SPECTRAL GEOMETRY ON ROUGH SPACES

JOINT WORK WITH

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AND MORE...

- (Ω, g) - Riem. manifold w/ boundary

- REGULARITY OF THE METRIC?
- REGULARITY OF THE BOUNDARY?
- DIMENSION? 2

• (Ω, g) - Riem. surface
w/ boundary

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• DIMENSION?

STEKLOV PROBLEM

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma f & \text{on } \partial\Omega \end{cases}$$

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \dots \nearrow \infty$$

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Weak formulation:

$$\sigma_j(\Omega) = \inf_{E_{j+1}} \sup_{f \in E_{j+1} \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, dV}{\int_{\partial\Omega} |f|^2 \, dA}$$

- $E_{j+1} \subset W^{1,2}_\partial(\Omega)$, $(j+1)$-dimensional.

· (Ω, g) – Riem. surface w/ boundary

· REGULARITY OF THE METRIC?

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STEKLOV PROBLEM

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma \beta f & \text{on } \partial\Omega \end{cases}$$

$0 \neq \beta : \partial\Omega \to [0, \infty)$

$0 = \sigma_0(\Omega, \beta) < \sigma_1(\Omega, \beta) \leq \sigma_2(\Omega, \beta) \dots \nearrow \infty$

Weak formulation:

$$\sigma_j(\Omega, \beta) = \inf_{E_{j+1}} \sup_{f \in E_{j+1} \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, dV}{\int_{\partial\Omega} |f|^2 \beta \, dA}$$

· $E_{j+1} \subset W^{1,2}(\Omega, \beta, 1)$

$$\boxed{\begin{array}{c}\text{EIGENVALUE ASYMPTOTICS} \\ \text{Weyl's law}\end{array}}$$

What is the behaviour of

$\sigma_j(\Omega, \beta)$ as $j \to \infty$?

SPECTRUM OF DISKS

$$S(L) = \left\{ \frac{2\pi}{L}|j| : j \in \mathbb{Z} \right\} \rightsquigarrow \left\{ \begin{array}{l} \text{spectrum of disk} \\ \text{with circumference } L \end{array} \right\}$$

$$S(L_1, \ldots, L_b) = \bigcup_{k=1}^{b} S(L_k)$$

$$= \{\text{spectrum of union of disks}\}$$

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as multisets, and we reorder.

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$$= \{\text{spectrum of union of disks}\}$$

Theorem A: Let Ω be a surface with smooth boundary, $\partial\Omega = \bigcup_{k=1}^{b} \Sigma_k$, $\beta \in L\log L(\partial\Omega)$,

$$L_k = \int_{\Sigma_k} \beta \, dA .$$

Theorem A: Let Ω be a surface with smooth boundary, $\partial\Omega = \bigcup_{k=1}^{b} \Sigma_k$, $\beta \in L\log L(\partial\Omega)$,

$$L_k = \int_{\Sigma_k} \beta \, dA.$$

$$\Longrightarrow \quad \sigma_j(\Omega;\beta) = \mathcal{S}(L_1,\dots,L_b)_j\,(1+o(1))$$

Ingredient #1 : True for β smooth

(with way better error estimates)

- Rozenblum '78-79
- Edward '90

$\Big\}$ Simply connected

- Girouard, Parnovski, Polterovich, Sher '14

$\Big\}$ Any finite topology

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Proofs are pseudo-differential in nature (*)

plus a clever reduction to simply connected (**)

Ingredient #2 : Counting functions

controlled by β :

$$\#\{ j \in \mathbb{N} : \sigma_j(\Omega;\beta) \le X \} \le C_\Omega \|\beta\|_{L\log L} X$$

In various forms by

- Solomyak '95
- Rozenblum - Shargorodsky '21
- Sukochev - Zanin '16
- Ponge '20
- . . . . . .

Ingredient #3 : Approximation of singular weights with smooth weights

- Ingredient #2 implies that Weyl law for smooth approximation $\Rightarrow$ Weyl law for the limit via the Weyl - Ky Fan inequality.

- Inspired by ideas of

$$\left.\begin{array}{ll} \text{Birman - Solomyak} & (\text{'80s}) \\ \text{Suslina} & (\text{'98}) \\ \text{Agranovich} & (\text{'06}) \end{array}\right\} \quad \beta \in L^p,\ p > 1.$$

Theorem B: Let Ω be a surface with Lipschitz boundary $\partial\Omega = \bigcup_{k=1}^{b} \Sigma_k$ with $L_k = \text{Length}(\Sigma_k)$

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$$\Longrightarrow \sigma_j(\Omega) = S(L_1, \dots, L_b)_j \left(1 + o(1)\right)$$

Ingredient #1 : Uniformisation.

$$\varphi : \Omega_e \longrightarrow \Omega \qquad \text{conformal}$$

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$$\varphi : \Omega_{\mathbb{C}} \longrightarrow \Omega \qquad \text{conformal}$$

Isospectrality

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma f & \text{on } \partial\Omega \end{cases} \iff \begin{cases} \Delta f = 0 & \text{in } \Omega_{\mathbb{C}} \\ \partial_\nu f = \sigma |\varphi'| f & \text{on } \partial\Omega_{\mathbb{C}} \end{cases}$$

Ingredient #1 : Uniformisation.

$$\varphi : \Omega_{\epsilon} \longrightarrow \Omega \qquad \text{conformal}$$

⚠ Isospectrality

Need φ to induce isomorphism of Sobolev spaces

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma f & \text{on } \partial\Omega \end{cases} \quad \longleftrightarrow \quad \begin{cases} \Delta f = 0 & \text{in } \Omega_\epsilon \\ \partial_\nu f = \sigma |\varphi'| f & \text{on } \partial\Omega_\epsilon \end{cases}$$

Ingredient #1 : Uniformisation.

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⚠ Isospectrality

OK if $|\varphi'| \in L \log L$

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma f & \text{on } \partial\Omega \end{cases} \quad \longleftrightarrow \quad \begin{cases} \Delta f = 0 & \text{in } \Omega_{e} \\ \partial_\nu f = \sigma |\varphi'| f & \text{on } \partial\Omega_{e} \end{cases}$$

Ingredient #2

Boundary regularity of conformal maps.

META-THEOREM

Boundary regularity result for compact, simply connected surface	$\Longrightarrow$	Boundary regularity result for <u>any</u> compact surface with boundary

Ingredient #2 — Boundary regularity of conformal maps.

If $\partial\Omega$ is Lipschitz, $\varphi : \Omega_c \to \Omega$ conformal

$\Longrightarrow$ $|\varphi'| \in L^p(\partial\Omega)$ for some $p > 1$.

- Porism of Bourgeois - Baratchard - Leblond '16 + meta theorem
- $L^1 \supsetneq L \log L \supsetneq L^p$, $p > 1$.

Ingredient #1 : Uniformisation.

$$\varphi : \Omega_{\mathbb{C}} \longrightarrow \Omega \qquad \text{conformal}$$

✓ Isospectrality ?

Yes! $|\varphi'| \in L^p$, $p > 1$.

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \partial_\nu f = \sigma f & \text{on } \partial\Omega \end{cases} \quad \longleftrightarrow \quad \begin{cases} \Delta f = 0 & \text{in } \Omega_{\mathbb{C}} \\ \partial_\nu f = \sigma |\varphi'| f & \text{on } \partial\Omega_{\mathbb{C}} \end{cases}$$

Final Step

Apply Weyl law for the problem on Ω_{ℓ} with weight $\beta = |\varphi'|$.

- $L_k = \text{Length}(\Sigma_k) = \displaystyle\int_{\varphi^{-1}(\Sigma_k)} |\varphi'| \, dA$

$\boxtimes$

Open questions

- What about higher dimensions
  - In development: ΨDO calculus with rough metrics for the Laplacian (joint w/ Alix Deleporte)
  - Hope: Also applies to DtN map with Lipschitz boundary

Open questions

- What about $C^{k,\alpha}$, $k+\alpha > 1$?

$$\sigma_j(\Omega) = S(L_1, \dots, L_b)_j \left(1 + o(j^{-s(k,\alpha)})\right)$$

- Shamma '71, $-C^4$, $o(j^{-1})$
- Girouard - Karpukhin '22 $- C^{2,\alpha}$
  Levitin - Polterovich $O(j^{-1})$
- Causley '22 $-C^{k,\alpha}$, $k \geq 5$

Work in progress with L. Parnovski (all K, α), upper and lower bounds.