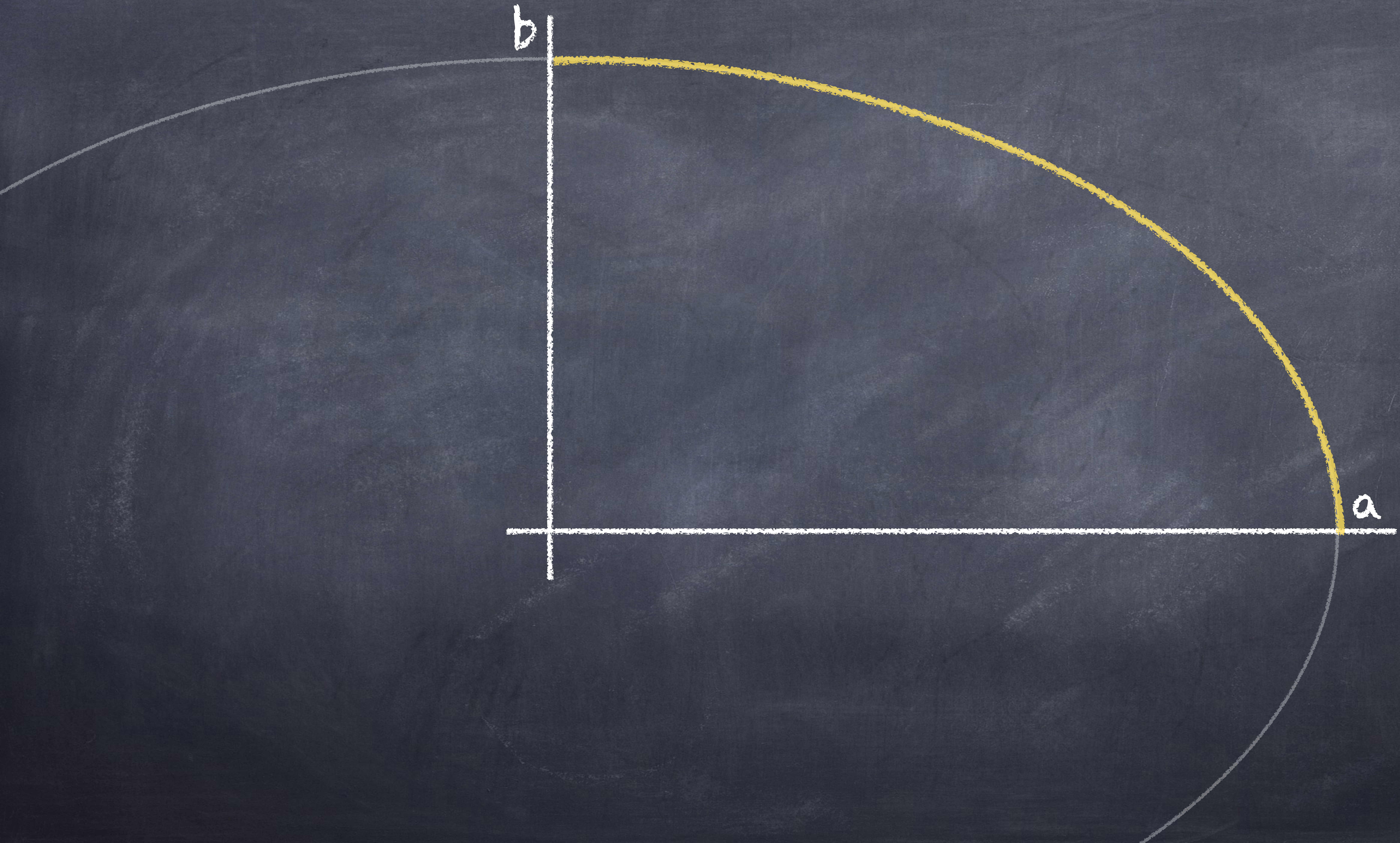
b
a

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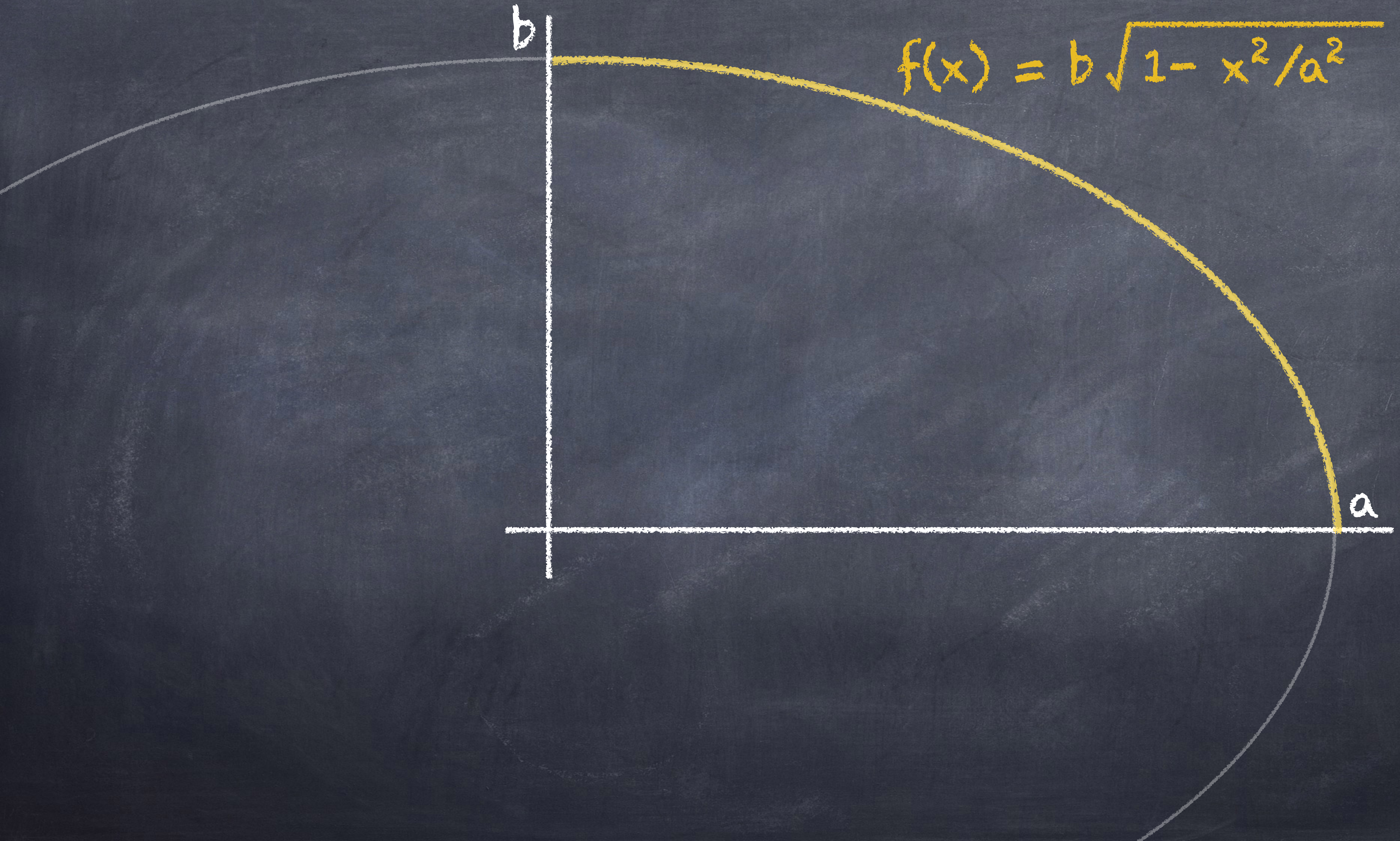

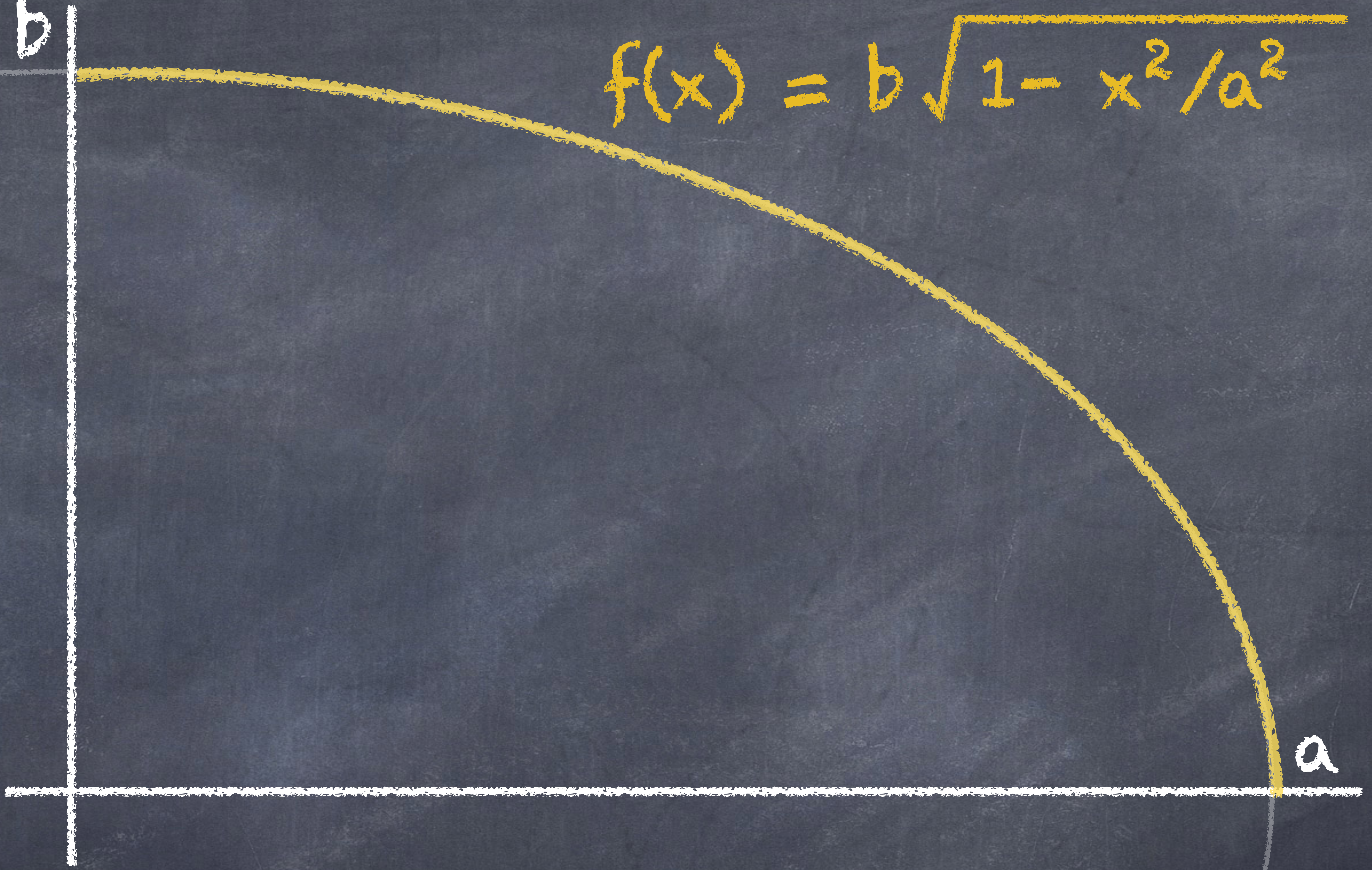

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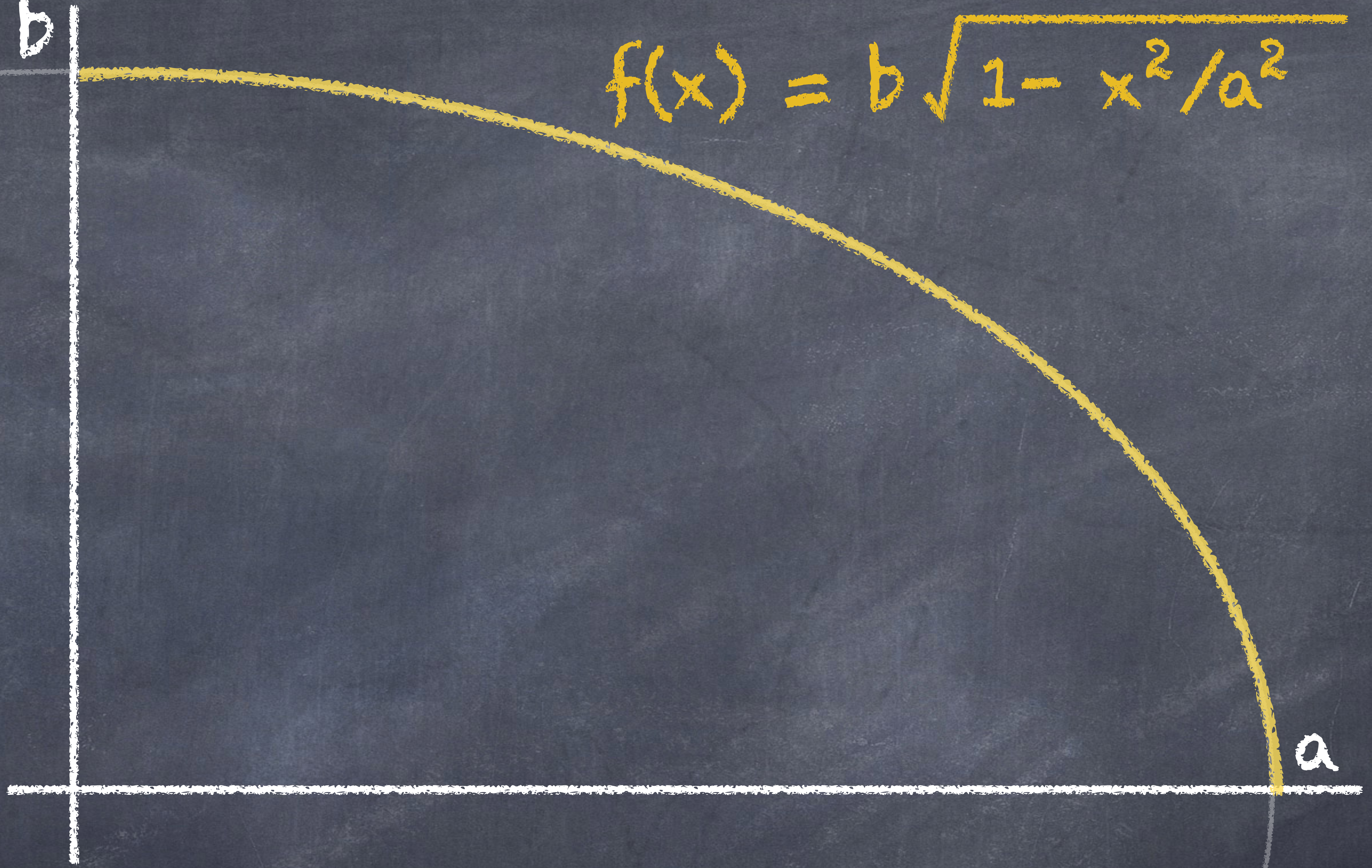
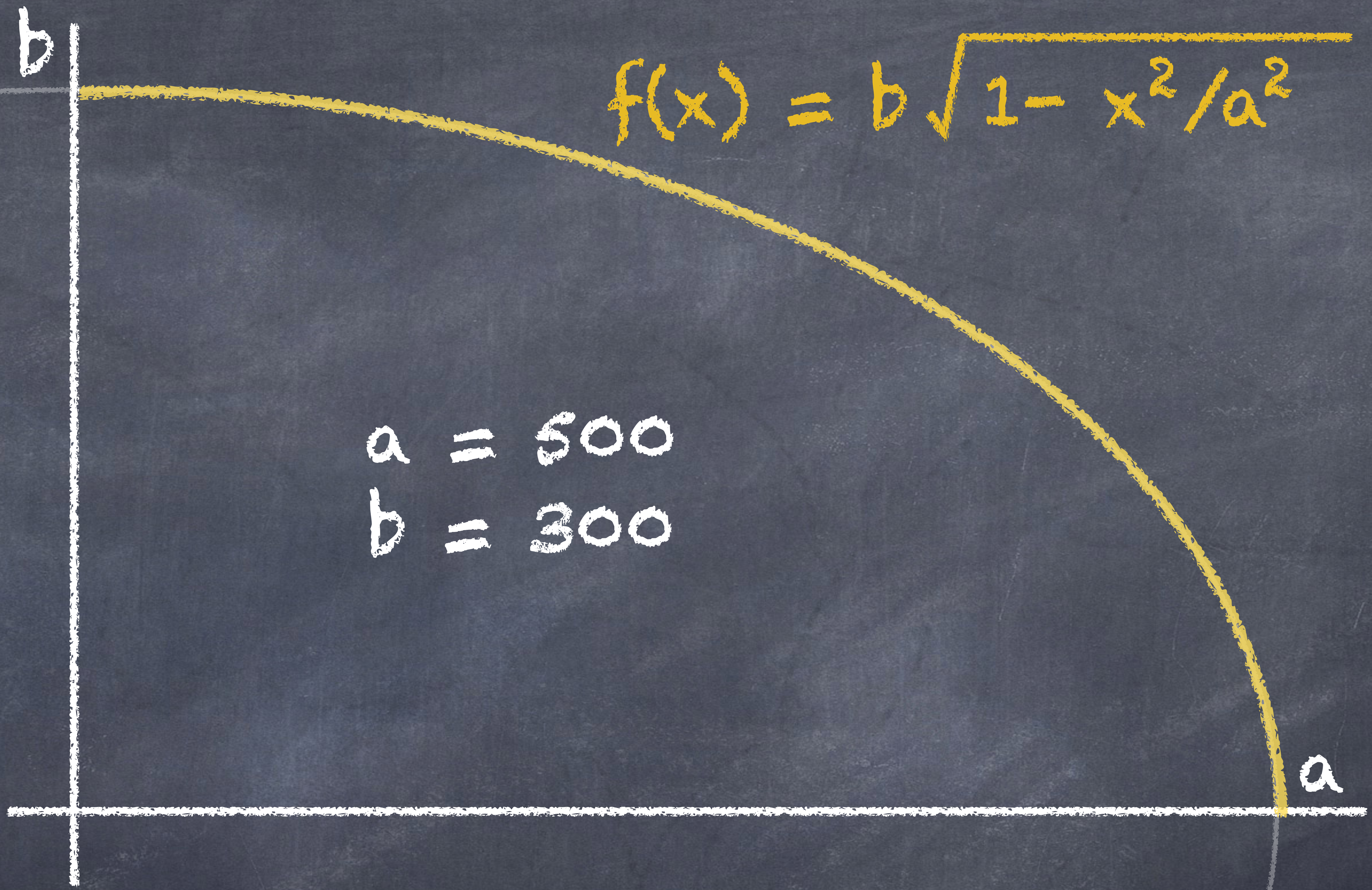
$$\int_0^a \sqrt{1 - f'(x)^2}\, dx$$



$$\int_0^a \sqrt{1 - f'(x)^2}\; dx = \int_0^1 \sqrt{\frac{a^2 - b^2x^2}{1 - x^2}}\; dx$$



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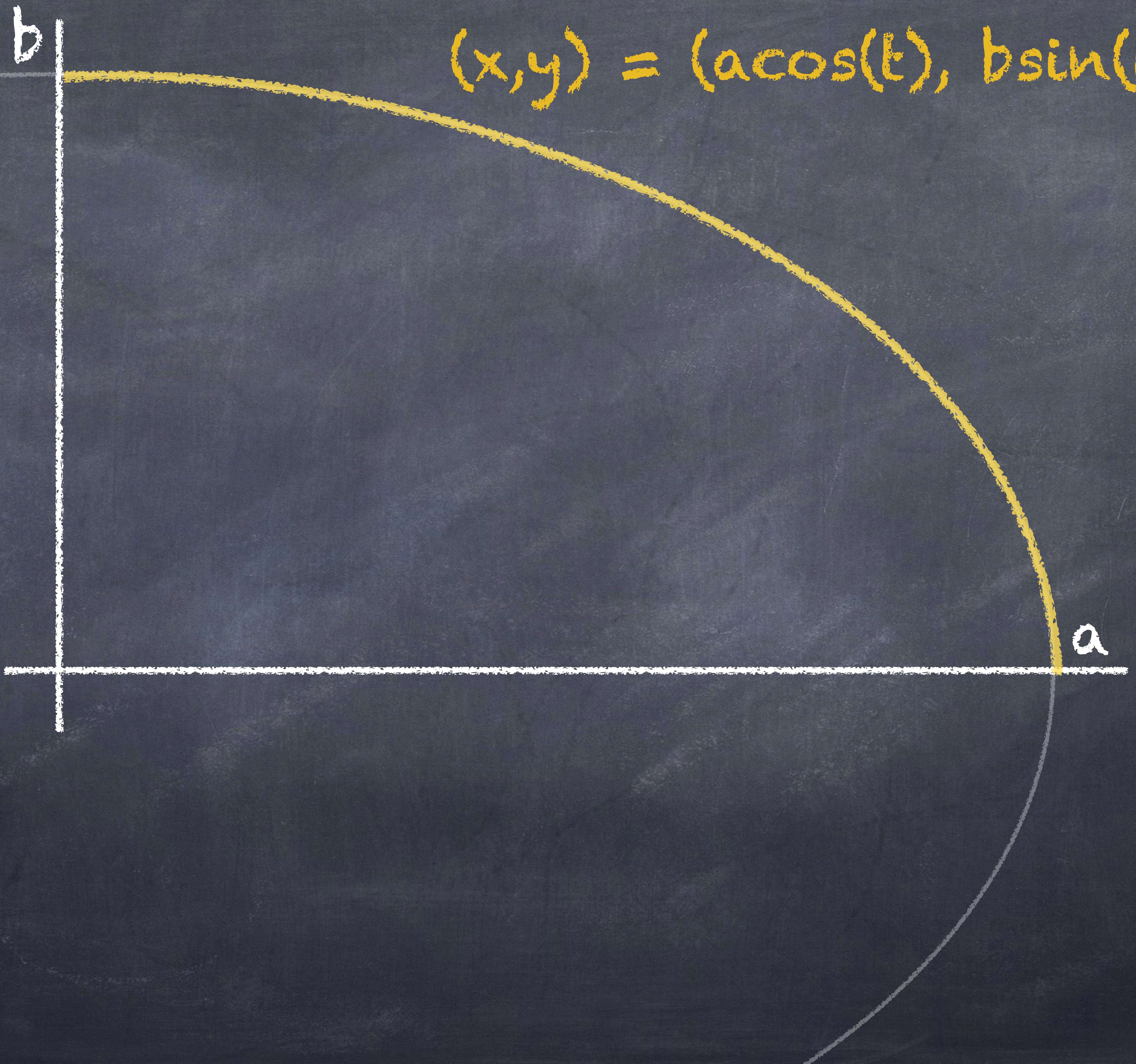

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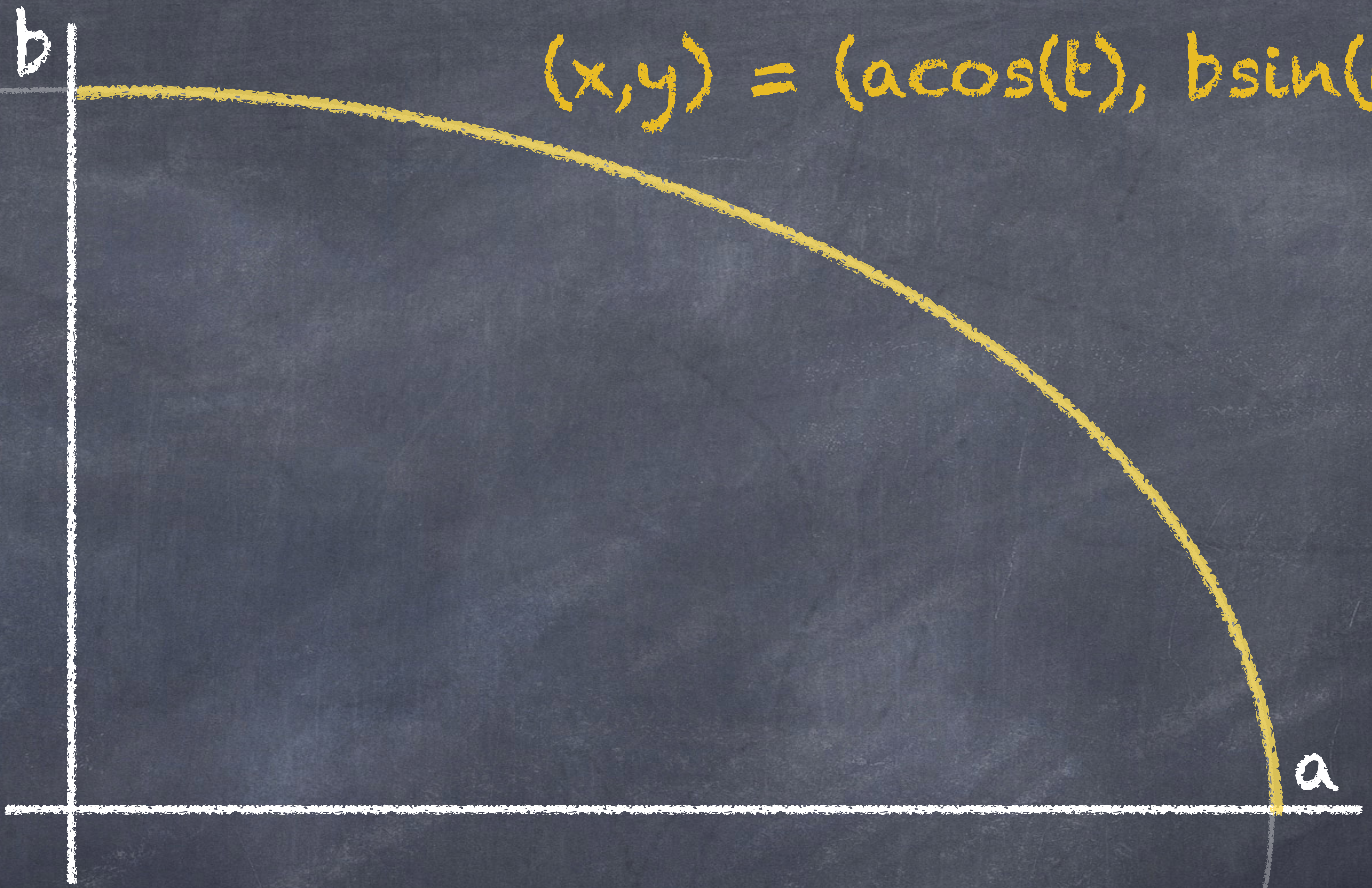

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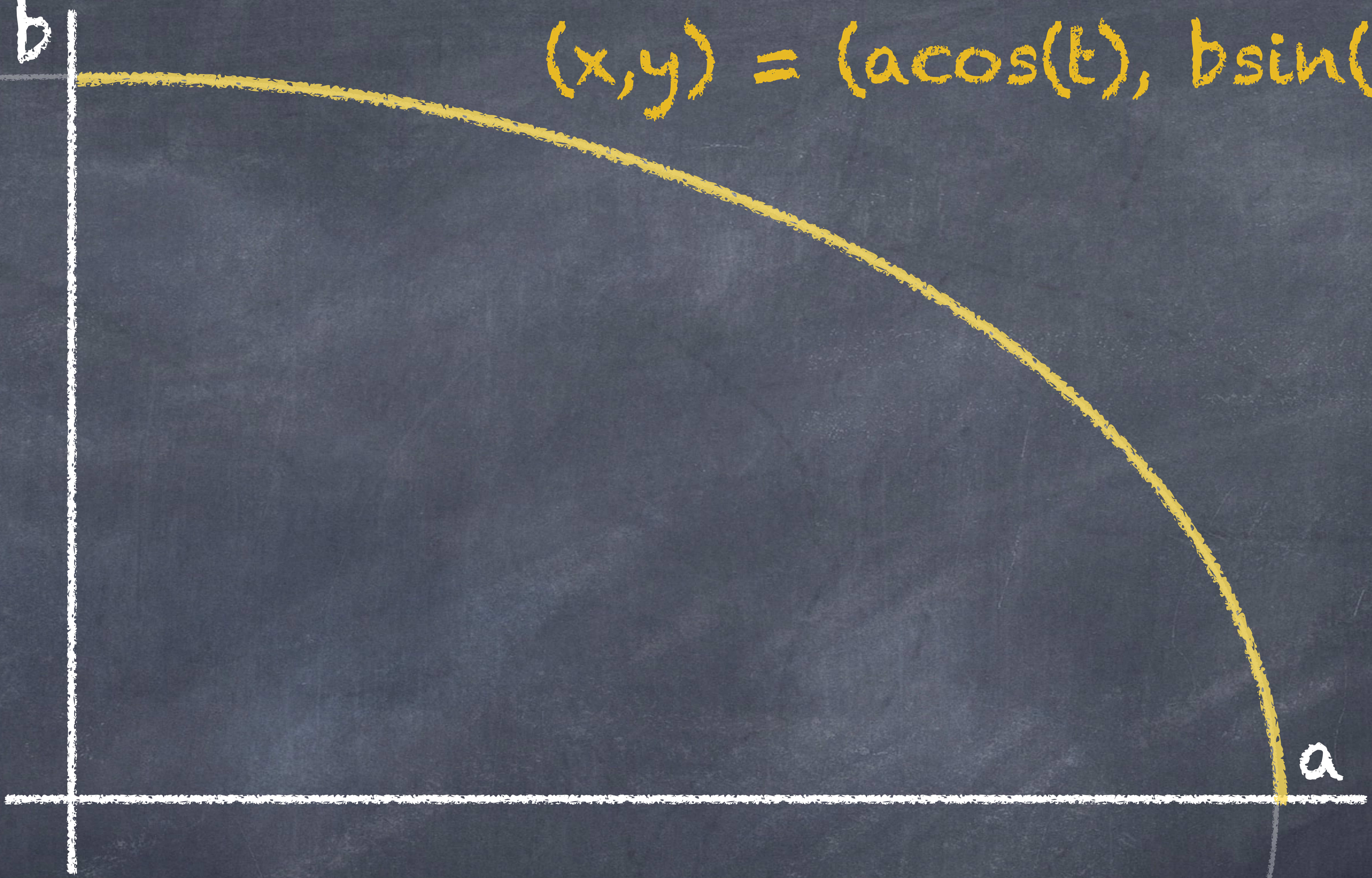

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Elliptic Integrals

$$E(a,b) = \int_0^{\pi/2} \sqrt{a^2 \sin(t)^2 + b^2 \cos(t)^2}\, dt$$

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first kind

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second kind

$$K(a,b) = \int_0^{\pi/2} \frac{dt}{\sqrt{a^2 \sin(t)^2 + b^2 \cos(t)^2}}$$

substitute $t = \arctan(x/a)$

$$K(a,b) = \int_0^\infty \frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}}$$

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$$\mathrm{AGM}(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

The Arithmetic-Geometric Mean

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Quadratic convergence

Variant for $E(a,b)$:

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n	AGM, a_n	AGM, b_n	N, a_n	N, b_n	N, c_n
0	500	300	250000	90000	0
1	400	387.298334620742	170000	150000	-150000
2	393.649167310371	393.597934253086	160000	159838.667696593	-459838.667696593
3	393.623550781728	393.623549948183	159919.333848297	159919.328598644	-1079596.66399183
4	393.623550364956	393.623550364956	159919.33122347	159919.33122347	-2319112.6592071
5	393.623550364956	393.623550364956	159919.33122347	159919.33122347	-4798144.64963774
6	393.623550364956	393.623550364956	159919.33122347	159919.33122347	-9756208.63049894

fx G8 + SQRT((E8 − G8) × (F8 − G8))

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Proposition: $$E(a,b) = \frac{\pi \; N(a^2, b^2)}{2 \; M(a, b)}$$

$a = 500$
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$$\frac{\pi \times 159919.33122347}{2 \times 393.6235503649} = 638.17497158$$

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Circumference = 2552.6998863

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Richard P. Brent, Fast Multiple precision evaluation of elementary functions, 1976

Eugene Salamin, Computation of π Using Arithmetic-Geometric Mean, 1976.