From Königsberg to Sudoku: Discrete Problem Solving via Graph Theory

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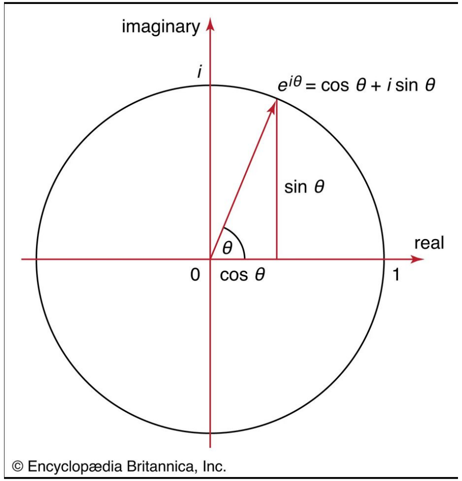
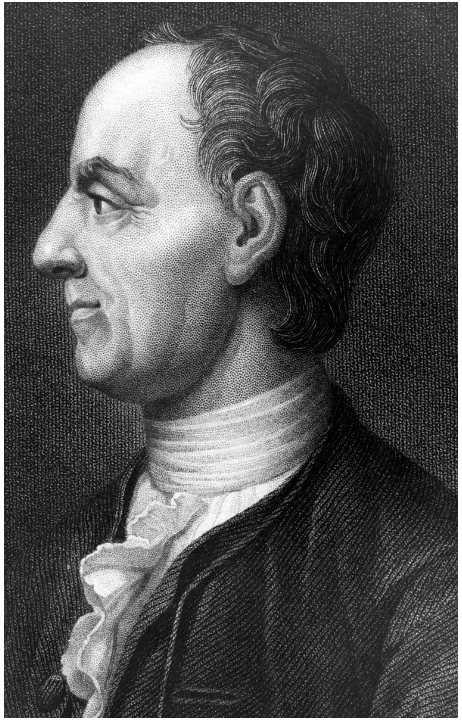
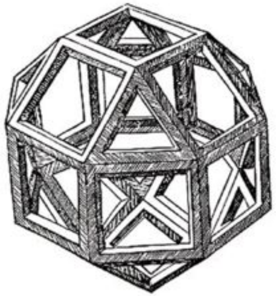
Leonhard Euler (1707-1783)

- Maths, Physics, Astronomy, Navigation
- Notation: i, e, f(x), Σ, trig functions
- Euler's formula: e^(iϴ) = cosϴ + i sinϴ
- Euler's polyhedron formula: V - E + F = 2





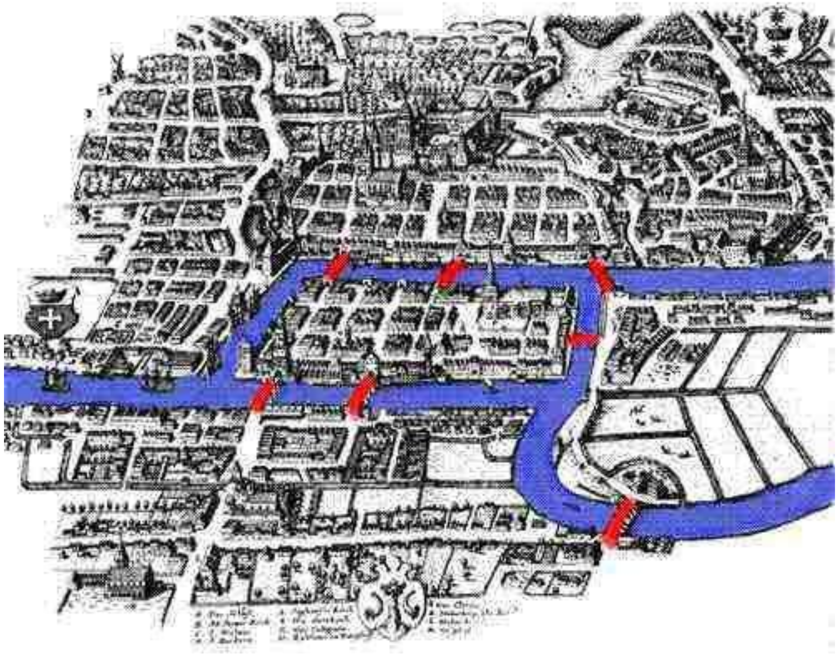
- And… **graph theory**!



https://www.britannica.com/biography/Leonhard-Euler
https://en.wikipedia.org/wiki/Leonhard_Euler
https://www.usna.edu/Users/math/meh/euler.html
https://www.britannica.com/science/Eulers-formula
https://en.wikipedia.org/wiki/Polyhedron

Euler’s Solution to the Bridges of Königsberg Problem (1735)

- Königsberg (now: Kaliningrad, Russia): city in Prussia with 7 bridges over the Pregel River
- Beautiful to walk through the city, but can all bridges be crossed without repeating?
  - Euler’s idea: careful counting argument. If a landmass touches an odd number N of bridges then it must be visited (N+1)/2 times during such a walk
  - But (3+1)/2 + (3+1)/2 + (3+1)/2 + (5+1)/2 = 9 > 8, so this is not possible!



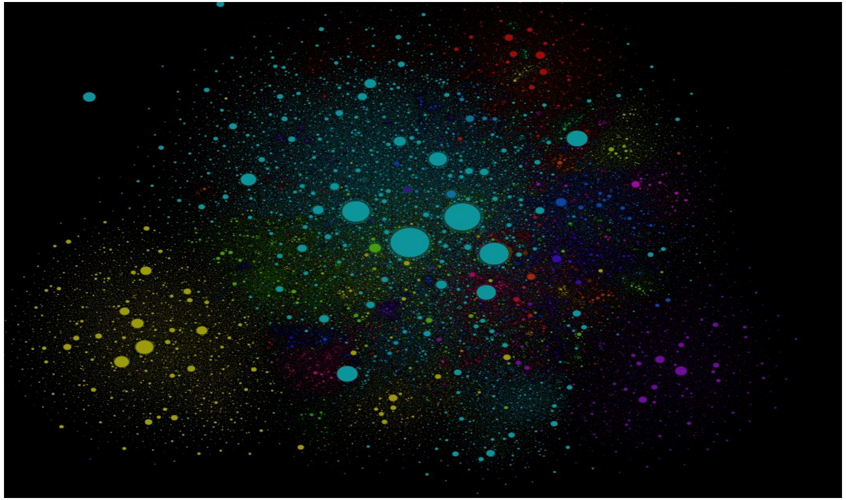
https://maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-problem
https://www.britannica.com/science/Konigsberg-bridge-problem

What are Graphs?

Vertices joined by edges:

V(G) = vertices of G, E(G) = edges of G

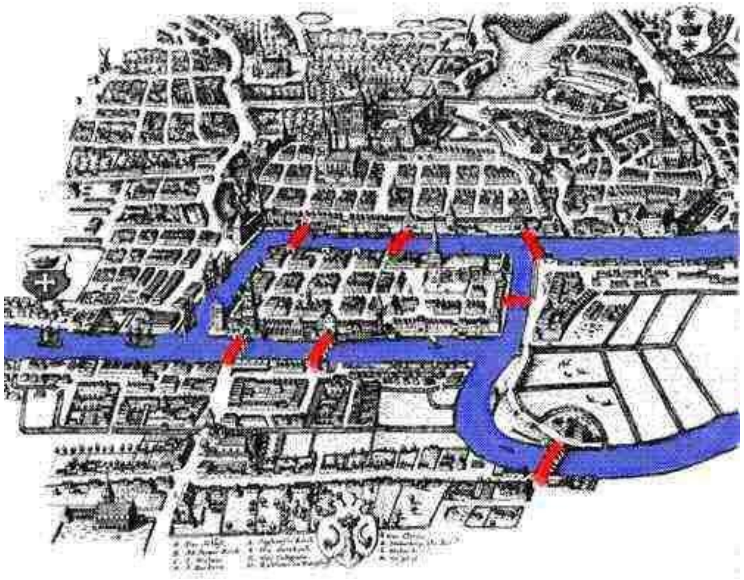
- Encode the *connectivity* of various objects
- Many real-world applications



https://content.tfl.gov.uk/standard-tube-map.pdf
https://www.scientificamerican.com/article/how-the-mind-emerges-from-the-brains-complex-networks/
http://internet-map.net/

Modern Formulation Using Graphs

- A graph can be used to encode the connectivity data of the Königsberg bridges:



- Does there exist an *Eulerian path*?
- Easy solution: Non-terminal vertices must have an even *degree* (i.e. number of edges attached)
- Additional bridge built (1875) creating an Eulerian path, but much of the city destroyed in WWII bombing

https://maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-problem
https://www.britannica.com/science/Konigsberg-bridge-problem

The Three Utilities Problem

<u>Problem:</u> **Is it possible to connect the three houses to each of the three utility stations such that the supply lines don't cross?**

<u>Answer:</u> No! The graph K_3 is non-planar… and this is easy to prove…

<u>Recall:</u> *Euler's polyhedron formula*

$$V \;-\; E \;+\; F \quad = \; 2$$

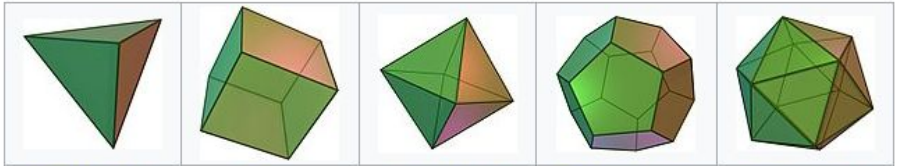
$$\| \qquad \| \qquad \leq$$

$$6 \qquad 9 \qquad 2E/4(=9/2)$$

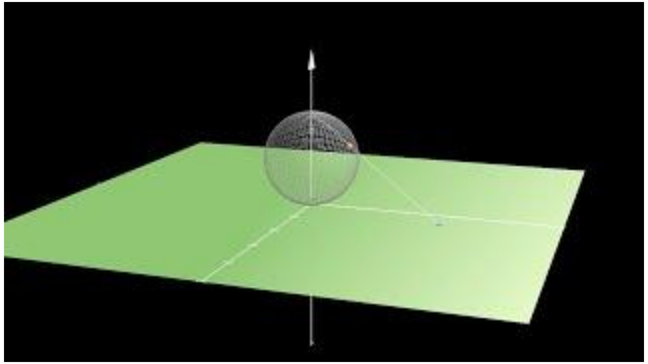
https://www.math.cmu.edu/~mradclif/teaching/228F16/Kuratowski.pdf

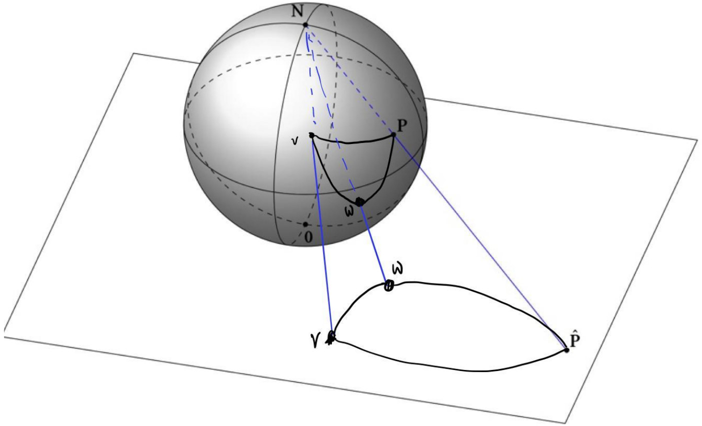
Euler’s Polyhedron Formula

$$V - E + F = 2$$



Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}
$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$





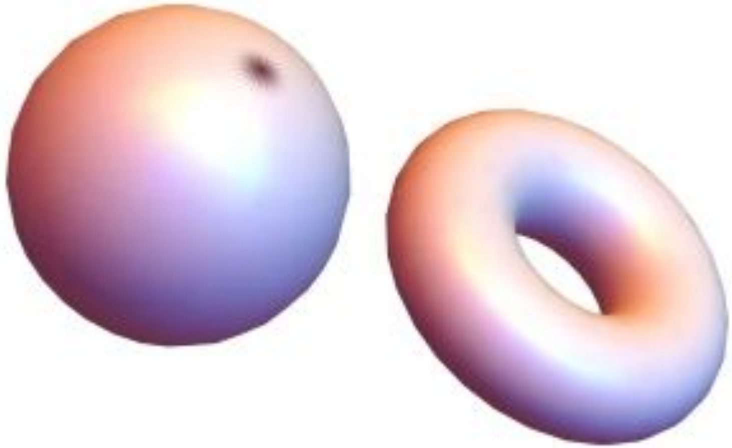
https://en.wikipedia.org/wiki/Regular_polyhedron
https://tex.stackexchange.com/questions/562590/stereographic-projection-of-graph-from-sphere-to-plane-tikz
https://www.youtube.com/watch?v=m9IsbVshRVE

Euler's Polyhedron Formula (con't)

$$V - E + F = \chi$$

Gauss-Bonnet Theorem:

$$\chi(M) = (1/2\pi) \int K \, dA$$



K=1

$\chi(M) = 2$

K=0

$\chi(M) = 0$

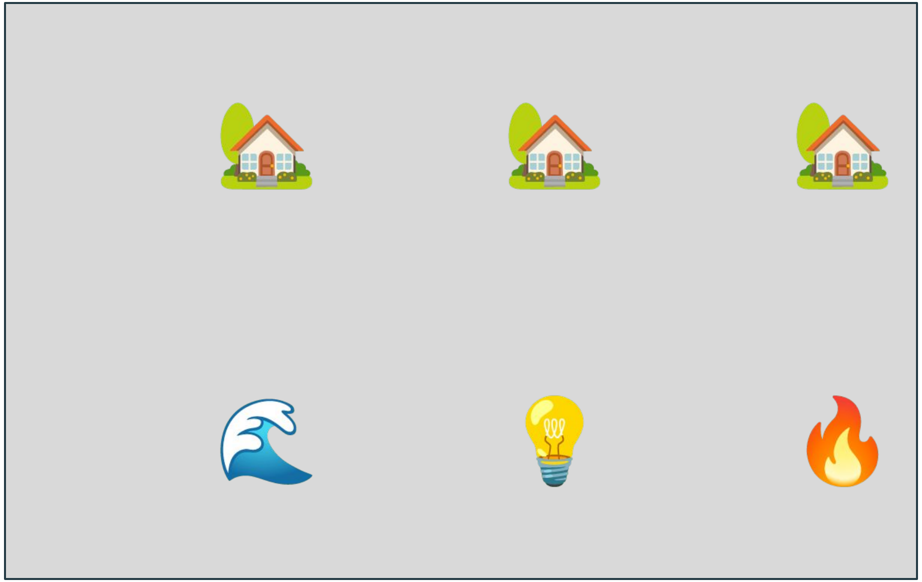
More generally: $\chi(M) = 2\text{-}2g$

https://thatsmaths.com/2020/09/24/doughnuts-and-dumplings-are-distinct-homopoty-101/sphere-torus/
https://en.wikipedia.org/wiki/Gauss%E2%80%93Bonnet_theorem

The Three Utilities Problem on a Torus

Euler Characteristic:

$$V + F - E = 0$$

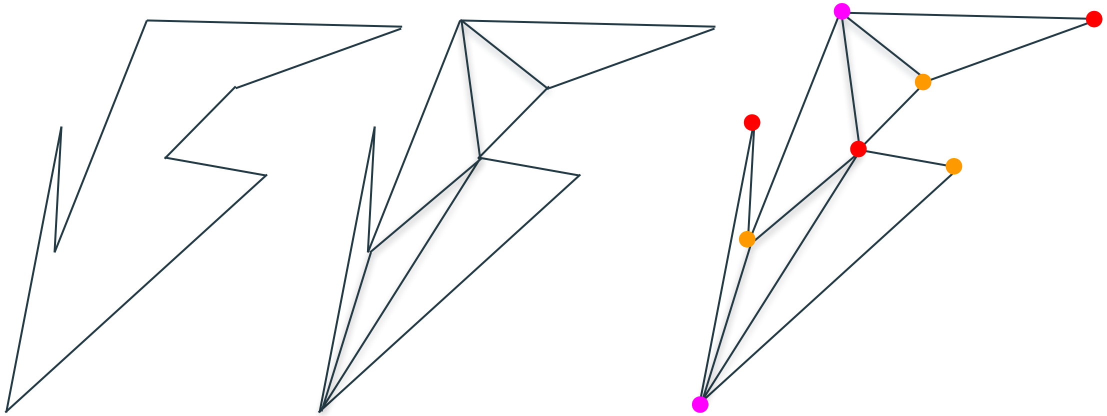


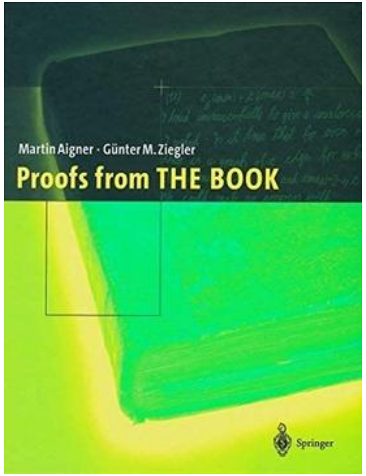
Challenge: Solve the "Four Utilities Problem" on a torus.

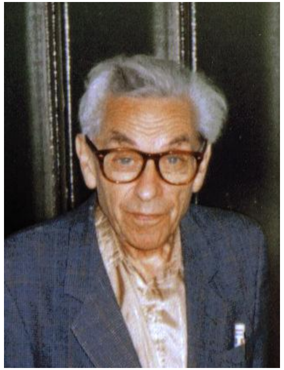
The Art Gallery Problem

<ins>Problem:</ins> **How many guards are needed to protect a polygonal art gallery?**

<ins>Answer (Chvátal 1975):</ins> ≤ Floor(n/3), where n=|V(G)|







https://en.wikipedia.org/wiki/Proofs_from_THE_BOOK
https://en.wikipedia.org/wiki/Paul_Erd%C5%91s
Fisk's Proof: http://www.ams.org/publicoutreach/feature-column/fcarc-diagonals4

Graph Colourings

Rules:

1. Every vertex must have a colour.
2. No adjacent vertices can have the same colour.

(Only makes sense for graphs without loops)

Chromatic number:

$\chi(G)$ = minimum number of colours needed

<u>Greedy Colouring Algorithm</u>

1. Choose an ordering of the vertices.
2. Colour the first vertex with Colour 1.
3. For each subsequent vertex, colour it with the smallest possible previously used colour, or, if this is impossible, introduce a new colour.

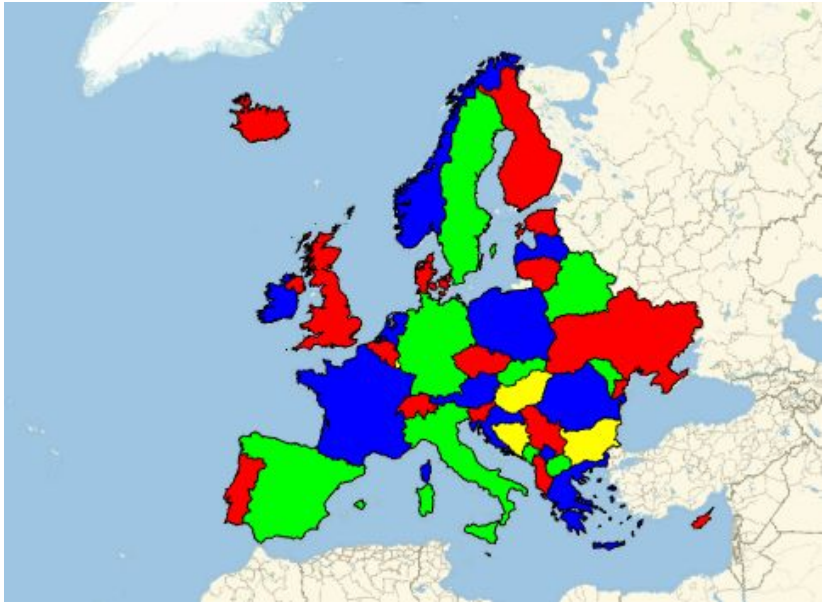
Note: It is always possible to choose an ordering of the vertices giving an optimal colouring (i.e. with the fewest number of colours)!

https://www.geeksforgeeks.org/graph-coloring-set-2-greedy-algorithm/

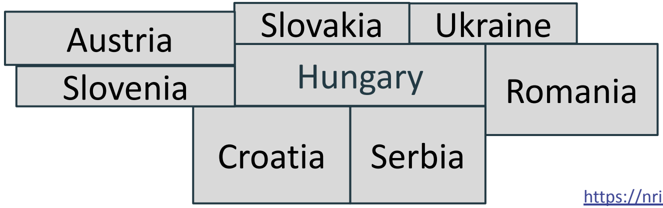
Four Colour Theorem

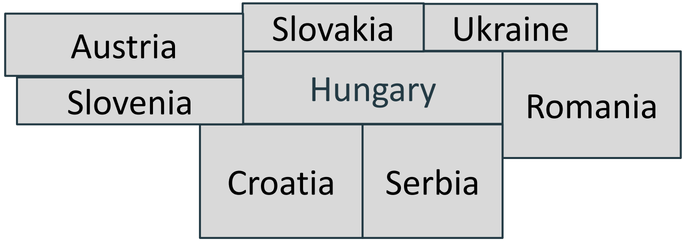
It is always possible to colour a (geographic) map using at most four colours (and there exist maps where four colours are required).

- First proposed by De Morgan–first maths prof at UCL!
- Extremely difficult problem–took over 100 years to prove.
- Early example of computer-assisted proof (Appel-Haken 1976)
- False proof by Kempe (1879), later pointed out by Heawood (1890).



Map Graph:





https://nrich.maths.org/6291
https://www.wolfram.com/mathematica/new-in-10/entity-based-geocomputation/find-a-four-coloring-of-a-map-of-europe.html

Five Colour Theorem (Heawood 1890)

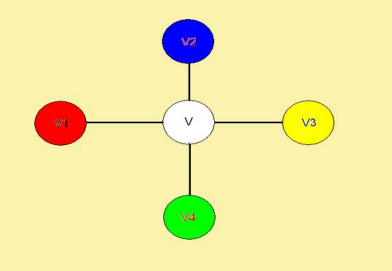
The Four Colour Theorem is difficult, but it becomes much easier if we allow one additional colour

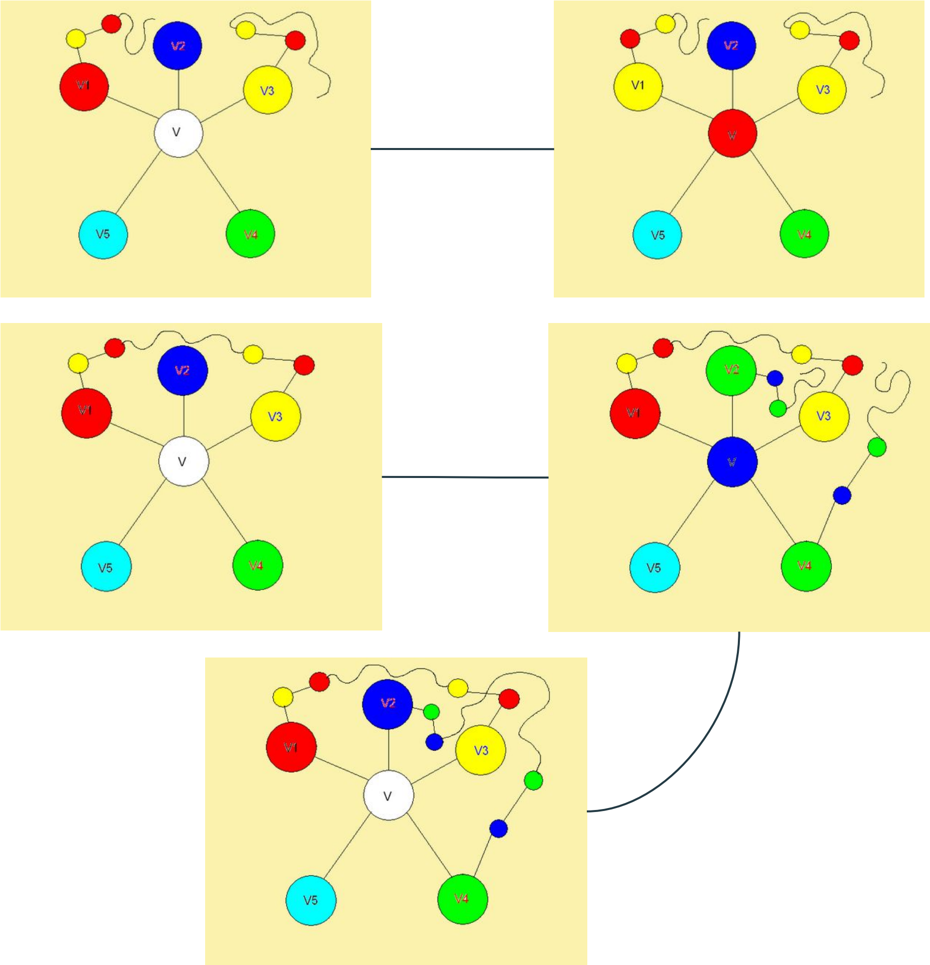
Key ingredient: Euler’s polyhedron formula

$$V - E + F = 2$$

Corollary: A map graph (or, in fact, any simple planar graph) has a vertex of degree at most 5.

Proof:



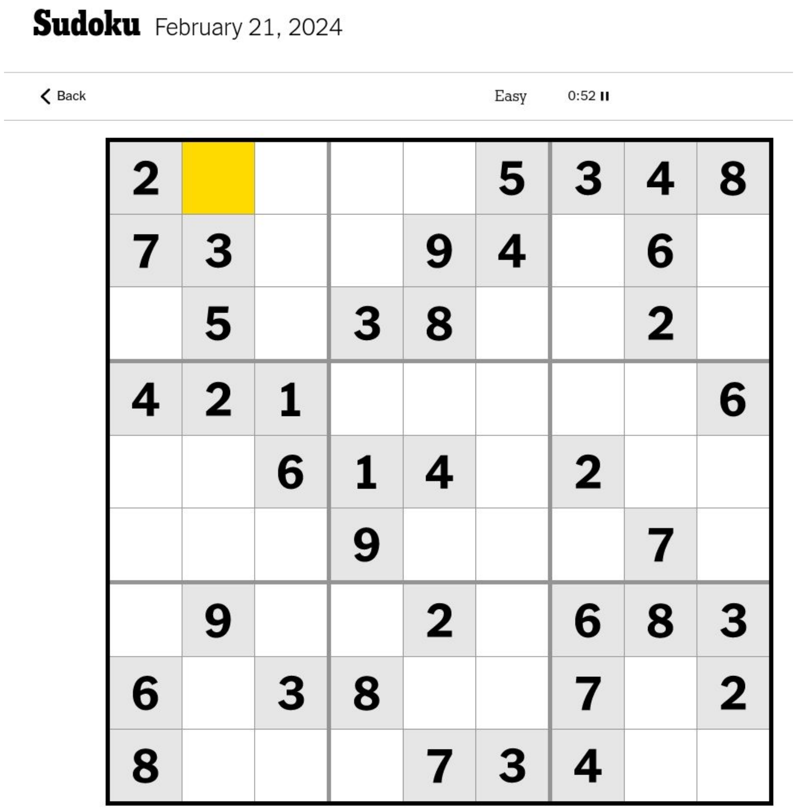


http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

Sudoku

Rule: Each row, column, and box must have exactly one copy of each integer 1,2, ..., 9.

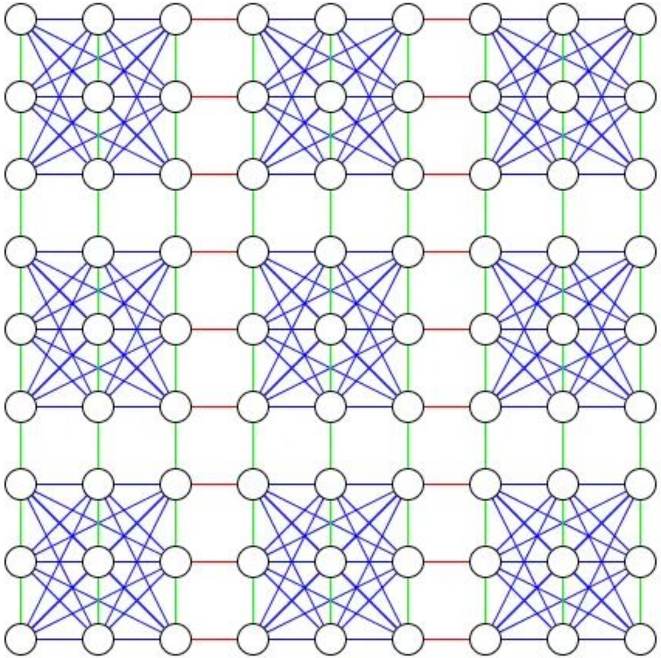
Note: This rule is essentially a statement about "**adjacency**" of the numbers in the grid… We have already encountered this idea with the Four Colour Theorem!



https://www.nytimes.com/puzzles/sudoku

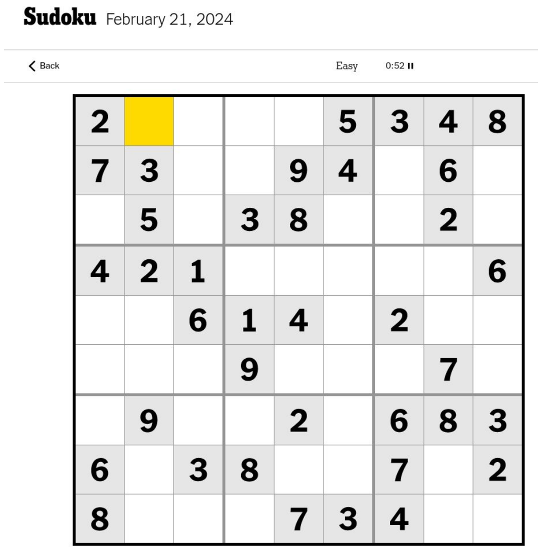
Sudoku (con't)

1. Introduce vertices into every position in the Sudoku board
2. Connect each vertex to all other vertices which share a row, column, or box with it.



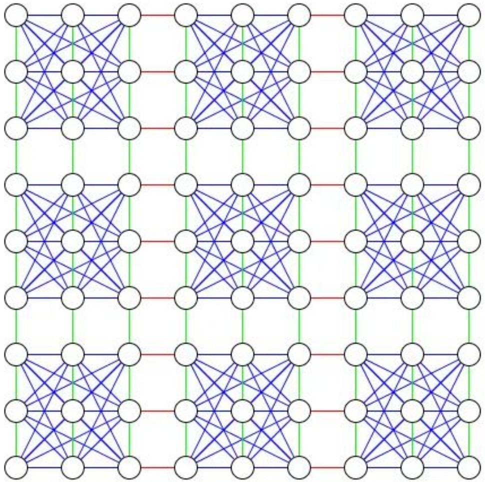
https://www.reddit.com/r/dataisbeautiful/comments/6ty4vf/visualizing_the_sudoku_connectivity_graph_more_in/
https://medium.com/code-science/sudoku-solver-graph-coloring-8f1b4df47072
https://community.wolfram.com/groups/-/m/t/2983903

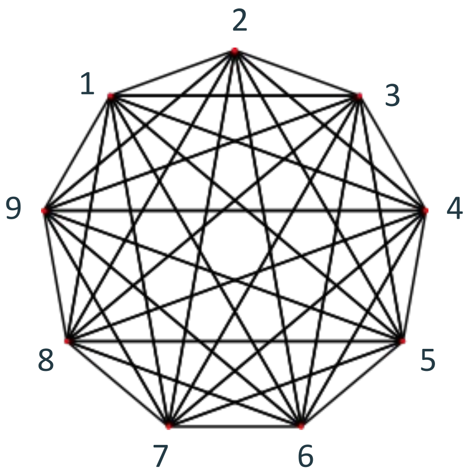
Sudoku (con’t)



How do we encode the “**hints**”?

1. Introduce a *complete graph* K_9 labelled 1,...9.
2. Connect each hint vertex to all vertices in K_9 with different numbers than it.





https://www.reddit.com/r/dataisbeautiful/comments/6ty4vf/visualizing_the_sudoku_connectivity_graph_more_in/
https://medium.com/code-science/sudoku-solver-graph-coloring-8f1b4df47072
https://commons.wikimedia.org/wiki/File:Complete_graph_K9.svg
https://community.wolfram.com/groups/-/m/t/2983903