Calculi your maths professors may not tell you about: from shadowy to quite random (or it is all about Hermite polynomials)

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Hermite Polynomials

Definition

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \\ &\cdots \end{aligned}$$

(Umbral $H_n(x) = e^{-\frac{D^2}{2}} x^n$.)

Alternative definition with generating function

Definition

$$H_n(x) = \frac{d^n}{du^n}\ e^{ux-\frac{u^2}{2}}\Big|_{u=0}$$

The Generating Function

$$e^{ux-\frac{u^2}{2}} = \sum_{n=0}^{\infty} H_n(x)\frac{u^n}{n!}.$$

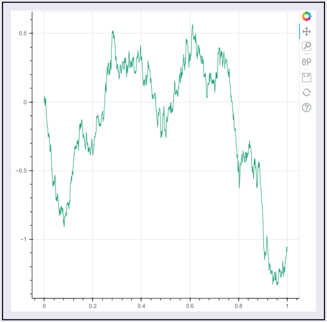
$$\frac{d}{dx}H_n(x) = nH_{n-1}(x).$$

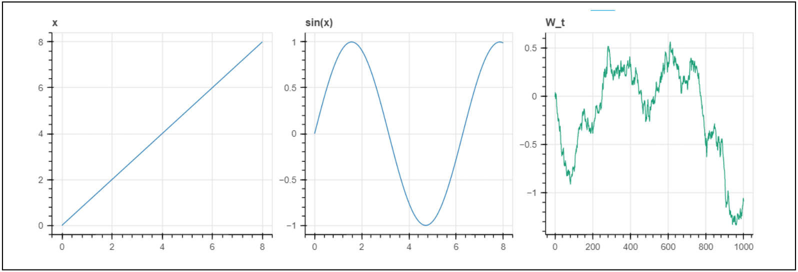
$$H_n(x) = \int_{-\infty}^{\infty} \delta^{(n)}(u)e^{ux-\frac{u^2}{2}}\,du.$$

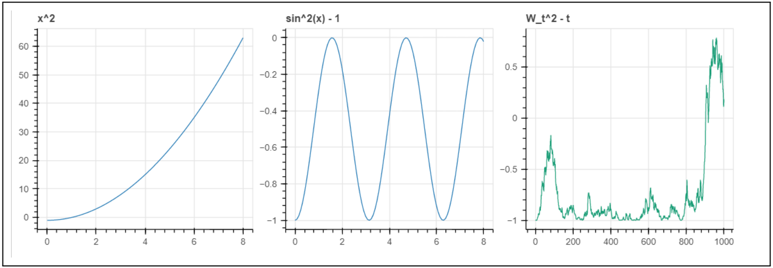
Wiener process

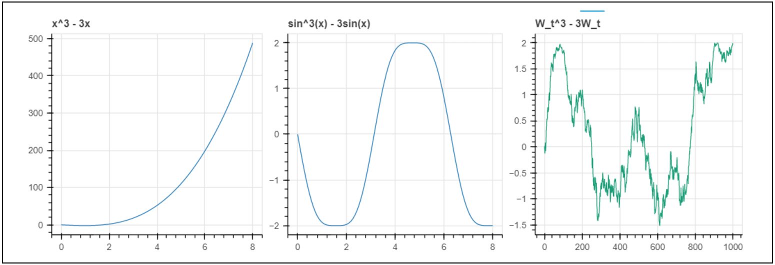
Definition

- $W_0 = 0$
- $W_t \sim N(0, t)$
- *Independent increments* $(W_t - W_s)$, $(W_u - W_v)$ *for all* $v < u < s < t$.




x
0
2
4
6
8
sin(x)
1
0.5
0
−0.5
−1
W_t
0.5
−0.5
200
400
600
800
1000


x^2
sin^2(x) - 1
W_t^2 - t


x^3 - 3x
sin^3(x) - 3sin(x)
W_t^3 - 3W_t

Differentiation?

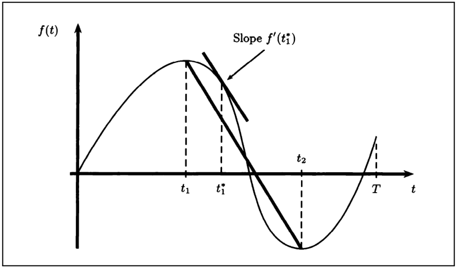
$$
\begin{aligned}
df &= f'(x)dx \\
df &= f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + \dots
\end{aligned}
$$

Taylors?

Variation

$$FV_T = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f(t_{j+1} - f(t_j| = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j)$$

$$= \int_0^T |f'(t)| dt$$



Second variation

What about second variation?

$$[f,f](T) \quad = \quad \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f(t_{j+1} - f(t_j)|^2 = 0$$

$$[W,W](T) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |W(t_{j+1} - W(t_j)|^2 = T.$$

Ito's formula

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2$$

and $(dW_t)^2 = dt$.

What is a martingale? How is it related to Hermite polynomias?

$\mathcal{H}_n(W_t, t) = t^{\frac{n}{2}} H_n\left(\frac{W_t}{\sqrt{t}}\right)$ *is a martingale!*

$$\mathbf{E}\left(\mathcal{H}_n(W_t, t) | \mathcal{F}_{\int}\right) = \mathcal{H}_n(W_s, s).$$