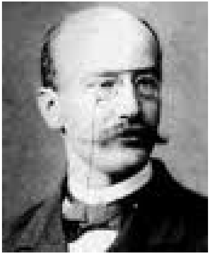

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π is transcendental $\implies$ one cannot “square the circle" (P.L. Wantzel 1837).

Hilbert's seventh problem (1900): If z is algebraic, $z \neq 0, 1$, and α is irrational and algebraic, then z^α is transcendental.
(This is a stronger statement than Euler's claim of 1744.)

Proof: A.O. Gelfond (1934) and, independently, T. Schneider (1934).

Transcendental numbers

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The Gelfond-Schneider theorem is the main ingredient in the proof of the Amick-Fraenkel theorem on the roots of

$$\sqrt{3}\,(1+z) = \tan(z\pi/2).$$

Alan Baker
19 August 1939 – 4 February 2018
Fields medal 1970

Algebraic independence

Definition.

Complex numbers $z_1, \dots, z_n$ *are called* ***algebraically independent*** *if for any polynomial* $P(x_1, \dots, x_n) \not\equiv 0$ *with algebraic coefficients,* $P(z_1, \dots, z_n) \neq 0$.

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Lindemann's theorem:
If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

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If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

Schanuel's conjecture (1960's)

If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent complex numbers, then among the $2n$ numbers $\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}$ at least n are algebraically independent.

Schanuel's conjecture

Schanuel's conjecture is believed to include all known transcendence results as well as all reasonable conjectures on the values of the exponential function.

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It is not known whether any of the following numbers is irrational:

$$\pi + e, \quad \pi e, \quad \pi^\pi, \quad e^e, \quad \pi^e, \quad \log \pi, \quad \pi \log 2, \quad 2^{\log 2}.$$

Schanuel's conjecture implies that they are algebraically independent (and hence transcendental).

Yu.V. Nesterenko (1996):

1. π, e^{π}, and $\Gamma\left(\frac{1}{4}\right)$ are algebraically independent;
2. π, $e^{\pi\sqrt{3}}$, and $\Gamma\left(\frac{1}{3}\right)$ are algebraically independent;
3. for every $n \in \mathbb{N}$, the numbers π and $e^{\pi\sqrt{n}}$ are algebraically independent.

Schanuel's conjecture: methods from model theory

A. Macintyre, A.J. Wilkie, B. Zilber, J. Pila, J. Kirby ...

Back to $\sqrt{3}\,(1+z) = \tan(z\pi/2)$

Consider a more general equation:

$$\beta^z = \frac{az+b}{cz+d}\,, \qquad (2)$$

where a, b, c, d, β are algebraic numbers, $\beta \neq 0, 1$,
$\beta^z := e^{z\ln\beta}$ and ln denotes a fixed branch of the logarithm.

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If

$$a = i\sqrt{3}\,,\;\; b = 1+i\sqrt{3}\,,\;\; c = -i\sqrt{3}\,,\;\; d = 1-i\sqrt{3}\,,\;\; \beta = e^{i\pi} = -1,$$

then (2) takes the form

$$\sqrt{3}\,(1+z) = \tan(z\pi/2).$$

Rational roots

Reminder: $\sqrt{3}\,(1+z) = \tan(z\pi/2)$ is a special case of (2) with $a, b, c, d \in \mathbb{Q}(\sqrt{-3})$:

$$a = i\sqrt{3}\,, \quad b = 1 + i\sqrt{3}\,, \quad c = -i\sqrt{3}\,, \quad d = 1 - i\sqrt{3}\,.$$

Notation: $\mathbb{Q}(\sqrt{r})$, $r \in \mathbb{Z}$ is the quadratic field

$$\mathbb{Q}(\sqrt{r}) := \left\{x + y\sqrt{r} : \; x, y \in \mathbb{Q}\right\}.$$

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$$\mathbb{Q}(\sqrt{r}) := \left\{ x + y\sqrt{r} : \; x, y \in \mathbb{Q} \right\}.$$

Question: Is there anything special about $\sqrt{-3} = i\sqrt{3}$ here?

Yes!

Rational roots

$$\beta^z = \frac{az+b}{cz+d} \quad (2)$$

Theorem. (ES, 2014)

Let $\beta = -1$, $a, b, c, d \in \mathbb{Q}(\sqrt{r})$, where r is a square free integer. Then (2) may have a rational root $z = \ell/q$, $\ell \in \mathbb{Z}$, $q \in \mathbb{N}$ only in the following cases:
(i) $q = 1$,
(ii) $r = -1$, $q = 2$,
(iii) $r = -3$, $q = 3$.

The theorem is an immediate corollary of a classical result on units of quadratic fields.

Rational roots

$$\sqrt{3}\,(1+z)=\tan(z\pi/2)$$

does not have rational solutions,

but a similar looking equation

$$\sqrt{3}\,(2+z)=\tan(z\pi/2)$$

has a rational root $z=-5/3$.

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Neither $\sqrt{5}\,(1+z) = \tan(z\pi/2)$ nor $\sqrt{5}\,(2+z) = \tan(z\pi/2)$ can have rational roots.

$$\beta^z = \frac{az+b}{cz+d} \tag{2}$$

Theorem. (ES, 2014)

Let $z_1, \ldots, z_n \in \mathbb{C} \setminus \mathbb{Q}$ be distinct solutions of (2). Then
(i) each number z_j is transcendental;
(ii) if

$$\beta^{\frac{ad+bc}{ac}} \neq \frac{c^2}{a^2},$$

then $1, z_j, z_k$ with $j \neq k$ are linearly independent over $\mathbb{Q}$;
(iii) if

$$\beta^{\frac{ad+bc}{ac}} = \frac{c^2}{a^2},$$

then $\frac{ad+bc}{ac}$ is rational and $-\frac{ad+bc}{ac} - z_j$ is also a solution of (2); on the other hand, if $k \neq j$ and $z_k \neq -\frac{ad+bc}{ac} - z_j$, then $1, z_j, z_k$ are linearly independent over $\mathbb{Q}$.

$$\sqrt{3}\,(1+z) = \tan(z\pi/2) \tag{1}$$

We have $\frac{ad+bc}{ac} = 2$, $c^2/a^2 = 1$ and $\beta = -1$. Hence

$$\beta^{\frac{ad+bc}{ac}} = \frac{c^2}{a^2}$$

and part (iii) of the last theorem implies that if z_1 is a solution of (1) then so is $-2 - z_1$. This follows also directly from (1). This does not contradict the Amick-Fraenkel theorem which deals only with the positive solutions of (1).

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Theorem. (ES, 2014)

Let $z_1, \ldots, z_n \in \mathbb{C} \setminus \mathbb{Q}$ be distinct solutions of (2). If the numbers $1, z_j, z_k$ are linearly independent over $\mathbb{Q}$ for every $j \neq k$, then Schanuel's conjecture implies that $z_1, \ldots, z_n$ are algebraically independent.

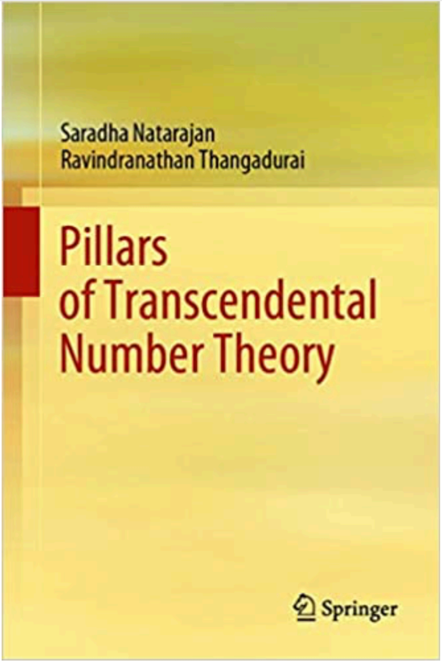
Stefan Steinerberger, A rigidity phenomenon for the Hardy-Littlewood maximal function. *Studia Math.* **229** (2015), no. 3, 263–278:

• if $x > 0$ and $\boxed{\tan x = x}$, then x is transcendental and x and π are linearly independent over $\mathbb{Q}$;

• any two elements of the set

$$\{z \in \mathbb{R}_+ : \ \tan z = z\}$$

are linearly independent over $\mathbb{Q}$.


Saradha Natarajan
Ravindranathan Thangadurai
Pillars of Transcendental Number Theory
Springer

Exercise

1. If $z \in \mathbb{C}$ is a non-rational zero of the equation

$$\sqrt{3}(1+z) = \tan(z\pi/2), \tag{3.19}$$

then show that z is transcendental.

2. Prove that the following statement is equivalent to Theorem 3.0.1. *Let α and β be algebraic numbers such that they are $\mathbb{Q}$-linearly independent. Then, for any $t \in \mathbb{C}\backslash\{0\}$, at least one of $e^{t\alpha}$ and $e^{t\beta}$ is transcendental.*

Notes

There are still different proofs of Gelfond–Schneider theorem available now, for instance, see [1] for a proof based on the method of interpolation determinants introduced in 1992 by M. Laurent. It is known that the roots of the Eq. (3.19) with $\Re z > -1$ is simple and real. The *Amick–Fraenkel conjecture* asserts that the set $\{1, z_1, z_2, \ldots\}$ of the zeros of (3.19) is linearly independent over $\mathbb{Q}$. This is known under Schanuel's conjecture. See [2].

References

1. Yu.V. Nesterenko, *Algebraic Independence*, vol. 14 (Tata Institute of Fundamental Research Publications, Mumbai, 2008), 157 p
2. E. Shargorodsky, On the Amick-Fraenkel conjecture. Quart. J. Math. **65**, 267–278 (2014)