
HIER RUHET IN GOTT
PROFESSOR ZU GOETTINGEN
GEBOREN IN BRESELENZ DEN 17. SEPTEMBER 1826
GESTORBEN IN SELASCA DEN 20. JULI 1866
DENEN DIE GOTT LIEBEN MUESSEN
ALLE DINGE ZUM BESTEN DIENEN.
GIULIO DELL'ORO
METILDE DELL'ORO
MARITATA DELL'ORO
GIORGIO DELL'ORO
ANTONIETTA DELL'ORO


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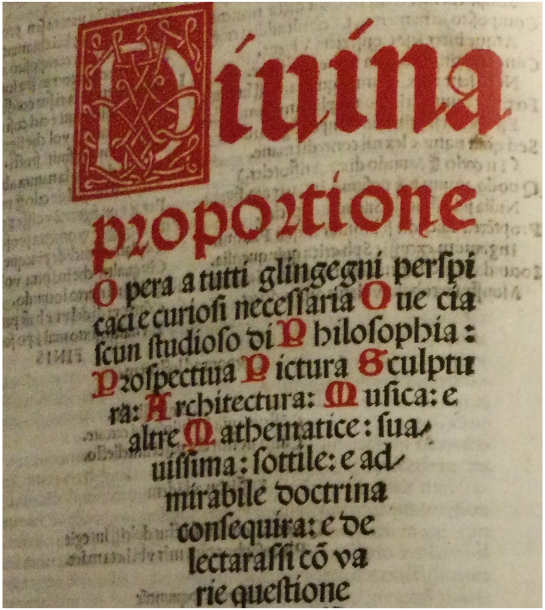

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proportione
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a book by Luca Pacioli
with contributions from
Piero della Francesca
Leonardo da Vinci

A study of golden geometry, 1498

Some highlights
Simon Salamon
Cumberland Lodge, 2024

The golden ratio

Consider a geometric progression a, b, c in which $c = a + b$, and set

$$\varphi = \frac{b}{a} = \frac{a+b}{b} = \varphi^{-1} + 1$$

$$\Rightarrow \quad \boxed{\varphi^2 = 1 + \varphi} \quad \Rightarrow \quad \varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

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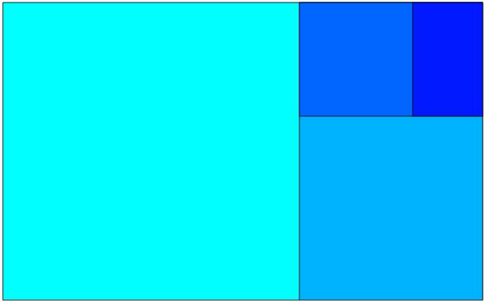

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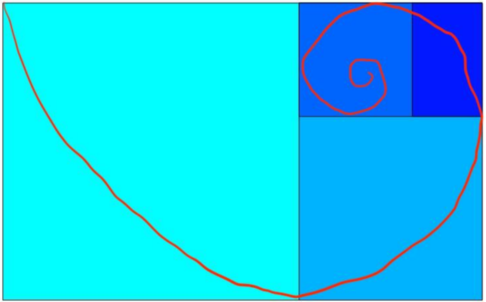
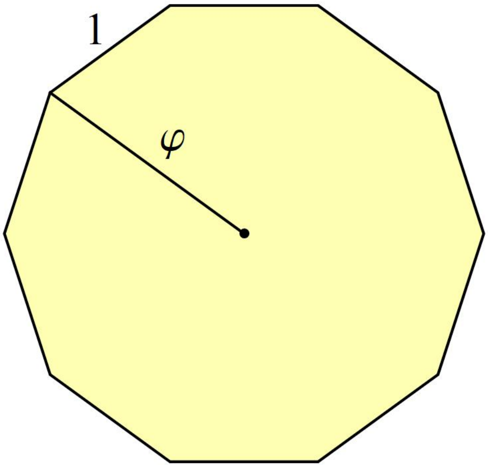


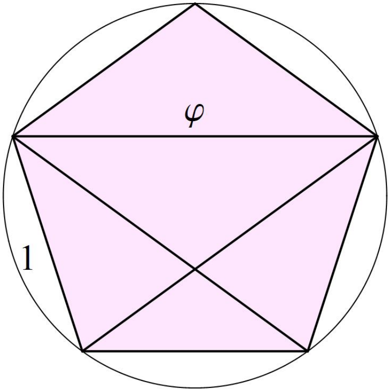
Decagons and pentagons

Proposition. $\varphi = 2\cos(\pi/5)$. Moreover,
the circumradius of a unit decagon equals φ:



Decagons and pentagons

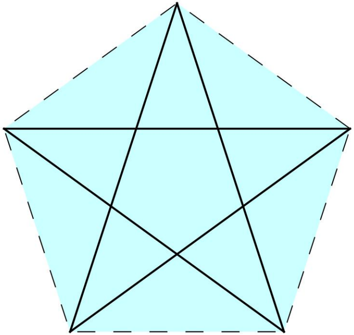
Proposition. $\varphi = 2\cos(\pi/5)$. Moreover, the diagonal of a unit pentagon has length φ:



Ptolemy's theorem for cyclic quadrilaterals $\Rightarrow$ $\varphi^2 = 1 + \varphi$

Decagons and pentagons

Proposition. $\varphi = 2\cos(\pi/5)$. Moreover,
the edges of a regular pentagram cut each other in golden sections:



Chebyshev polynomials: Let $x = \cos(\pi/5) = \varphi/2$ so $T_5(x) = \cos(\pi) = -1$.
Therefore x is a root of $T_5(x) + 1 = x(4x^2 - 2x + 1)^2$.

The golden spiral

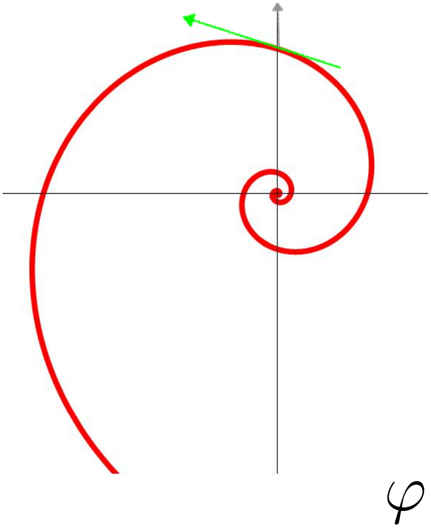
This is a logarithmic spiral whose radius grows by a factor of φ each quarter turn. Its equation in polar coordinates is

$$r = \varphi^{2\theta/\pi}.$$

Any logarithmic spiral $r = e^{\lambda\theta}$ has the property that its tangent makes a fixed angle α with the radial direction, where

$$\cos\alpha = \frac{\lambda}{\sqrt{\lambda^2 + 1}}.$$

Here $\lambda = 2\ln\varphi/\pi$ and $\alpha = 73^{\mathrm{o}}$.



phi for Phidias?

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}$$

$$= \frac{21}{13}\ldots$$

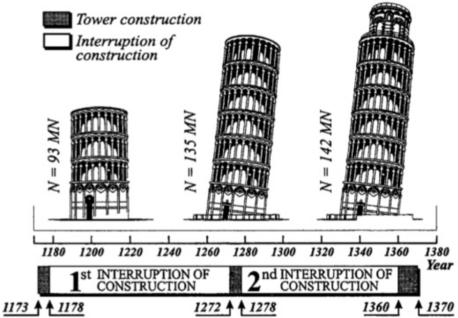
Continued fraction notation: $\varphi = [1; 1, 1, 1, \ldots]$, compare $\sqrt{2} = [1; 2, 2, 2, \ldots]$.

The fact that the division of rectangles never ceases shows that both are irrational.

Leonardo da Pisa = Fibonacci, 117?–124?

Never saw the tower leaning.

Travelled widely, educated in Algeria, where he learnt the Hindu-Arabic decimal system pioneered by Al-Khwarizmi, author of Al-Jabr, around 820 CE.



Author of *Lber abaci*, promotes the decimal system.

The sequence $1, 2, 3, 5, 8, 13, \ldots$ probably arose from Sanskrit grammar. Known to the poet Pingala in 200 BCE, revived by Virahanka around 600 CE, and then Hemachandra in 1135 CE.

(François) Édouard Lucas, 1842–91

Coined the name ‘Fibonacci sequence’.

The equation $\varphi^2 = \varphi + 1$ is used to solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$ and introduce related sequences:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	842	1363

Curiosity: $mile/km \sim 1.609$, so $55\,mph \sim 89\,kph$ and $76\,mph \sim 123\,kph$.

Conjectured that $\sum_{n=1}^{N} n^2 = M^2$ only has non-trivial solution $(N, M) = (24, 70)$.

Proved by hand that $2^{127} - 1$ is prime.

Invented Tower of Hanoi problem: number of moves needed to relocate n discs is $2^n - 1$’:

Mario Merz installation in Turin

Treating 0 as a number dates back at least to Brahmagupta 630 CE.
Peano's axioms (Turin 1894) regard 0 as a natural number: $0 \in \mathbb{N}$.

Recurrence and insertion

The Fibonacci, Lucas and other rows of Wythoff's game-theoretic array:

1	2	3	5	8	13	21	34	55	89
4	7	11	18	29	47	76	123	199	322
6	10	16	26	42	68	110	178	288	466
9	15	24	39	63	102	165	267	432	699
12	20	32	52	84	136	220	356	576	932
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309

The array contains each positive integer exactly once.
Each entry $A_{i,j}$ inserts between $A_{i-1,j}$ and $A_{i-1,j+1}$, $i \geqslant 3$.
Each pair $(A_{2i-1,j}, A_{2i,j})$ is a cold square of Wythoff's chessboard.

Euclid's algorithm

With $F_0 = 0$ and $F_1 = 1$, we have

$$\begin{aligned}\gcd(F_{12}, F_8) = \gcd(144, 21) &= \gcd(21, 18)\\ &= \gcd(18, 3)\\ &= \gcd(3, 0)\\ &= 3\\ &= F_4.\end{aligned}$$

Theorem. The Fibonacci numbers satisfy the 'strong divisibility property':

$$\boxed{\gcd(F_m, F_n) = F_{\gcd(m,n)}} \quad \text{for all } m, n \geqslant 1.$$

But so do the Mersenne numbers $2^n - 1 = 0, 1, 3, 7, 15, 31, 63, \ldots$.

Luca Pacioli, 1445?–1517

A financial mathematician, lecturer and magician, born in Sansepolcro.

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Pacioli's bibliography

Includes:

Tractatus mathematicus. Lecture notes on merchant arithmetic, taught in Perugia.

Summa de arithmetica, geometria. Plus more accounting and the double bookkeeping (income and expenditure) method.

De viribus quantitatis, magic and recreational mathematics, rediscovered by David Singmaster (promoted by Lucy McDonald), so a link to Cumberland Lodge.

De Divina Proportione, illustrated by Leonardo da Vinci. Many geometrical propositions to reveal the golden ratio in 2 and 3 dimensional figures.
Incorporates *Libellus* by Piero della Francesca without acknowledgment.

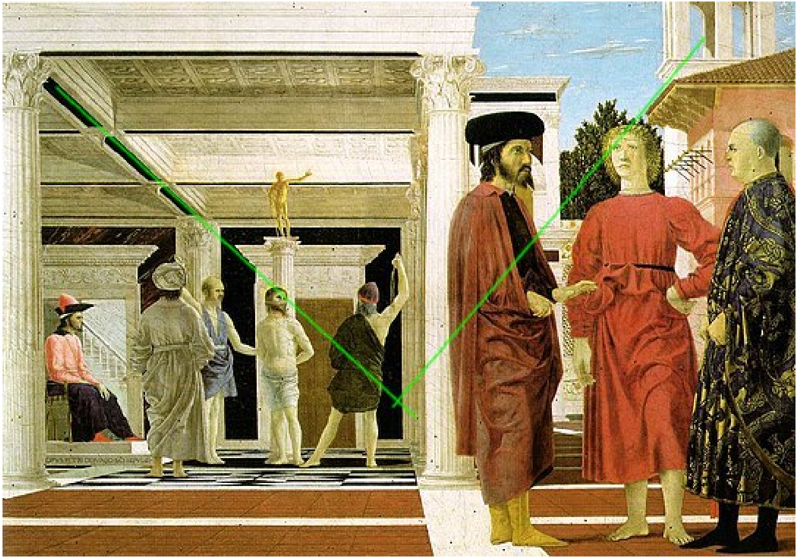
Pacioli's calculations

versione dall'arabo in latino degli Elementi di Euclide, usatissima nel medioevo e nel Rinascimento. Il problema concerne il solido a 72 facce, del quale Piero riesce a calcolare superficie e volume. La superficie è la somma delle superfici di 24 triangoli isosceli e il 48 trapezi isosceli; il volume è la somma dei volumi delle corrispondenti 24 piramidi a base triangolare e 48 a base trapezioidale - tutte col vertice nel centro della sfera circoscritta - delle quali Piero trova l'altezza in perfetta modalità Euclidea. Volendo parlare di numeri come fa Piero, il corpo ha come "equatore" e "meridiani" dodecagoni regolari di dato 2; diametro della sfera circoscritta e calcolato in $\sqrt{32+\sqrt{768}}$; la superficie vale:

$$\sqrt{540}+\sqrt{2160+\sqrt{248\,832}+\sqrt{2\,239\,488}}+\sqrt{3996+\sqrt{3\,048\,192}+\sqrt{5\,038\,848}}.$$

Questo è il risultato così come lo riferisce Piero, a parte la modernizzazione dei simboli. All'epoca non si usava la notazione decimale né per i numeri razionali, né per gli irrazionali. Risparmiamo al lettore il risultato del volume che vede sotto radice una trentina di numeri interi e frazionari, il tutto senza errori di calcolo. Il problema del corpo 72 facce a un interesse architettonico particolare, e che si collega la costruzione delle cupole; corpi analoghi erano usati anche per la costruzione di globi geografici

Piero della Francesca, 1415?-1492



De Divina Proportione incorporates Libellus

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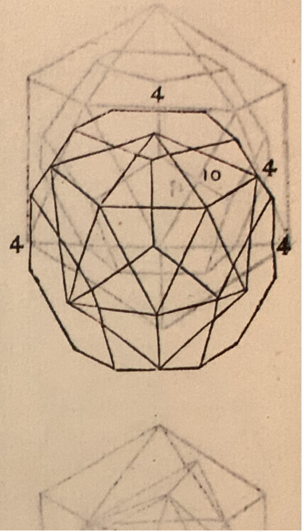
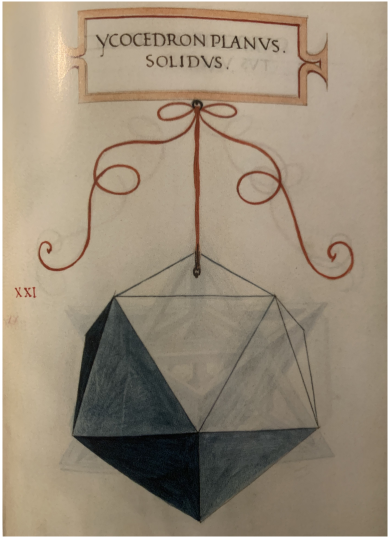
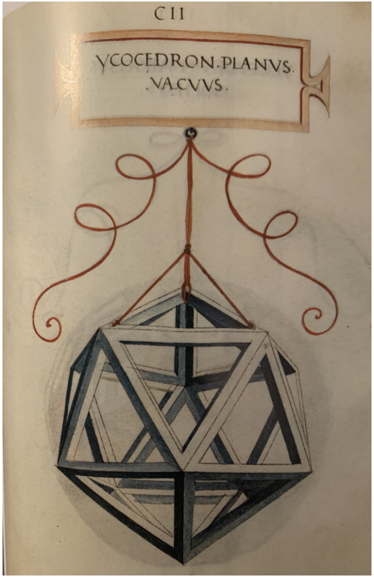


Leonardo da Vinci, 1452–1519

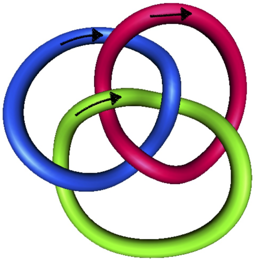


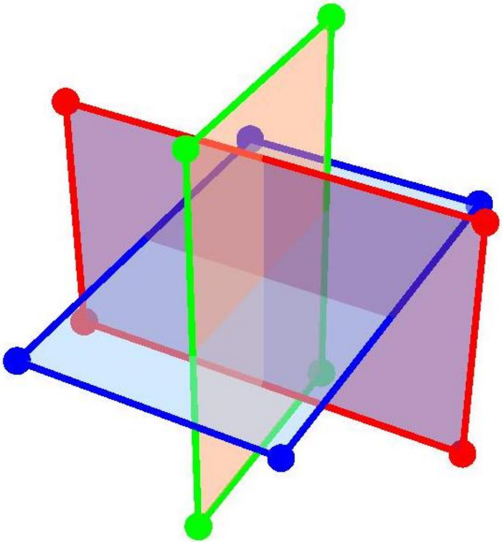


Vertices of a regular icosahdron

$$(0,\ \pm 1,\ \pm\varphi)$$
$$(\pm 1,\ \pm\varphi,\ 0)$$
$$(\pm\varphi,\ 0,\ \pm 1)$$

are the corners of 3 golden rectangles forming Borromean 'rings':





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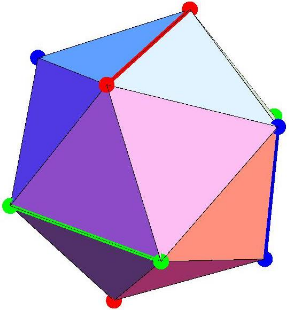
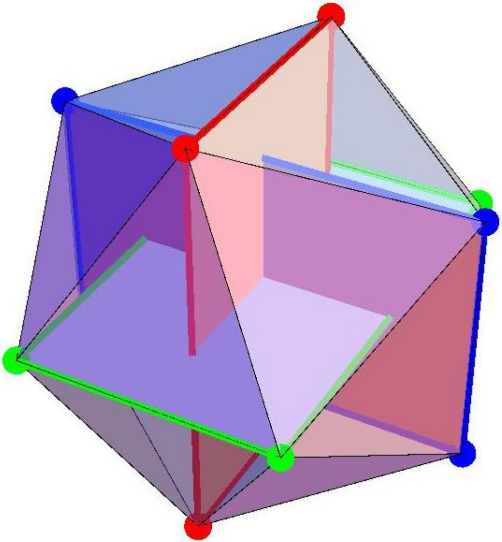


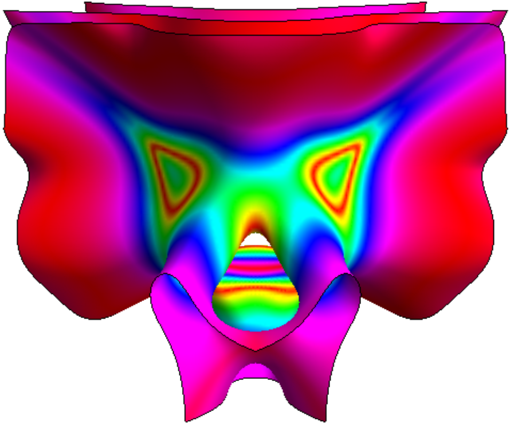


More cyclic symmetry

The ood surface (and its colouring) is determined by the torsion of the vector field

$$\mathbf{V}(x, y, z) = \left(\sqrt{2} - x^3 + yz,\ \sqrt{2} - y^3 + zx,\ \sqrt{2} - z^3 + xy\right)$$

computed by Serret-Frenet:



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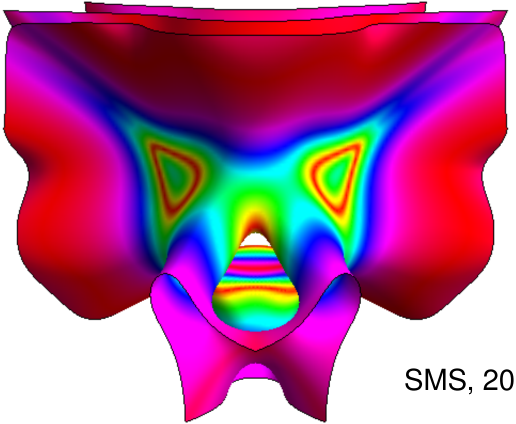
computed by Serret-Frenet:

created by Matt Jones, 2006



SMS, 2008

Chebyshev polynomials

Expand the cosine of $n\theta$, with properties analogous to the Fibonacci numbers:

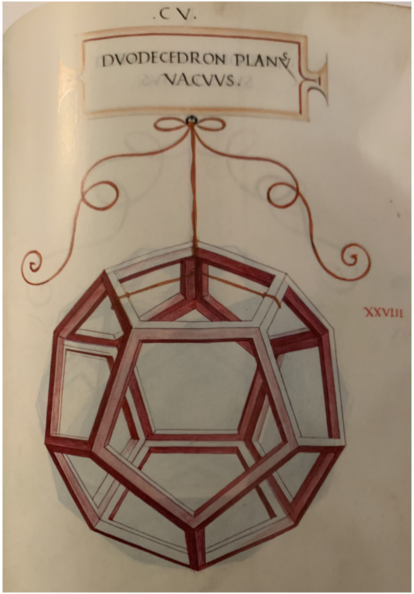
Definition. $T_n(x) = \cos(n\theta)$, where $x = \cos\theta$.

Proposition. $T_n(x) = \frac{1}{2}\,\mathrm{trace}\begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^n$

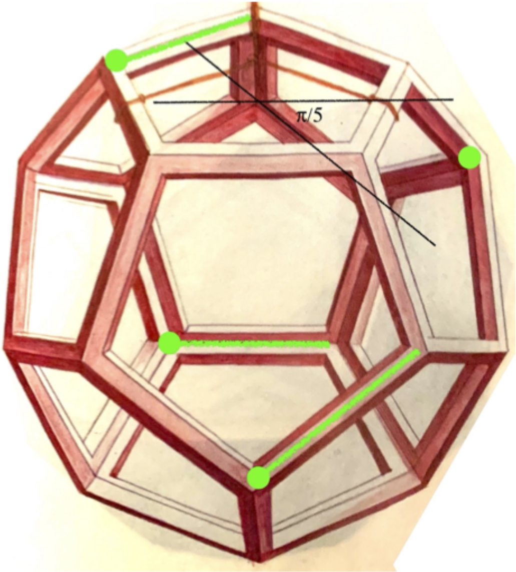
$$\begin{aligned} T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 - 5x. \end{aligned}$$

Theorem. $\boxed{\gcd(T_m(x), T_n(x)) = T_{\gcd(m,n)}}$ provided m and n are odd.

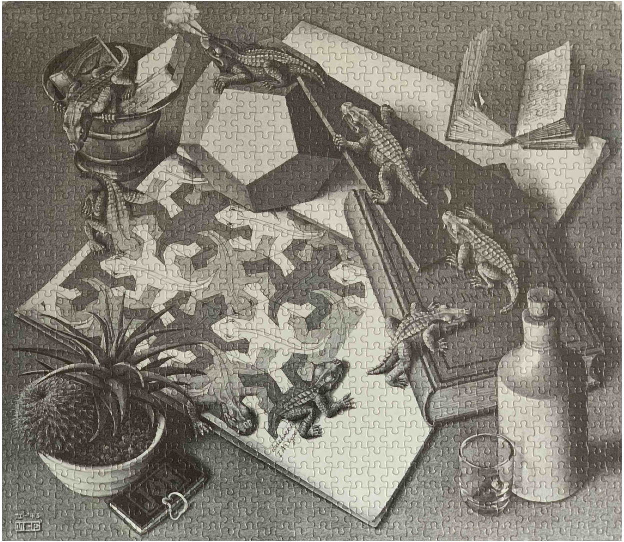
Leonardo da Vinci



(ii) Deduce that $\mathscr{M}$ has a 'subdivision' for which $V - E + F - 1 = 0$. The final '1' represents the contractible 3-dimensional interior of $\mathscr{M}$.



MC. Escher, 1898–1972



With Penrose tessellation

Salvator Dalí, 1904–1989

Dimensions $267/166.7 = 1.602 \sim 8/5$

Leonardo da Vinci