Around a black hole

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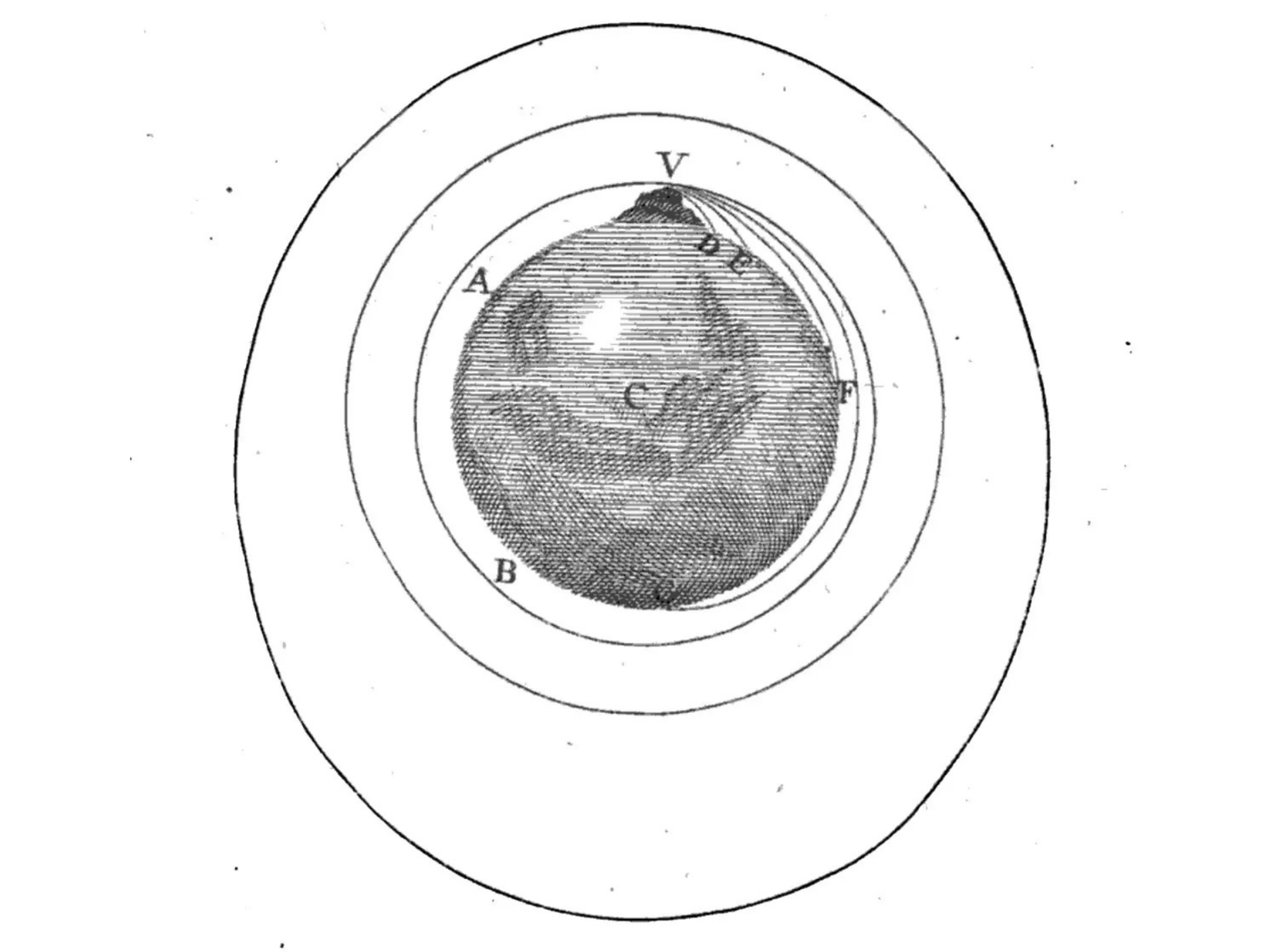
Are black holes the vacuum cleaners of the universe?

What is a black hole?

Dark stars

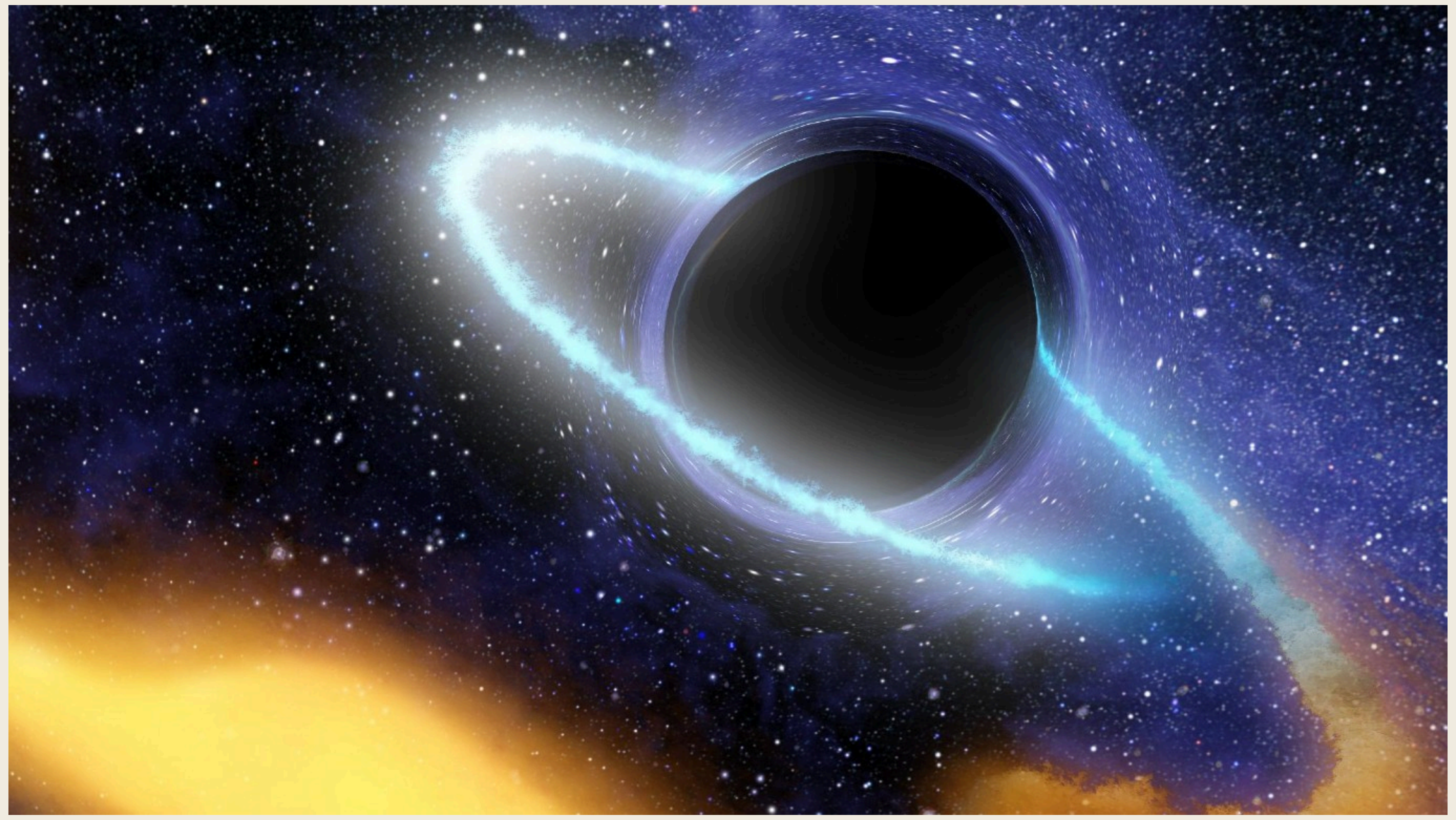


Minimum speed to escape a planet: escape speed

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

What happens if the escape speed is the speed of light?

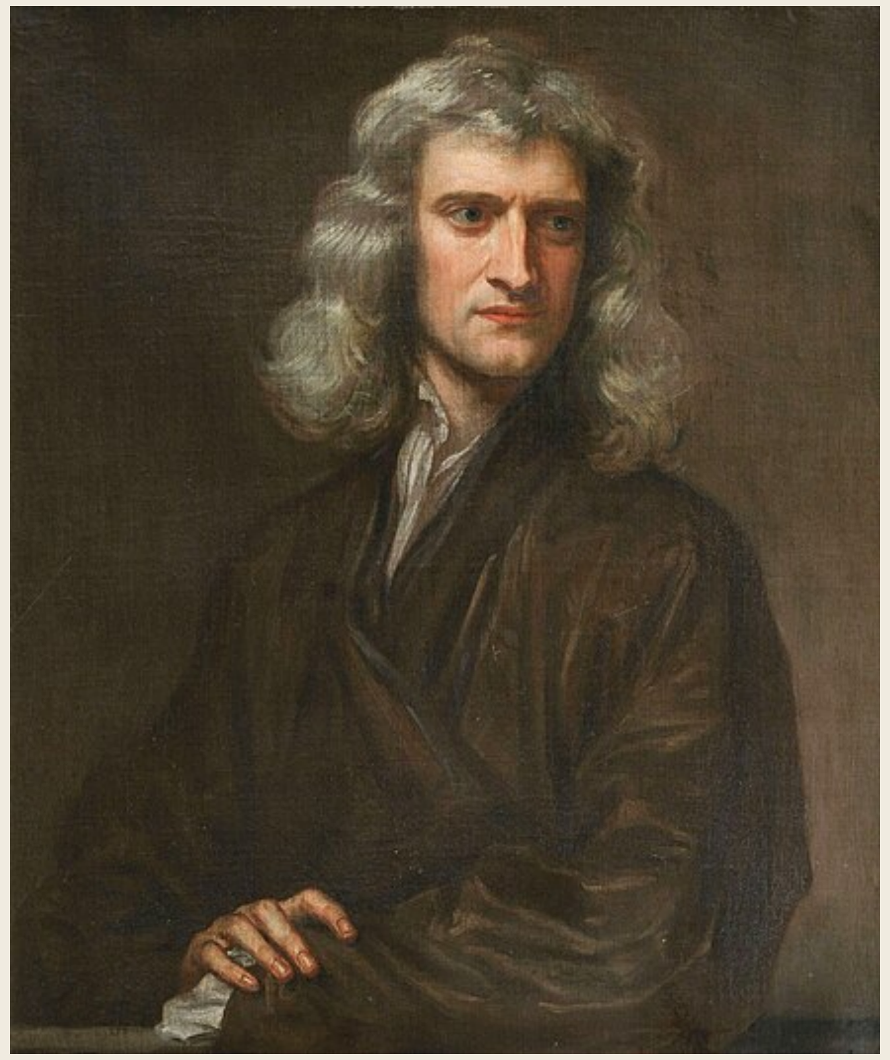
No light can escape! Dark star



$$v_{esc} = c$$

$$R = \frac{2GM}{c^2}$$

Special relativity



Newton: Time is absolute

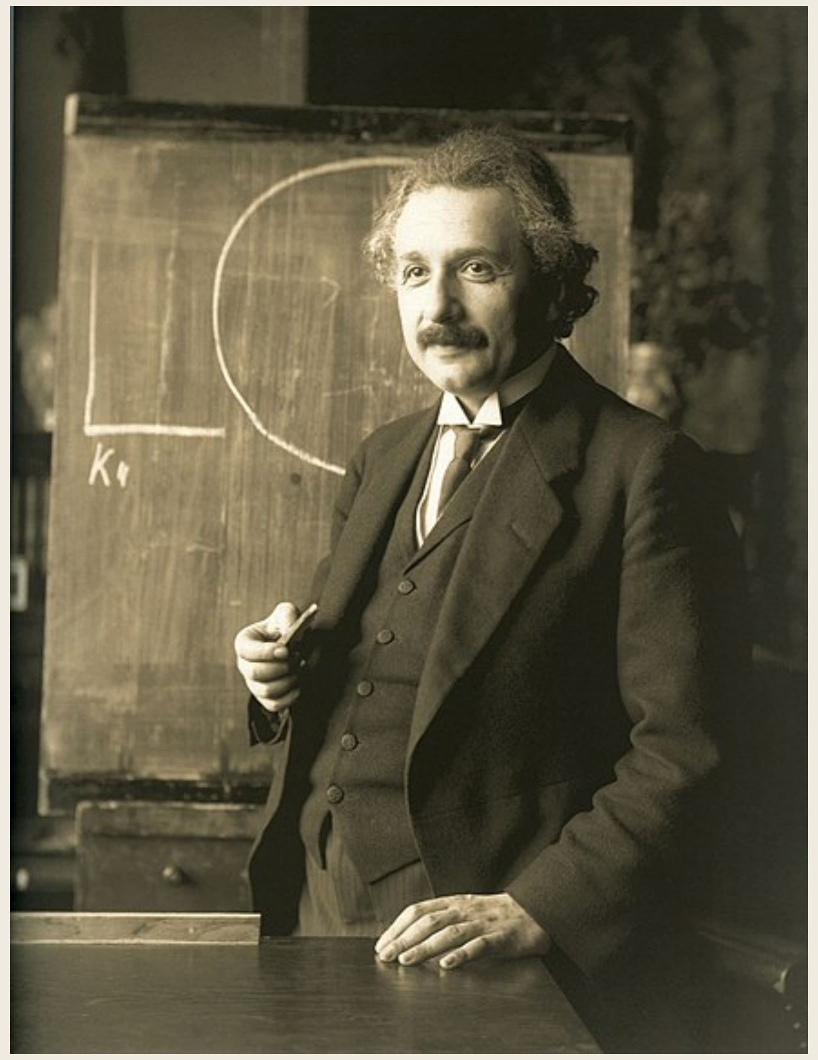
Every observer measures the same time

Einstein: Time is relative

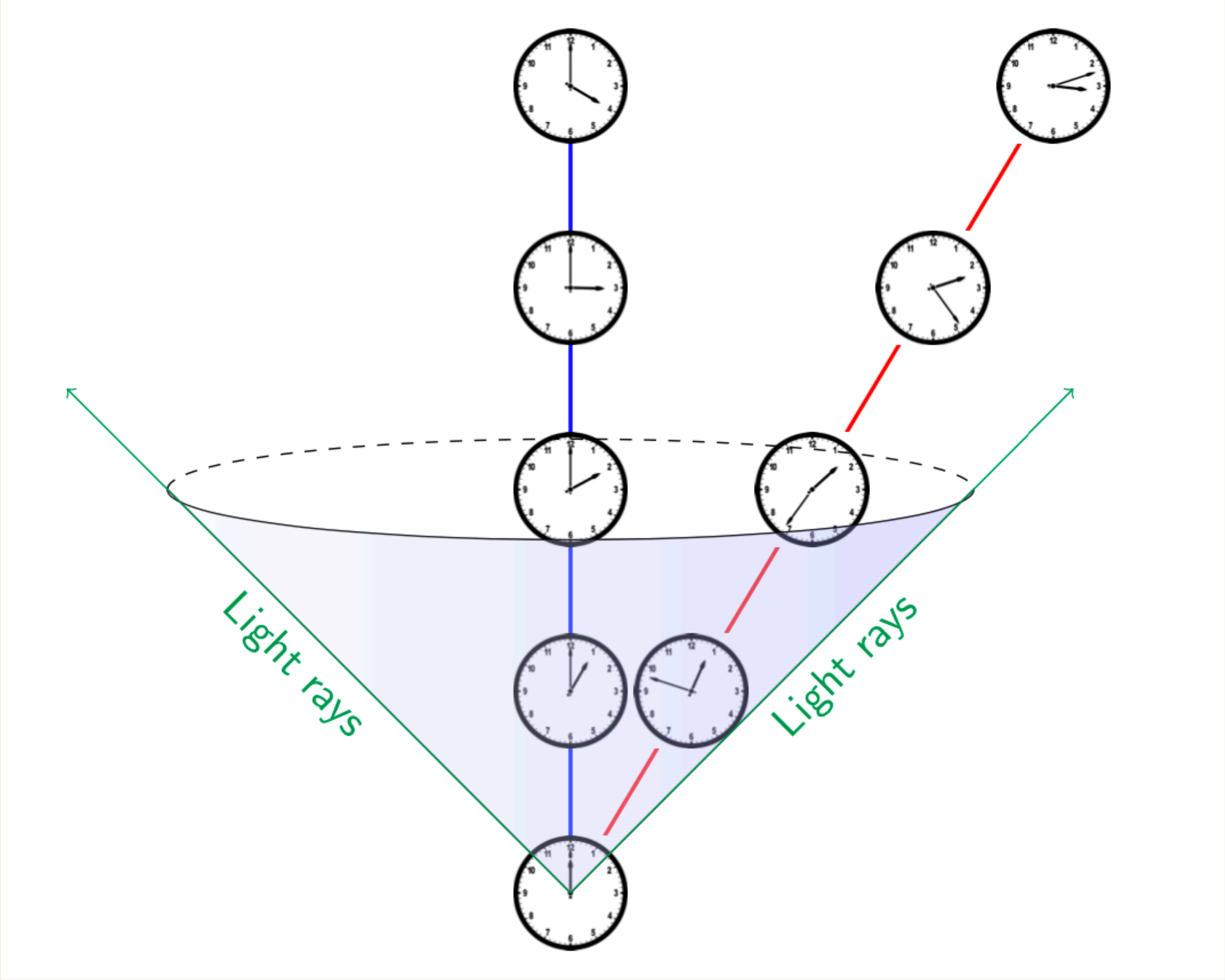
Every observer measures a different time

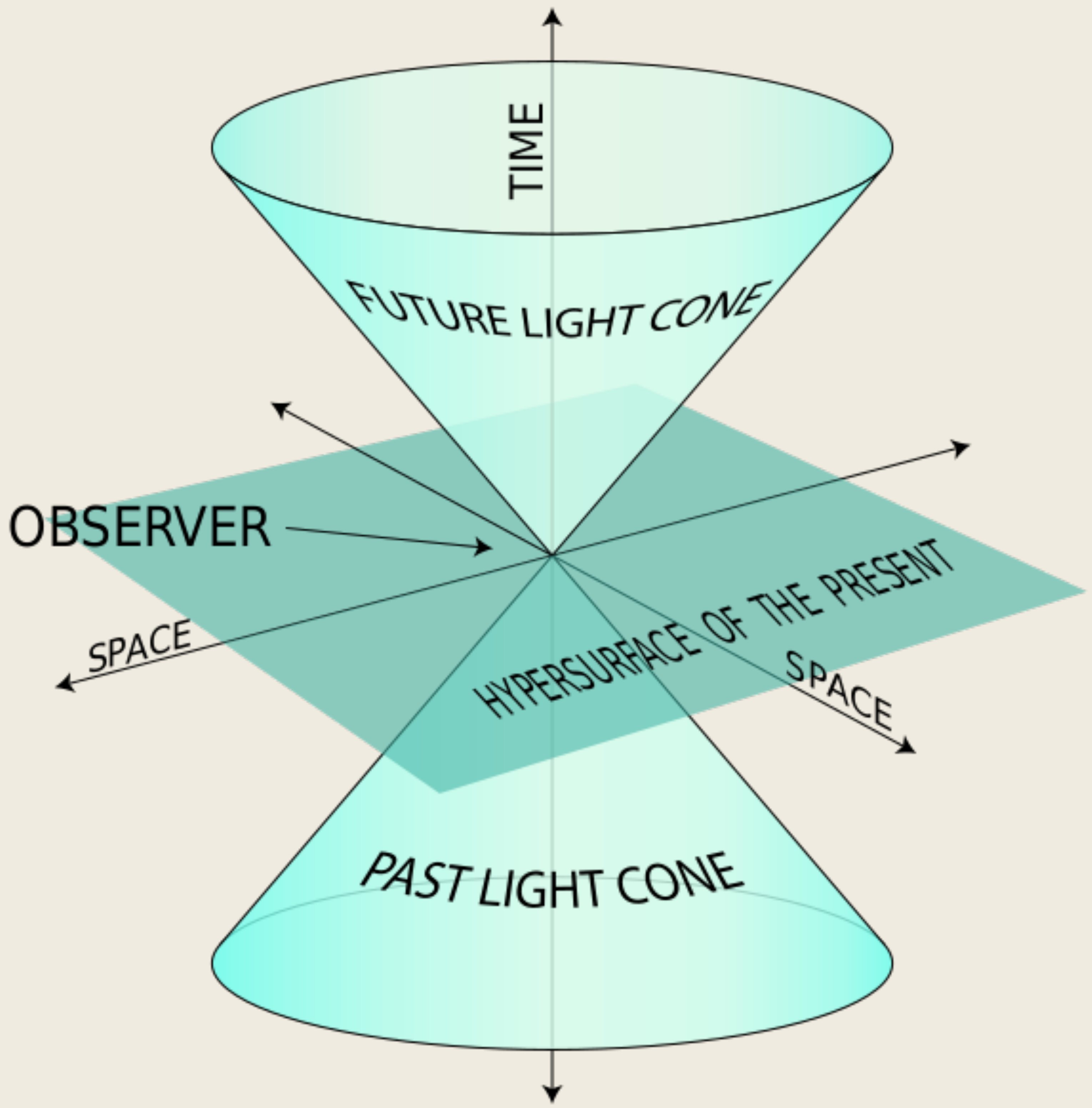
The speed of light in the vacuum

is the same as measured by every observer



Spacetime

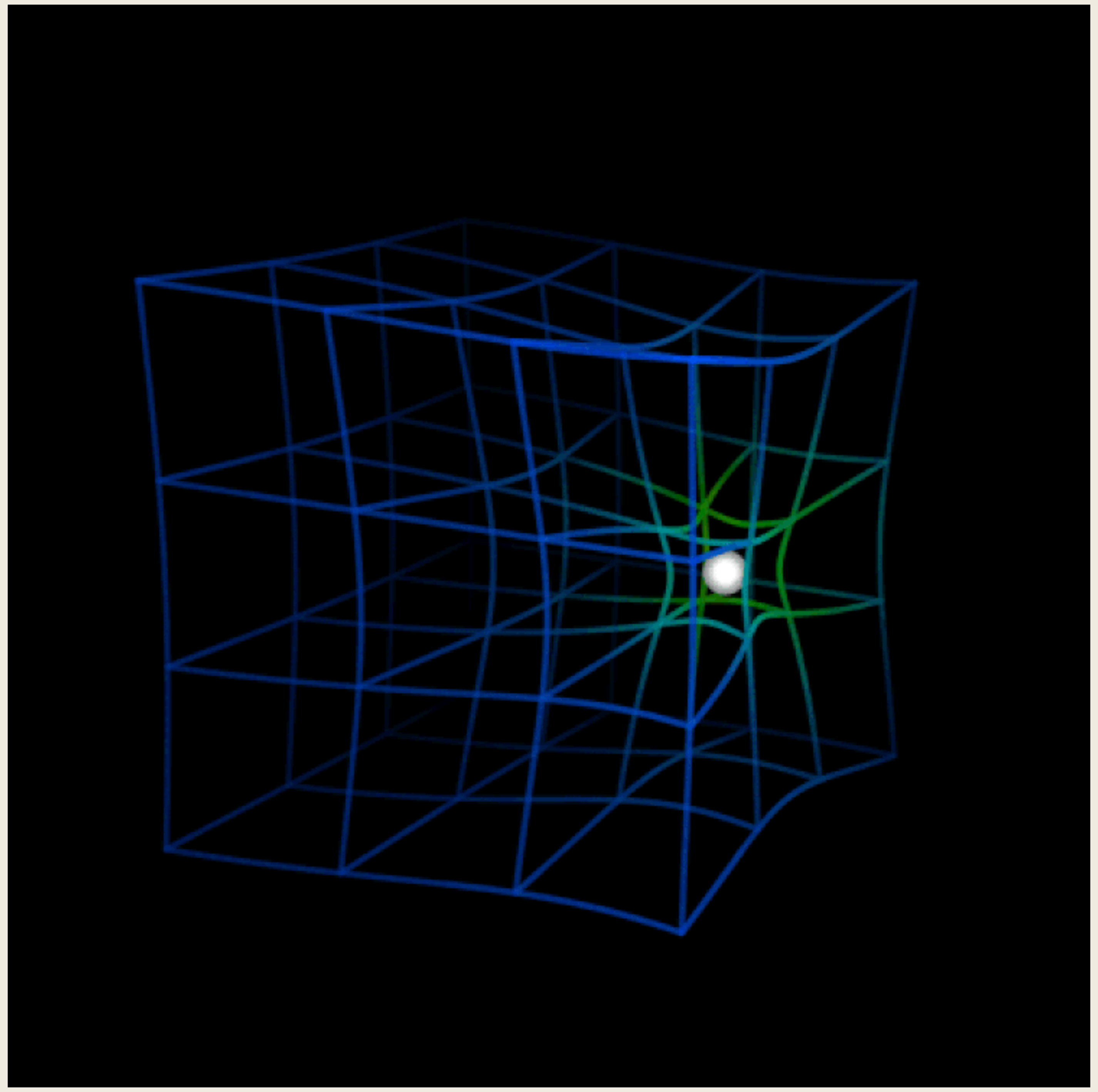

Light rays
Light rays

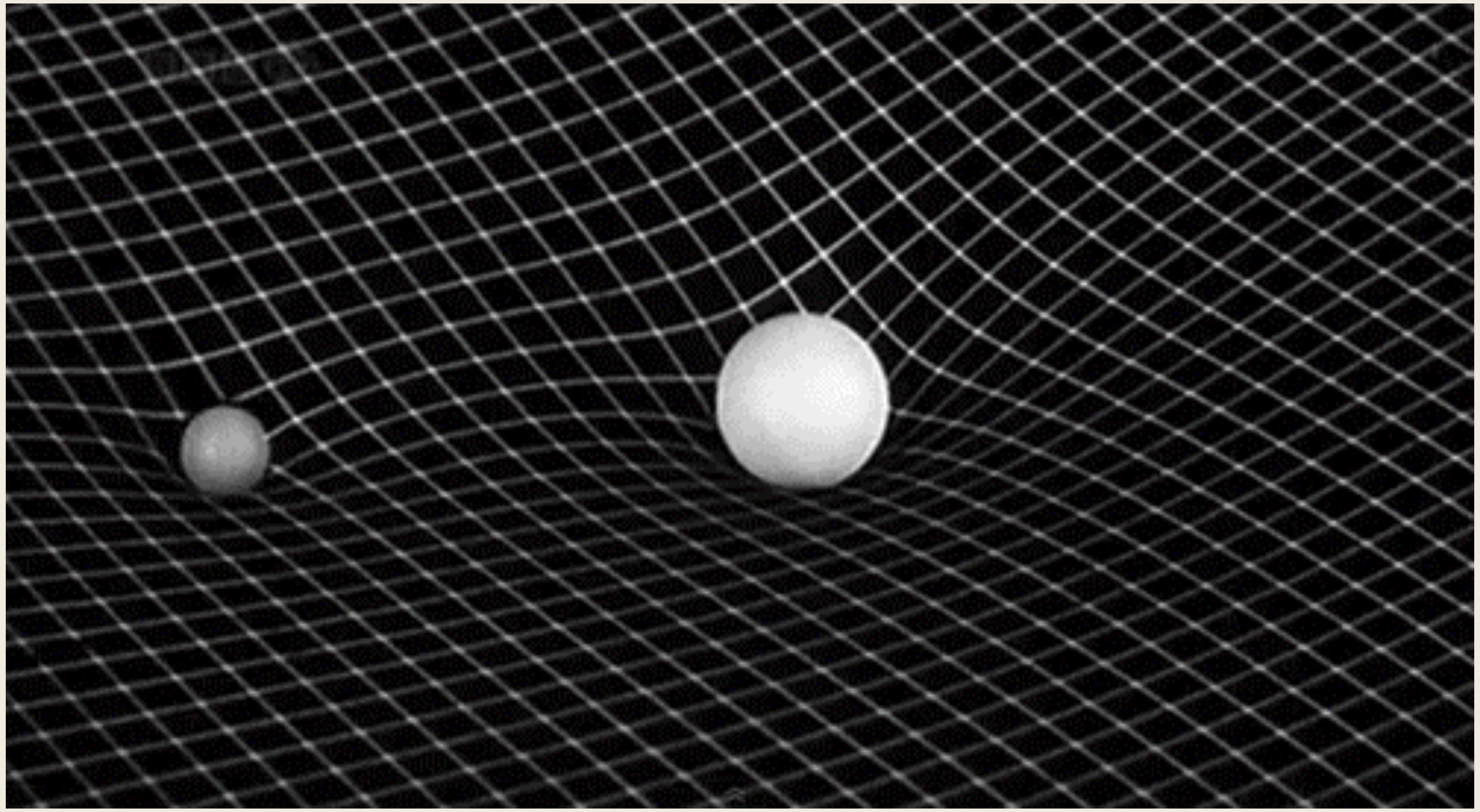

TIME
FUTURE LIGHT CONE
OBSERVER
SPACE
HYPERSURFACE OF THE PRESENT
SPACE
PAST LIGHT CONE

Theory of general relativity

“Matter tells spacetime how to curve

and curved spacetime tells matter how to move”

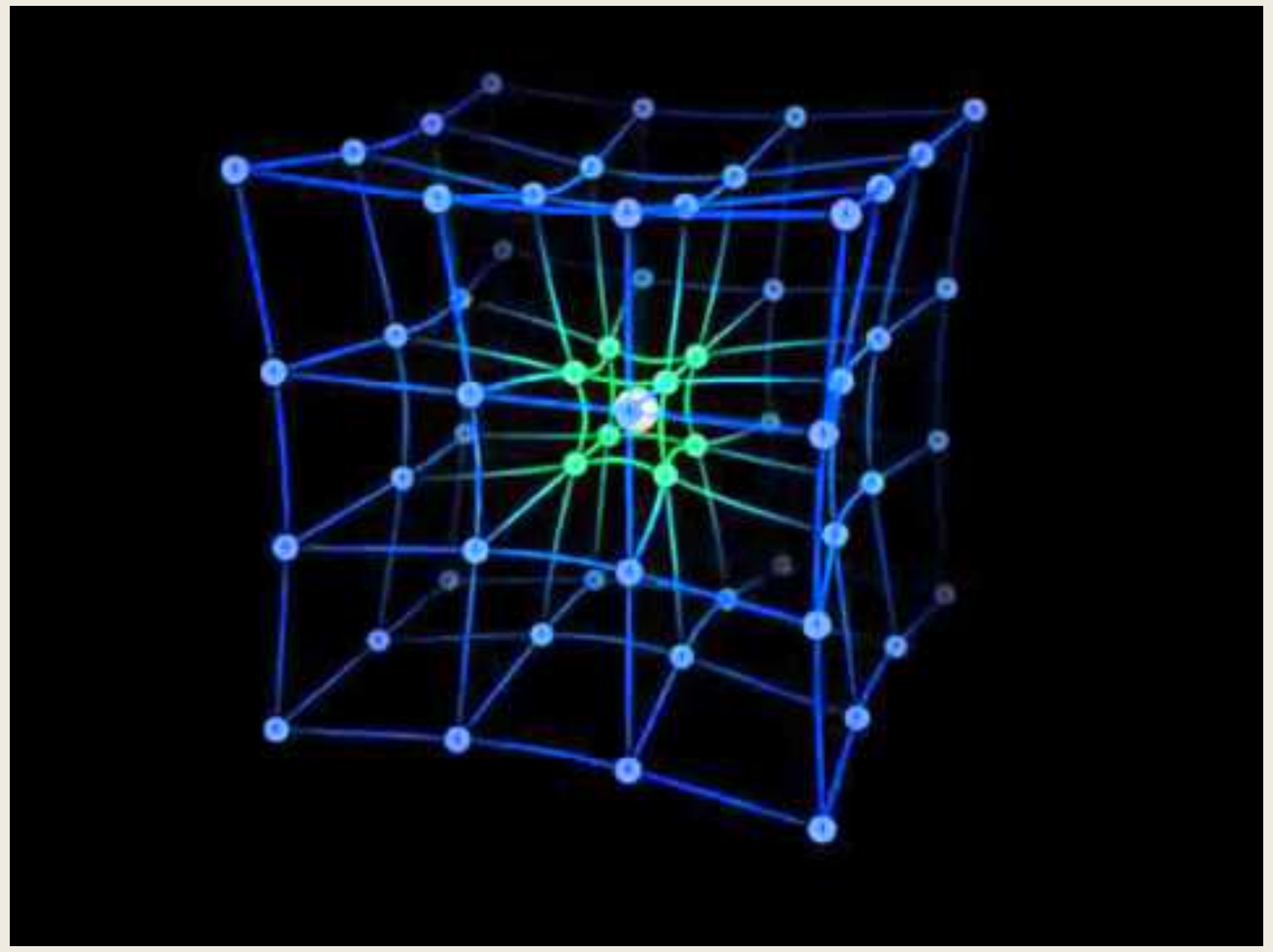




Einstein's equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Spacetime curvature

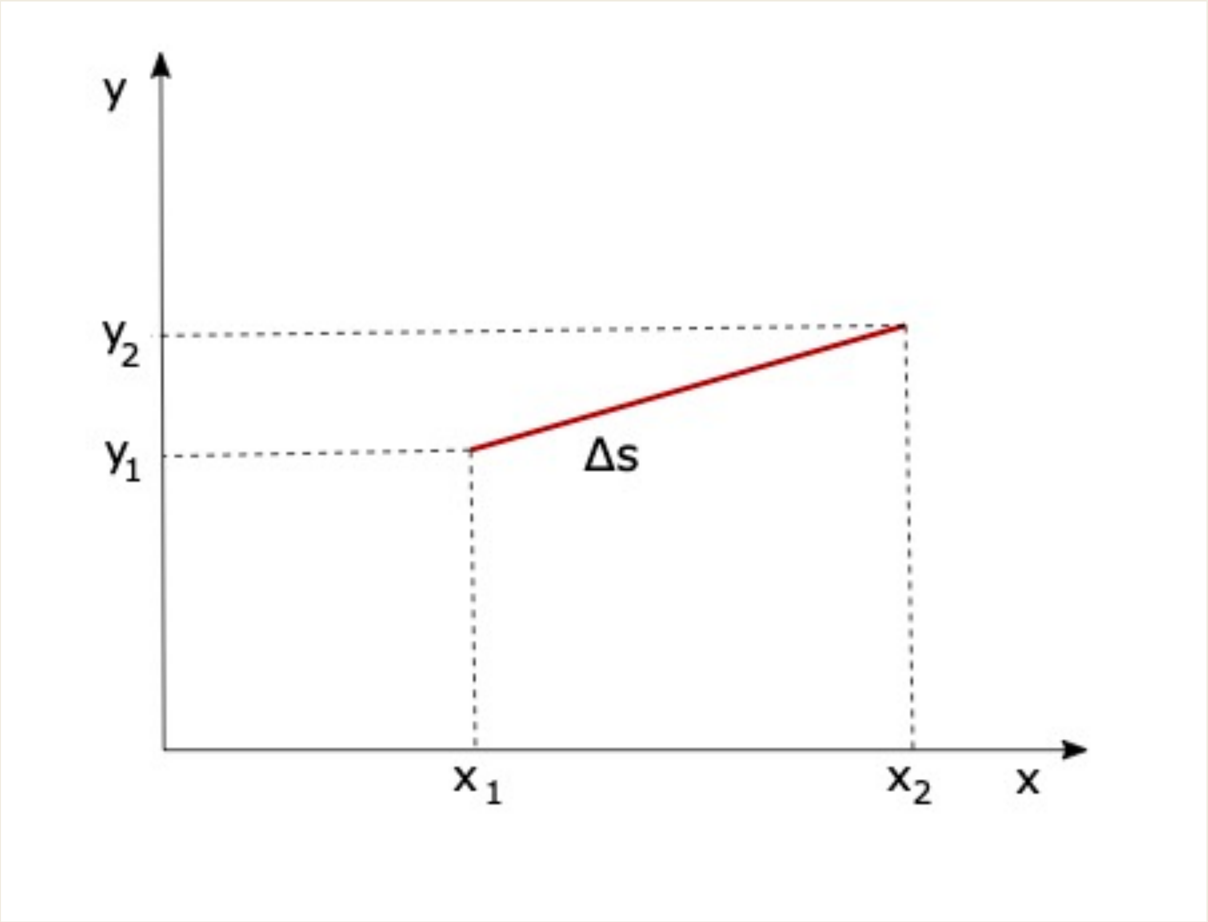


Matter, energy and pressure

Spacetime distance

How do we measure distances in spacetime?

Two-dimensional Space



$$(\Delta s)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (\Delta x)^2 + (\Delta y)^2$$

Four-dimensional flat spacetime

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Infinitesimal distance: Minkowski metric

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

In general curved spacetime

$$(ds)^2 = f_0(t,x,y,z)(dt)^2 + f_1(t,x,y,z)dtdx\ldots$$

Very difficult! What matter does it correspond to? Unknown!

Schwarzschild metric

Assumptions:

1. Spherical symmetry
2. Vacuum solution of the Einstein equation

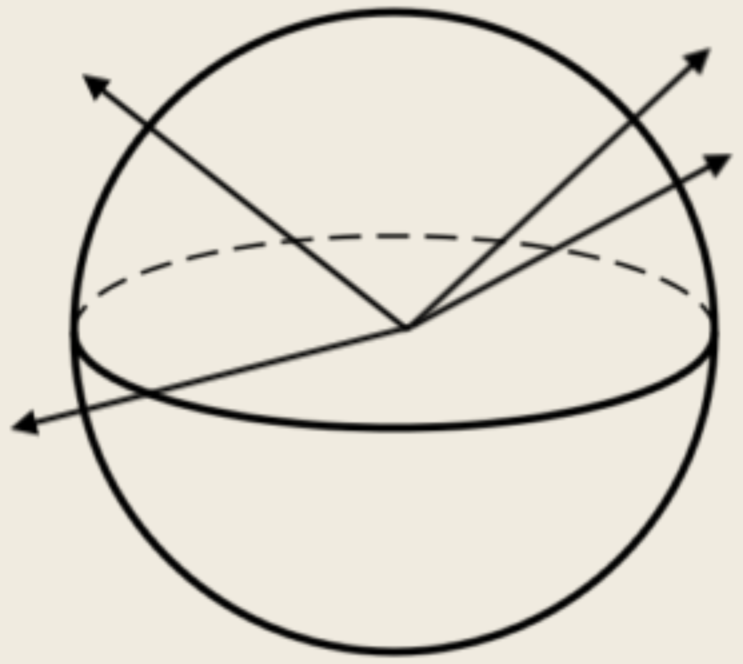
General spherically symmetric metric

$$ds^2 = -\alpha(r)dt^2 + \beta(r)dr^2 + \gamma(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

1. Minkowski metric in spherical coordinates:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$ds^2 = -c^2dt^2 + dr^2 + \boxed{r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$



2. Vacuum solution

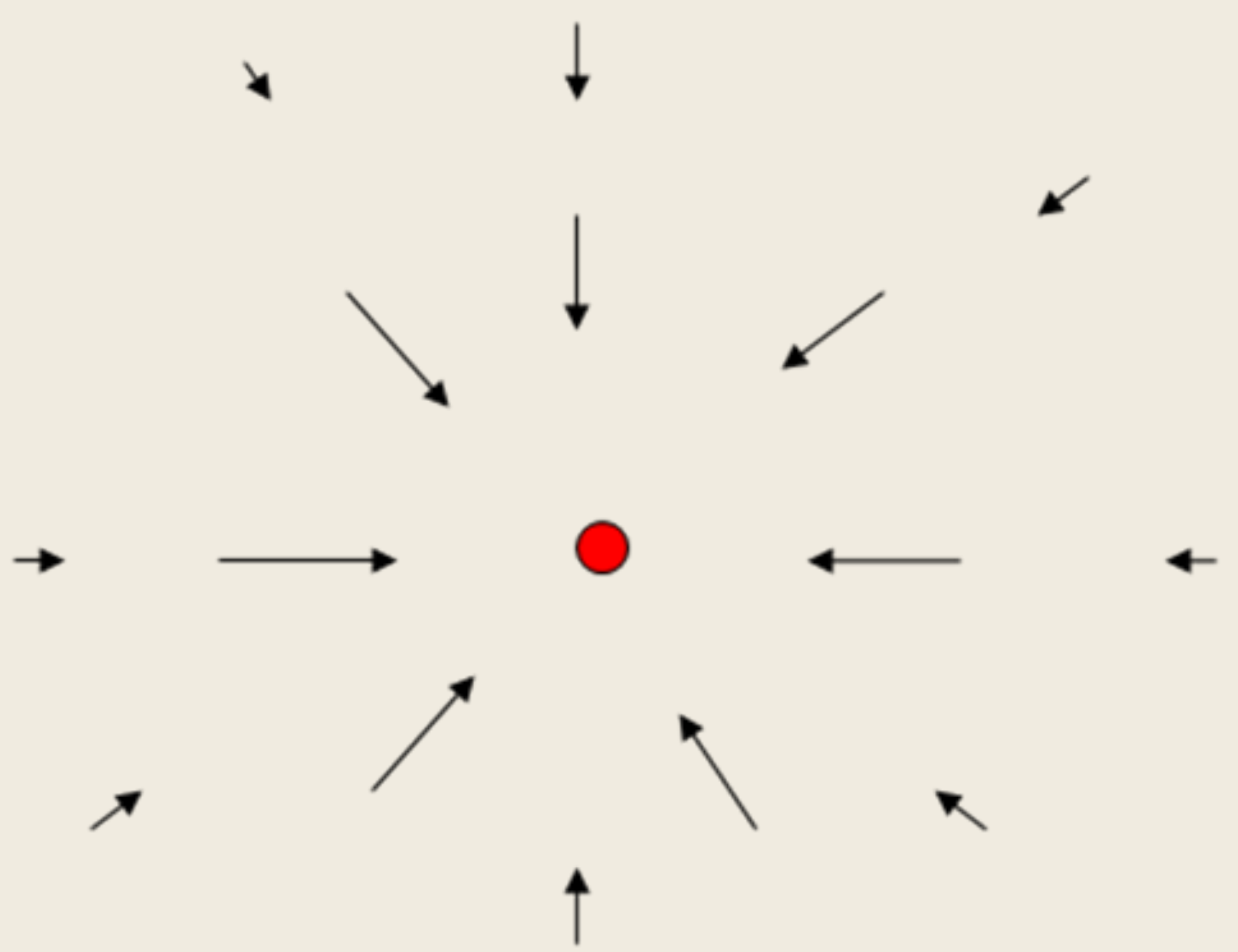
$T_{\mu\nu} = 0 \Rightarrow G_{\mu\nu} = 0$ No matter!

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

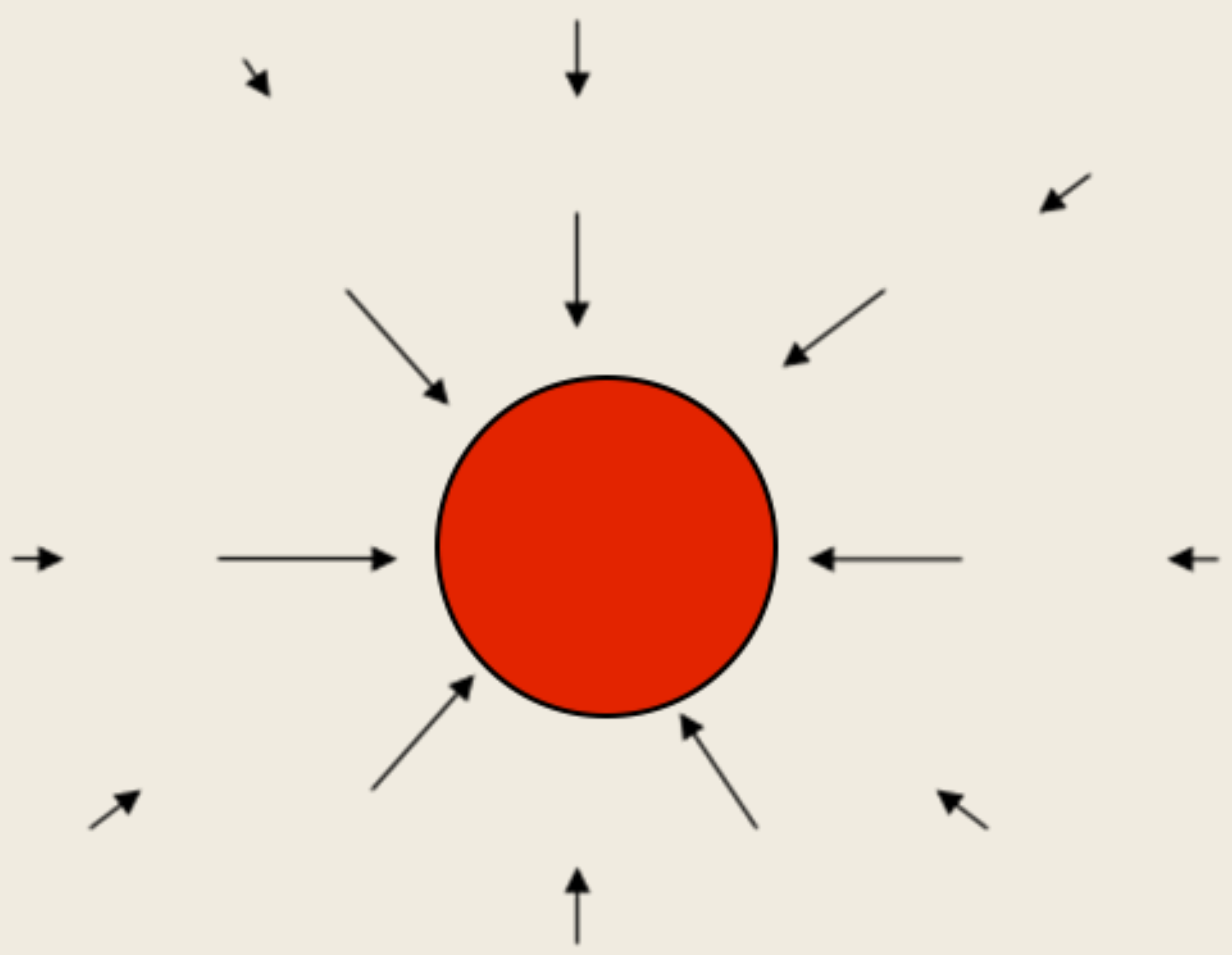
What is R_s ?

Schwarzschild metric

What kind of physical system is spherically symmetric and doesn't include any matter?



Point mass!



Spacetime outside a star

The limit of weak gravity should give us back Newtonian gravity $R_s = \frac{2GM}{c^2}$

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

Singularities

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

Metric becomes singular at $r = 0$ and $r = \dfrac{2GM}{c^2}$

General relativity: If we change coordinates the physics should be the same

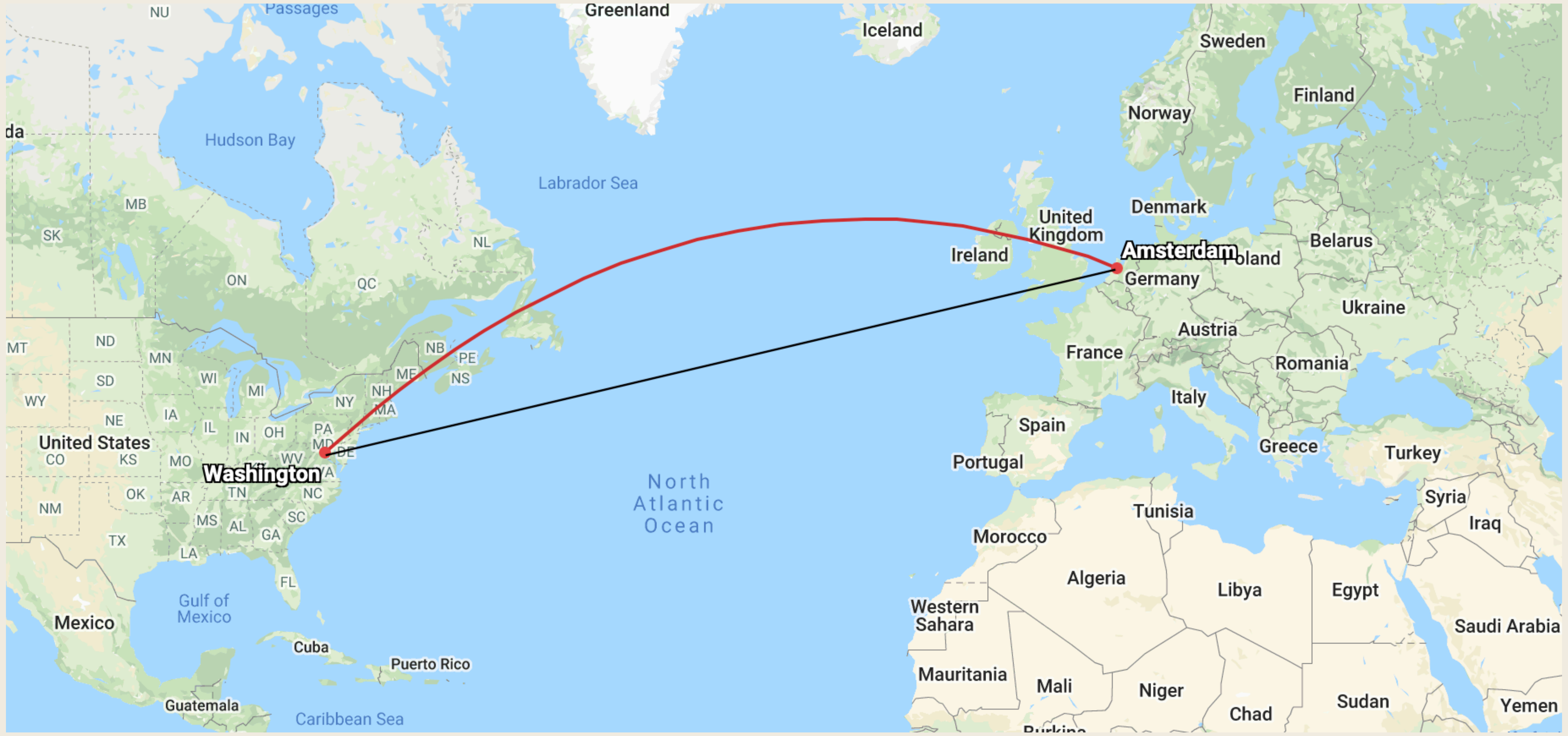
$r = \dfrac{2GM}{c^2}$ If we change coordinates, the singularity disappears: Nothing bad happens!

$r = 0$ If we change coordinates, the singularity remains: Something bad happens!

Spacetime curvature diverges: Physics?

Geodesics

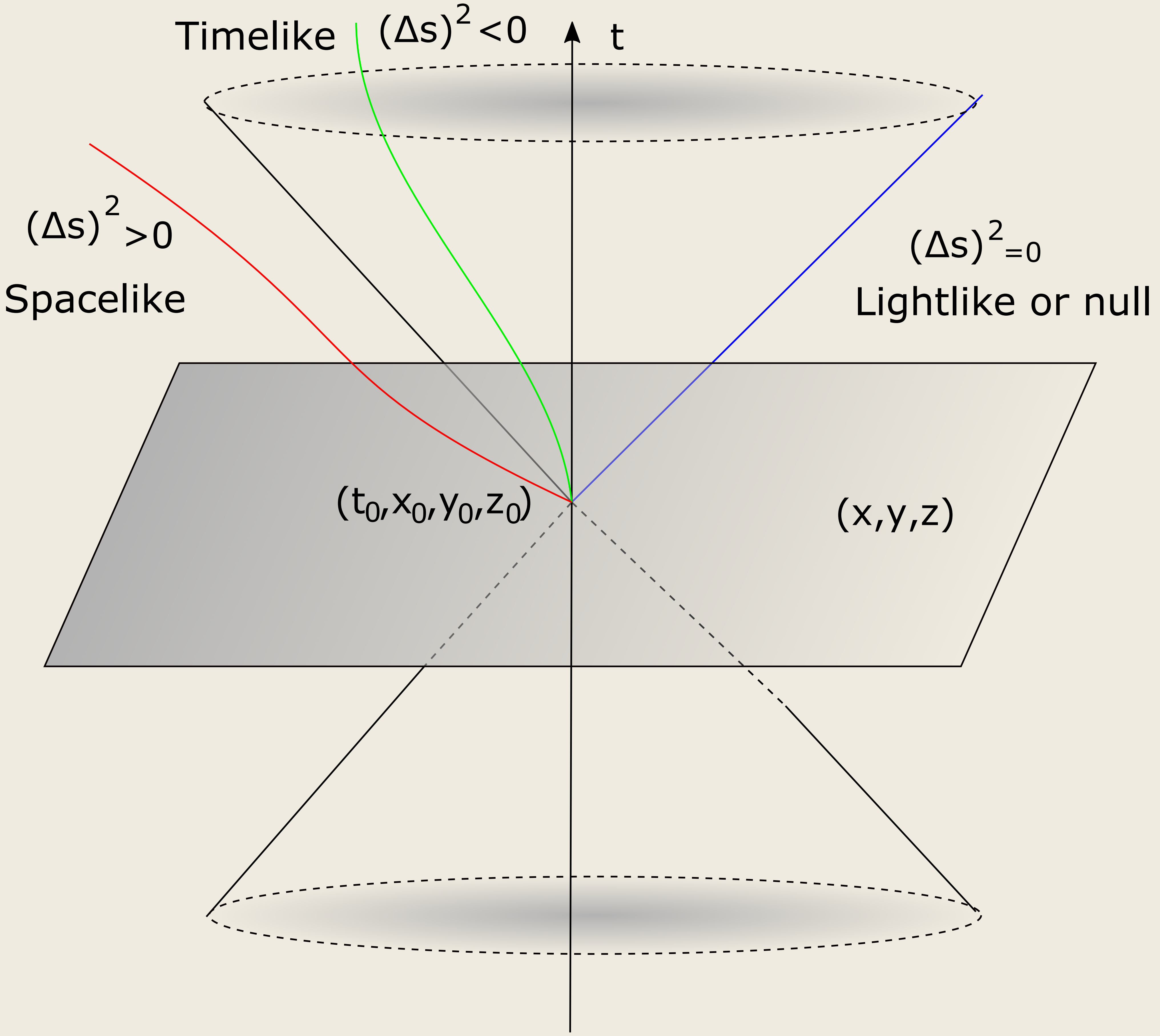
Geodesics are the shortest paths on space.



Geodesics

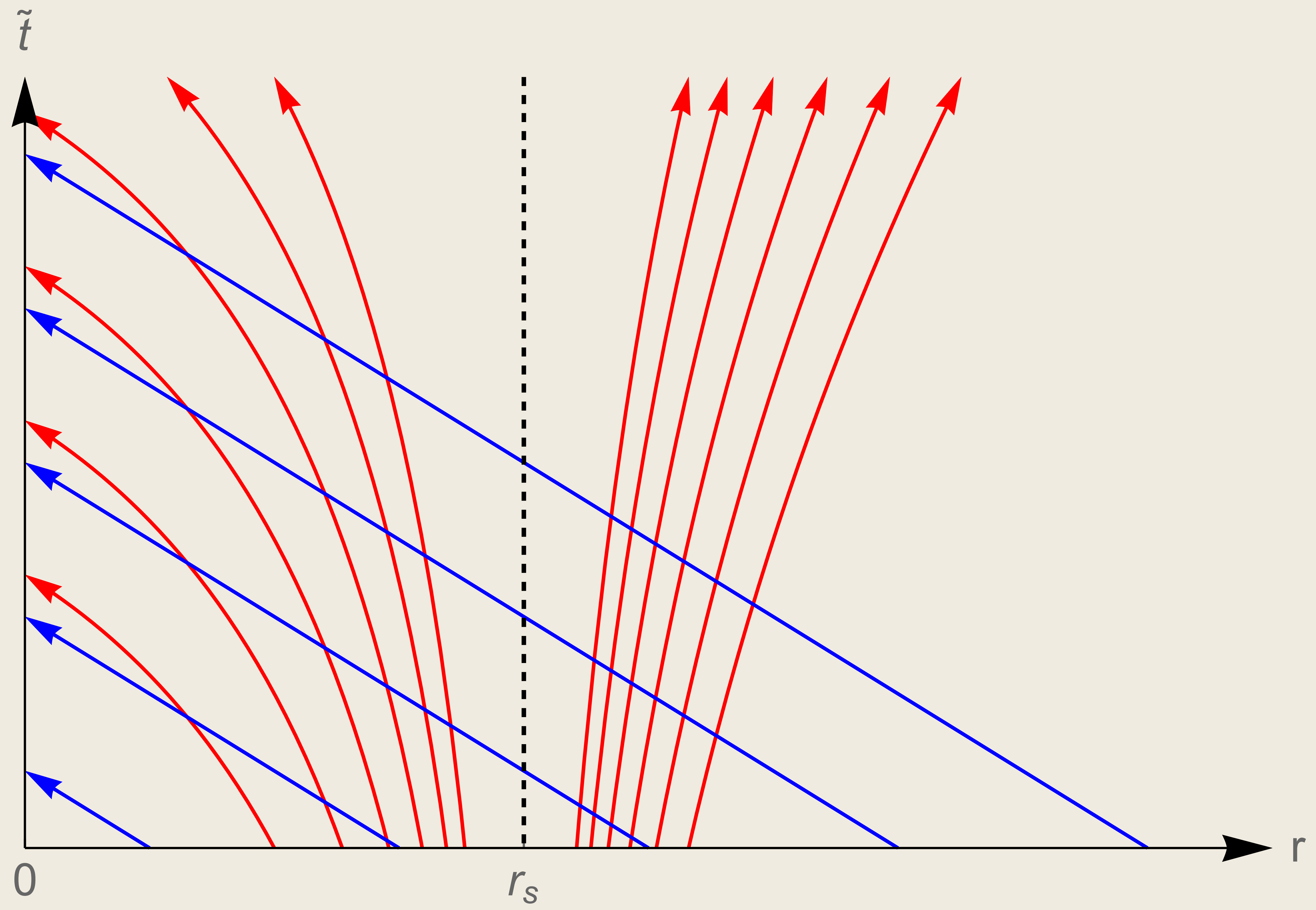
What about spacetime? (Timelike) geodesics maximize the time measured by the observer

$$(\Delta\tau)^2 = -(\Delta s)^2$$



Geodesics in Schwarzschild

Paths of light: we use coordinates that cover the whole spacetime



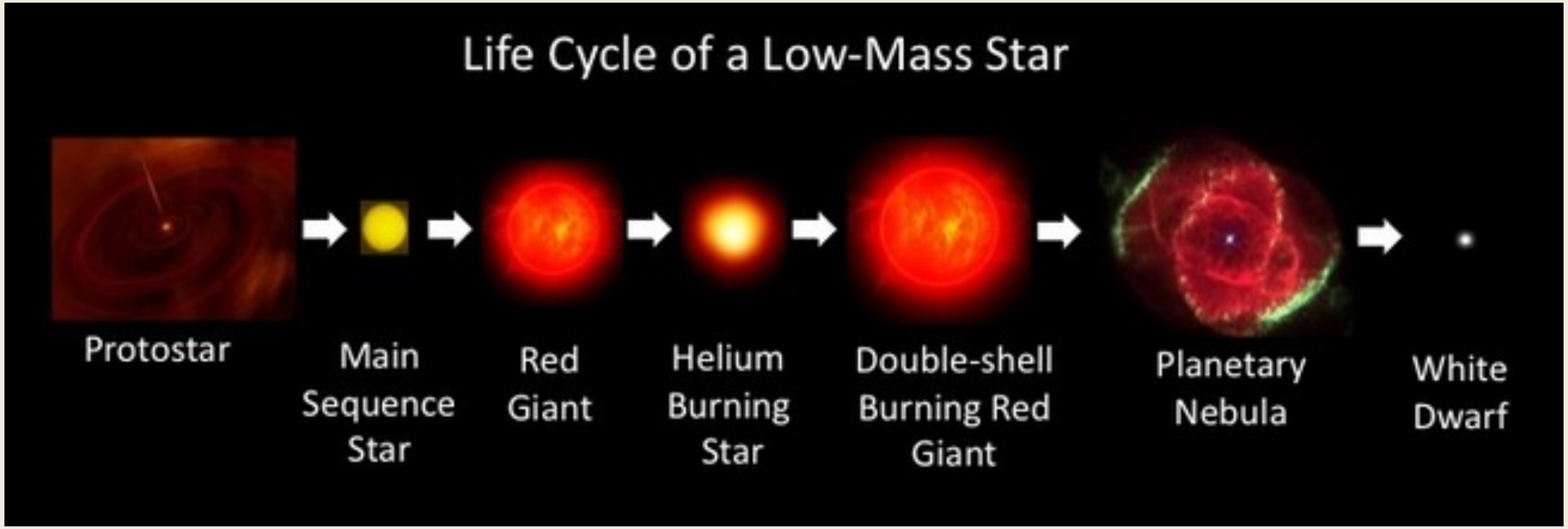
Light cannot escape from inside the Schwarzschild radius!

Is the Schwarzschild spacetime real?

Two concerns:

1. Maybe it just describes the outside of stars: There are no objects that collapse smaller than R_s
2. Could we have "dark stars"? Solid objects with radius smaller than R_s

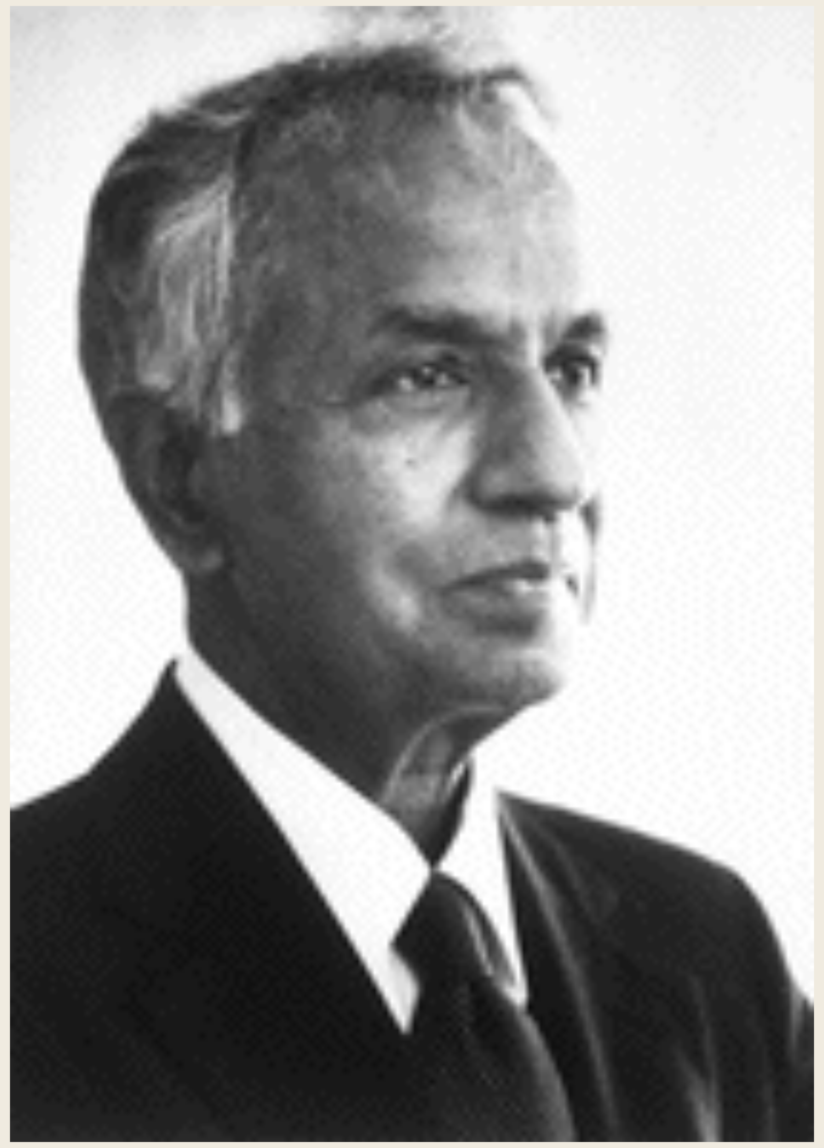
1. Collapse of stars



White dwarf: a compact object that the pressure of electrons is keeping it stable

Is the Schwarzschild spacetime real?

Can that happen for any star mass?



Chandrasekhar: No!

If the star has a mass over 3 solar masses nothing can stop its collapse

2. Could we have solid objects with radius smaller than R_s?

No! Anything inside R_s moves towards the singularity

Trajectories around a black hole

λ

Geodesics parametrized by λ

$$E = \left(1 - \frac{2MG}{c^2 r}\right)\frac{dt}{d\lambda}$$

Energy (of light or particle)

$$L = r^2 \frac{d\phi}{d\lambda}$$

Angular momentum

Both conserved: these quantities do not change

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 + V(r) = E^2}$$

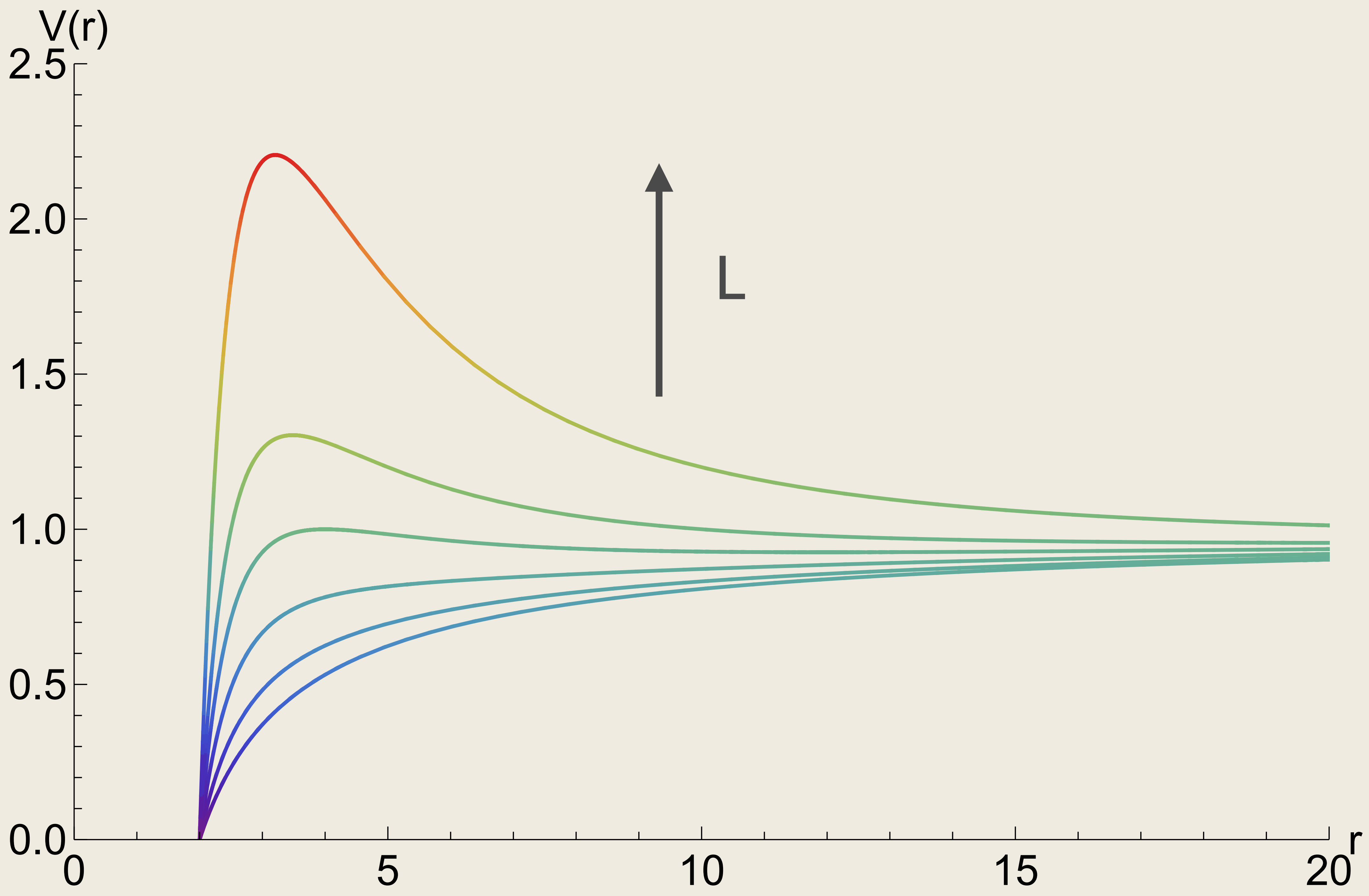
$$V(r) = 1 - \frac{2GM}{rc^2} + \frac{L^2}{r^2} - \frac{2GML^2}{c^2 r^3}$$

Paths of particles (timelike)

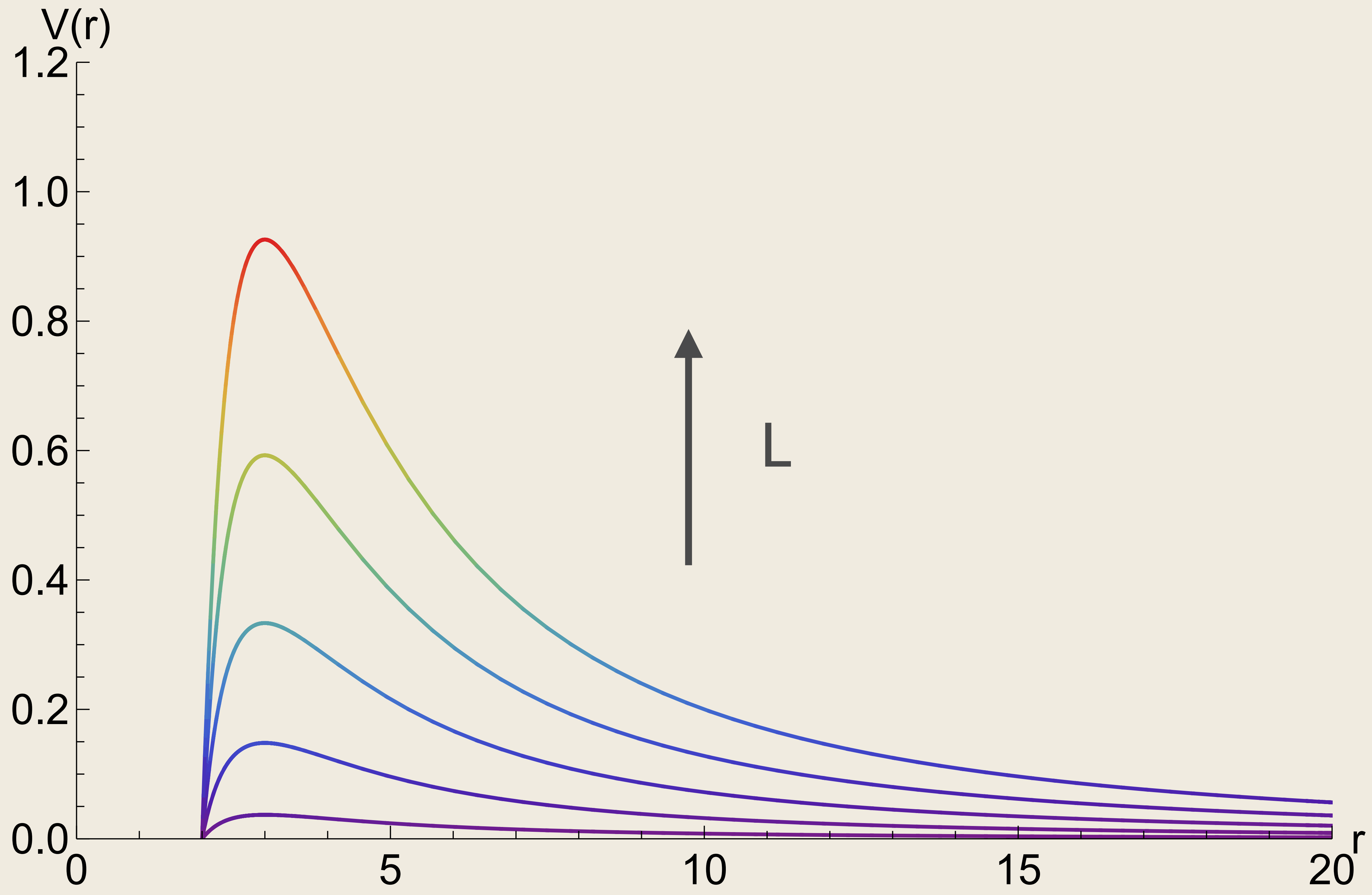
$$V(r) = \frac{L^2}{r^2} - \frac{2GML^2}{c^2 r^3}$$

Paths of light

Trajectories around a black hole



Paths of particles



Paths of light

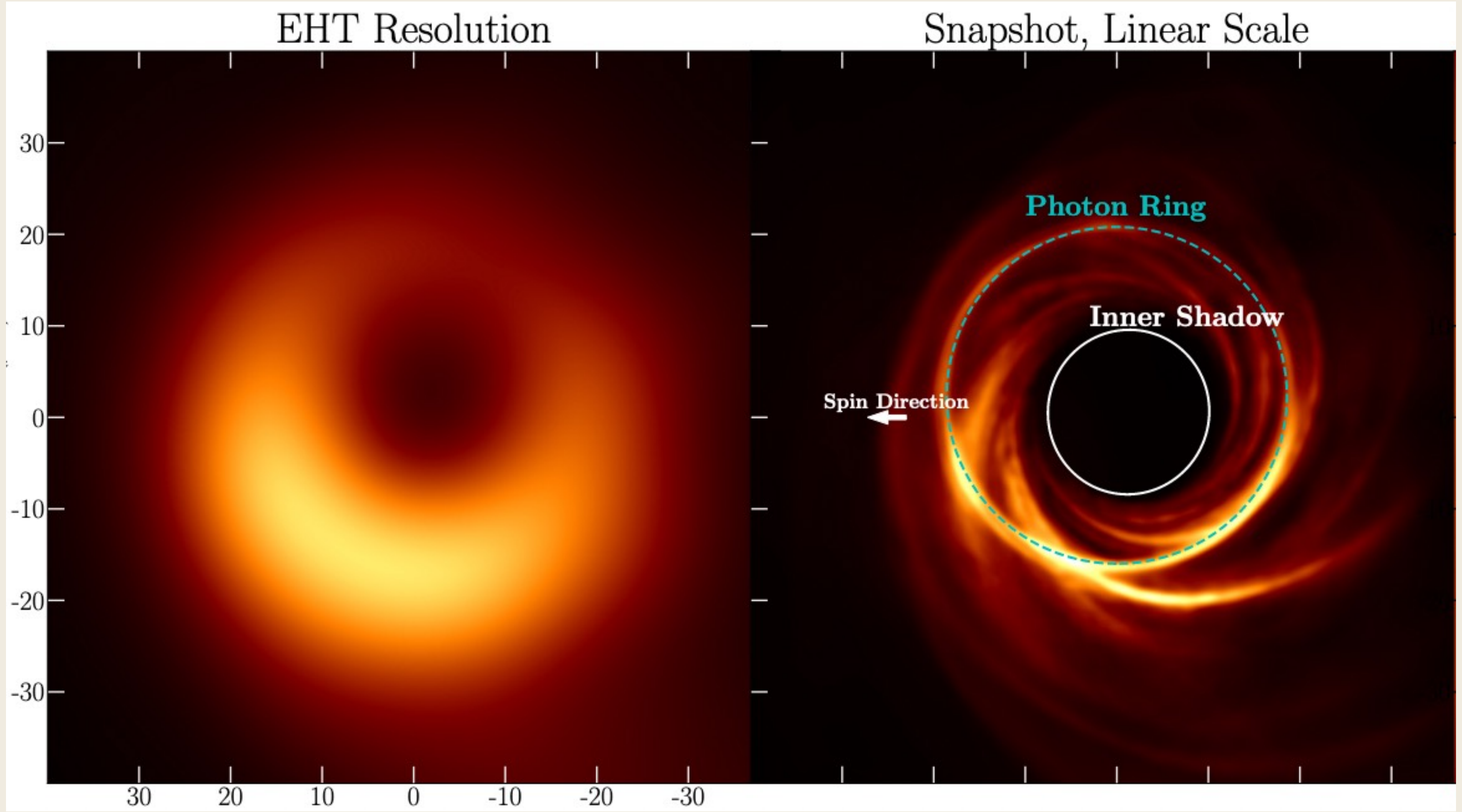
Circular orbits

Circular orbits (r is constant): $V'(r) = 0$

Paths of light:

$$V'(r) = 0 \Rightarrow r_c = \frac{3GM}{c^2}$$

Maximum: unstable orbit



Paths of particles:

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2/c^2}}{2GM/c^2}$$

Maximum and minimum: both stable and unstable orbits

Summary

- In general relativity black holes are described by a solution to the Einstein equation
- They theoretically exist as a result of the collapse of very massive stars (also experimental evidence!)
- Nothing, not even light can escape from the black hole horizon
- Outside the black hole we can have stable orbits of massive objects: planets!
- Outside the black hole we have an (unstable) orbit of light: photon ring