The twin prime conjecture

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Introducing prime numbers

Definition

An integer $p > 1$ is called a **prime number** if it has no positive divisors other than 1 and p.

- **First examples:** 2, 3, 5, 7, 11, 13, 17, 19, 23, ...
- **Largest known:** $2^{82,589,933} - 1$, which has 24,862,048 digits.

Euclid's theorem (c.300BC)

There are infinitely many primes.

- Famous example of proof by contradiction!

How are the primes distributed?

Definition

For a given number X, we define the **prime counting function** to be

$$\pi(X) := \#\{p \leq X : p \text{ is prime}\}.$$

- Gauss and Legendre indpendently studied the density of primes using tables.
- Gauss (late 1700s) conjectured that as $X \to \infty$, we have

$$\pi(X) \sim \mathrm{li}(X) := \int_2^X \frac{dt}{\log(t)} \sim \frac{X}{\log(X)}.$$

- Legendre (1808) conjectured that

$$\pi(X) \sim \frac{X}{\log(X) - 1.08366}.$$

How are the primes distributed?

- Finally, in 1896, Hadamard and de la Vallée Poussin independently proved:

Prime number theorem

As $X \to \infty$, we have

$$\pi(X) \sim \mathrm{li}(X)$$

- Proof uses complex analysis (contour integrals) and the **Riemann zeta function**.
- **Crucial:** information about the **zeros** of the Riemann zeta function.

The Riemann zeta function

Definition

Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then the **Riemann zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Defined through analytic continuation to the rest of the complex plane.
- **Trivial zeros:** -2, -4, -6, -8,...

Riemann Hypothesis

The real part of all non-trivial zeros of the Riemann zeta function is $1/2$.

Introducing twin primes

- Are there patterns among the primes?
- Simplest to consider are twin prime pairs: $(p, p+2)$.
- **First examples:**

$$(3,5),(5,7),(11,13),(17,19),(29,31)\dots$$

- **Largest known:**

$$(2,996,863,034,895 \times 2^{1,290,000} - 1,$$
$$2,996,863,034,895 \times 2^{1,290,000} + 1).$$

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The twin prime conjecture

Twin prime conjecture

There exist infinitely many primes p such that $p + 2$ is also prime.

Looking at the list of primes, we see other pairs over and over again:

$$(p, p + 4), (p, p + 6), (p, p + 8), (p, p + 10), \dots$$

So, naturally, we have the following generalisation:

Generalised twin prime conjecture

Let $h \geq 2$ be an even integer. There exist infinitely many primes p such that $p + h$ is prime.

Breakthroughs

- What do we know so far?
- Goldston, Pintz, Yıldırım, 2005: there exist infinitely many consecutive primes which are closer than any arbitrarily small multiple of the average spacing.
- Zhang, 2013: there exist infinitely many pairs of distinct primes which differ by no more than 70,000,000.
- Maynard, 2013: 600...
- Polymath, 2014: 246...
- Best known conditional result (Polymath): 6

Other prime patterns

Conjecture (Germain primes)

There exists infinitely many primes p such that $2p+1$ is prime.

Examples: $(2,5),(3,7),(5,11),(11,23),(23,47),(29,59),\ldots.$

Dirichlet's theorem (1837)

Let a, b be positive coprime integers. Then there exists infinitely many primes of the form $a+nb$.

Examples: $4n+1, 4n+3, 6n+1, 6n+5\ldots$

Other prime patterns

Dickson's conjecture

If a finite set of linear forms $a_1 + b_1 n, a_2 + b_2 n, \ldots, a_k + b_k n$ is **admissible**, then there are infinitely many positive integers n for which they are all prime.

- $k = 1$ is Dirichlet's theorem!
- **Admissible** means the forms are all positive at infinitely many integers n and for each prime p there exists an integer n s.t. $p \nmid \prod_i (a_i + b_i n)$.
- **Non-example:** $(n, n+2, n+4)$, one is always a multiple of 3!
- **Example:** $(10n+1, 10n+3, 10n+7, 10n+9)$.

Other prime patterns

Green-Tao Theorem (2004)

The sequence of prime numbers contains arbitrarily long arithmetic progressions.

Landau's fourth problem

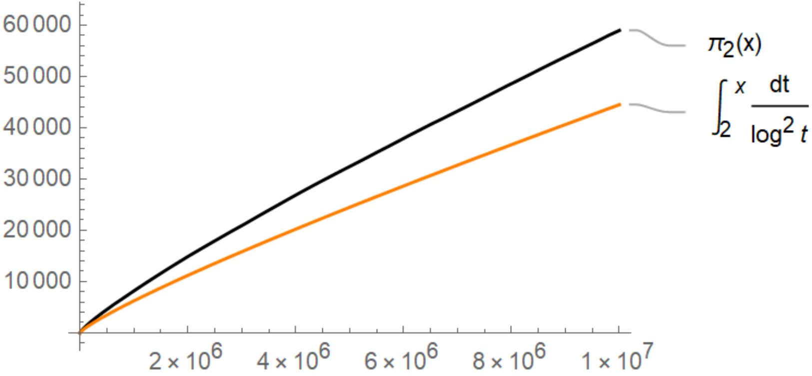
There exists infinitely many primes of the form $n^2 + 1$.

Bunyakovsky's conjecture generalises Dirichlet's theorem to higher degree polynomials.

The distribution of twin primes

- Could we propose an analogue of the prime number theorem for twin primes?
- PNT $\Rightarrow$ probability an integer $n \leq X$ is prime is $\sim \frac{1}{\log X}$.
- **Guess:** probability integers $n \leq X$ and $n+2$ are both prime is $\sim \frac{1}{\log^2 X}$.
- **Naive guess:**
$$\#\{n \leq X : n, n+2 \text{ both prime}\} \sim \frac{X}{\log^2 X}.$$
- This assumed "n is prime" and "$n+2$ is prime" are independent events - clearly false!

The twin prime conjecture



Revised prediction: For some constant $C > 0$ as $X \to \infty$

$$\#\{p \leq X : p, p+2 \text{ both prime}\} \sim C \frac{X}{\log^2 X}.$$

The twin prime constant

- For n and $n+2$ to both be prime, they must both be odd.
- For randomly chosen n, probability that both n and $n+2$ are odd is $1/2$.
- Probability two randomly chosen integers are both odd is $(1/2)^2$.
- **Correction:** add factor of
$$\frac{1/2}{(1/2)^2} = 2.$$

The twin prime constant

- For n and $n+2$ to both be prime they cannot both be divisible by any odd prime p.
- For odd prime p, $p \nmid n$ and $p \nmid n+2 \iff n \not\equiv 0, -2 \mod p$, which happens with probability $(1 - 2/p)$.
- Probability two randomly chosen integers are both not divisible by p is $(1 - 1/p)^2$.
- **Correction:** for each odd prime p add factor of

$$\frac{1 - 2/p}{(1 - 1/p)^2}.$$

The twin prime constant

Definition (Twin prime constant)

Define the *twin prime constant*

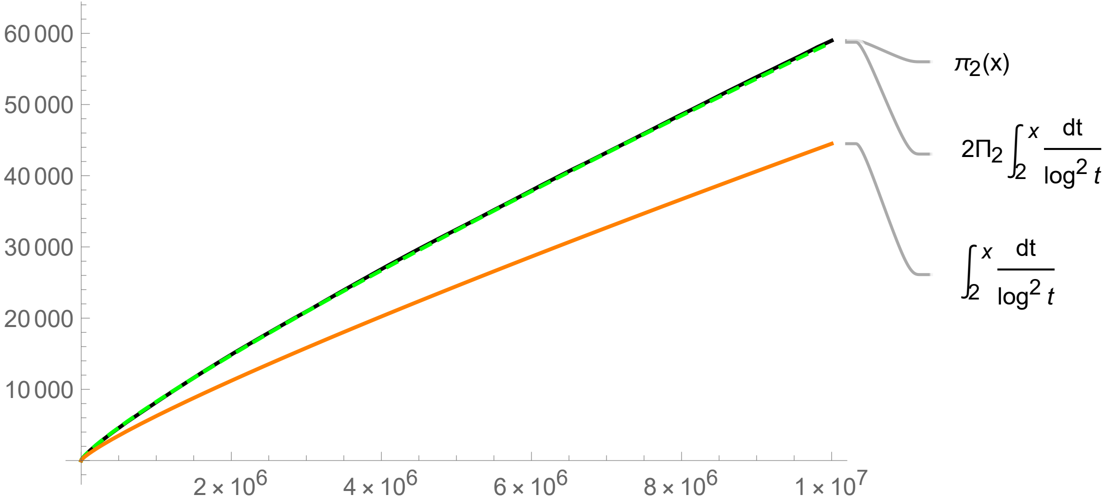
$$\Pi_2 := \prod_{p>2} \frac{(1-2/p)}{(1-1/p)^2} = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Hardy-Littlewood conjecture

As $X \to \infty$

$$\#\{p \le X : p, p+2 \text{ both prime}\} \sim 2\Pi_2 \frac{X}{\log^2 X}.$$

Compare the conjecture with our first guess:



Hardy-Littlewood conjecture

Let $h \neq 0$ be an integer. Then as $X \to \infty$

$$\#\{p \leq X : p, p+h \text{ both prime}\} \sim \mathfrak{S}(h)\frac{X}{\log^2 X}.$$

Define the correction factor

$$\mathfrak{S}(h) := 2\Pi_2 \prod_{p|h, p>2} \frac{p-1}{p-2}$$

if h is even and $\mathfrak{S}(h) = 0$ if h is odd.

Pairs of primes

Partial progress: several results showing Hardy-Littlewood conjecture holds on average over difference h. Best known:

Theorem (Matomäki, Radziwiłł, Tao, 2019)

Let $\varepsilon > 0$. Then

$$\#\{p \in (X, 2X] : p, p+h \text{ both prime}\} \sim \mathfrak{S}(h) \frac{X}{\log^2 X},$$

for "almost all" $0 < |h| \leq X^{8/33+\varepsilon}$.

Introducing almost primes

- Primes are tough! What if we consider "almost primes" instead?
- Almost prime = numbers with exactly two prime factors = more flexibility!
- Can we count the number of almost primes $n \leq X$ such that $n + h$ is also almost prime?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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91	92	93	94	95	96	97	98	99	100

An almost prime analogue

PNT $\Rightarrow$ a formula for the number of almost primes up to X:

Theorem

As $X \to \infty$, we have

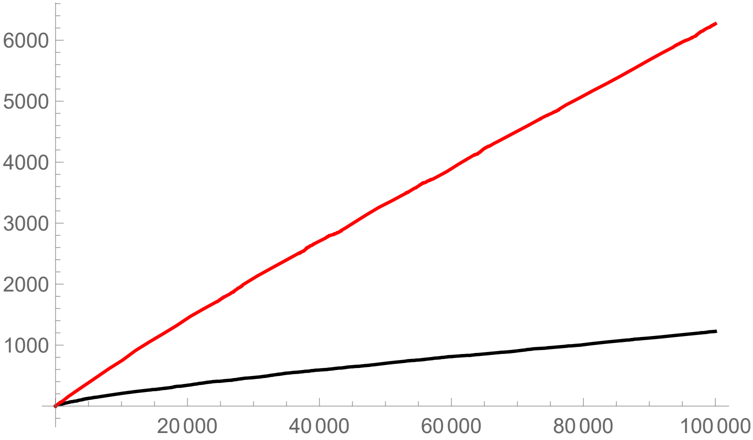
$$\#\{n \leq X : n \text{ has exactly two prime factors}\} \sim \frac{X \log\log X}{\log X}.$$

Conjecture

Let $h \neq 0$ be an integer. As $X \to \infty$, we have

$$\#\{n \leq X : n, n+h \text{ have exactly two prime factors}\} \sim \mathfrak{S}(h)\frac{X(\log\log X)^2}{(\log X)^2}.$$

Twin primes vs twin almost primes



Comparing $\#\{p \leq X : p, p+2 \text{ both prime}\}$ (black) with $\#\{n \leq X : n, n+2 \text{ have exactly two prime factors}\}$ (red).

My thesis: correlations of almost primes

We prove the conjecture holds on average over h:

Theorem

Let $\varepsilon > 0$. Then

$$\#\{n \in (X, 2X] : n, n+h \text{ have exactly two prime factors}\} \sim \mathfrak{S}(h)\frac{X(\log\log X)^2}{(\log X)^2}$$

holds for "almost all" values of $0 < |h| \leq \exp((\log X)^{1-\varepsilon})$.

Remark: Replacing typical almost primes with a specific factorisation, this theorem holds with average of length $\log^{19+\varepsilon} X$.