Chucking matrices into functions: why, when and how?

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Let $\lambda_1, \dots, \lambda_n \in \mathbb{C} \setminus \{0\}$.

Consider system:

$$(\partial_t V_i)(t) = -\lambda_i V(t), \quad V_i(0) = v_i. \tag{1}$$

Solution:

$$V_i(t) = \mathrm{e}^{-t\lambda_i} v_i. \tag{2}$$

Alternative perspective:

$$\Lambda := \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} \tag{3}$$

$$(\partial_t V)(t) = -\Lambda V(t), \quad v := V(0) = (v_1, \dots, v_n),$$

Immediate: (1) $\iff$ (3).

Write

$$e^{-t\Lambda} := \begin{pmatrix} e^{-t\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-t\lambda_n} \end{pmatrix},$$

and

$$V(t) = e^{-t\Lambda} v. \tag{4}$$

Then (4) $\iff$ (2).

A coupled system

$$
\begin{aligned}
(\partial_t V_1)(t) &= \pi V_1(t) + 7V_2(t) + \mathrm{e}^{10} V_3(t) + 29V_4(t) \\
(\partial_t V_2)(t) &= 17V_1(t) + 5V_2(t) - 3V_3(t) + \pi^2 V_4(t) \\
(\partial_t V_2)(t) &= 75V_1(t) - 10V_2(t) + 11V_3(t) + 4V_4(t) \\
(\partial_t V_2)(t) &= \pi^3 V_1(t) + 12V_2(t) + 75V_3(t) - 42V_4(t),
\end{aligned}
$$

such that

$$
V_1(0) = \mathrm{e}^3, \quad V_2(0) = -7, \quad V_3(0) = \sqrt{\pi}, \quad V_4(0) = 55.
$$

Generally: $a_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq n$ and consider:

$$
(\partial_t V_i)(t) = \sum_{j=1}^{n} a_{ij} v_j(t), \quad V_i(0) = v_i \tag{5}
$$

Vectorise!!!

$$
A := (a_{ij}), \quad (\partial_t V)(t) = AV(t), \quad V(0) = v = (v_1, \ldots, v_n). \tag{6}
$$

Assume: A diagonalisable. I.e., there exists invertible matrix P such that $A = P^{-1}DP$, where

$$D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}.$$

Then, equation 6

$$\begin{aligned}(\partial_t V)(t) = P^{-1}DPV(t) &\iff P(\partial_t V)(t) = DPV(t) \\ &\iff (\partial_t PV)(t) = DPV(t) \\ &\iff \partial_t W(t) = DW(t).\end{aligned}$$

I.e., (6) $\iff$

$$\begin{aligned} &V(t) = PW(t) \\ &(\partial_t W)(t) = DW(t), \quad W(0) = PV(0) = Pv := w \end{aligned} \tag{7}$$

Already know how to solve (7) by (3):

$$W(t) = \mathrm{e}^{-tD} w \iff PV(t) = \mathrm{e}^{-tD} Pv \iff V(t) = P^{-1} \mathrm{e}^{-tD} Pv.$$

Motivated by this: $A = P^{-1}DP$ and $f : \{\lambda_1, \dots, \lambda_n\} \to \mathbb{C}$,

$$f(A) := P^{-1} f(D) P \quad \text{where} \quad f(D) = \begin{pmatrix} f(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f(\lambda_n) \end{pmatrix}.$$

Suggestive:

- eigenvalues and eigenvectors are crucial;
- functions need to be defined on eigenvalues.

A $\mathbb{C}$omplex beginning

Let $\Omega \subset \mathbb{C}$ an open set. Recall $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* if for $z \in \Omega$,

$$\lim_{w \to z} \frac{f(z) - f(w)}{z - w}$$

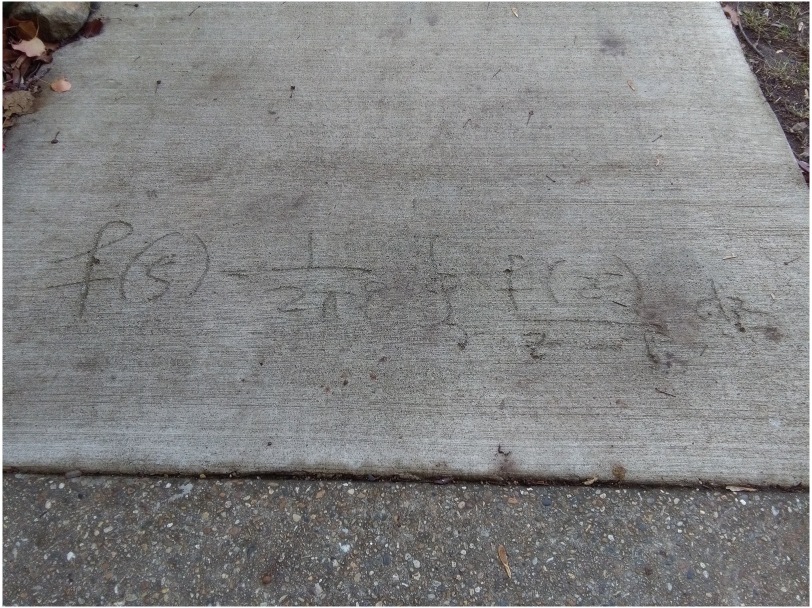
exists.

Such functions satisfy the (in)famous *Cauchy integral formula*:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

γ closed curve parametrised anti-clockwise enveloping z inside Ω.

Real life example



Turn this on its head: fix $z \in \mathbb{C}$, let Ω open with $z \in \Omega$. Let $\mathrm{Hol}(\Omega)$ be all $f : \Omega \to \mathbb{C}$ holomorphic.

Define $M_z : \mathbb{C} \to \mathbb{C}$ by

$$M_z\xi := z\,\xi.$$

M_z is a as 1×1 matrix.

From Cauchy integral formula:

$$M_z\xi = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\zeta}{\zeta - z}\,\xi\,d\zeta = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \zeta(\zeta - z)^{-1}\,\xi\,d\zeta.$$

$$(\zeta - z)\xi = \zeta\xi - z\xi = \zeta\xi - M_z\xi = (\zeta\mathrm{I} - M_z)\xi$$

$$\implies \frac{1}{\zeta - z} \cdot \xi = (\zeta - z)^{-1} \cdot \xi = (\zeta\mathrm{I} - M_z)^{-1}\xi$$

So,

$$M_z\xi = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \zeta(\zeta\mathrm{I} - M_z)^{-1}\xi \, d\zeta.$$

Perspective: M_z is 1×1 matrix,

$$\exists \xi \in \mathbb{C} \setminus \{0\} \quad (\lambda - M_z)\xi = 0 \iff \lambda = z.$$

Eigenvalues: $\mathrm{spec}(M_z) = \{z\}$.

Define for $f \in \mathrm{Hol}(\Omega)$, $f(M_z) : \mathbb{C} \to \mathbb{C}$,

$$f(M_z)\xi := \frac{1}{2\pi\mathrm{i}} \oint_\gamma f(\zeta)(\zeta\mathrm{I} - M_z)^{-1}\xi \, d\zeta,$$

γ parametrised anti-clockwise enveloping $\mathrm{spec}(M_z)$ in Ω.

Matrices!

Map $h = z \mapsto (h_1(z), \dots, h_N(z)) : \Omega \to \mathbb{C}^N$ holomorphic if each $h_i : \Omega \to \mathbb{C}$ holomorphic and

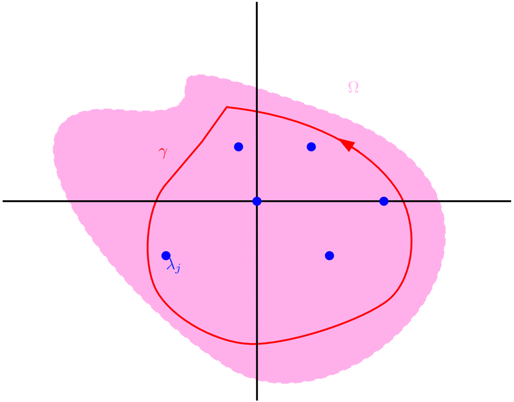
$$\oint_\gamma h(\zeta)\, d\zeta := \left(\oint_\gamma h_1(\zeta)\, d\zeta, \dots, \oint_\gamma h_N(\zeta)\, d\zeta \right).$$

Let $M : \mathbb{C}^N \to \mathbb{C}^N$ matrix.

$\operatorname{spec}(M) = \{\lambda_1, \dots, \lambda_N\}$ eigenvalues of M.

$\Omega \subset \mathbb{C}$ open with $\operatorname{spec}(M) \subset \Omega$.

Closed curve γ enveloping $\operatorname{spec}(M)$ in Ω.



For $f \in \mathrm{Hol}(\Omega)$, define $f(M) : \mathbb{C}^N \to \mathbb{C}^N$,

$$f(M)v := \frac{1}{2\pi \mathrm{i}} \oint_\gamma f(\zeta)(\zeta \mathrm{I} - M)^{-1} v \, d\zeta.$$

Properties:

- $1(M) = \mathrm{I}$, where $1(z) = 1$,
- $\mathrm{id}(M) = M$, where $\mathrm{id}(z) = z$,
- $R_\zeta(M) = (\zeta \mathrm{I} - M)^{-1}$ for $\zeta \notin \mathrm{spec}(M)$, where $R_\zeta(z) = (\zeta \mathrm{I} - z)^{-1}$,
- $f \mapsto f(M) : \mathrm{Hol}(\Omega) \to \mathrm{Matrices}(\mathbb{C}^N)$ homomorphism.

A familiar sight

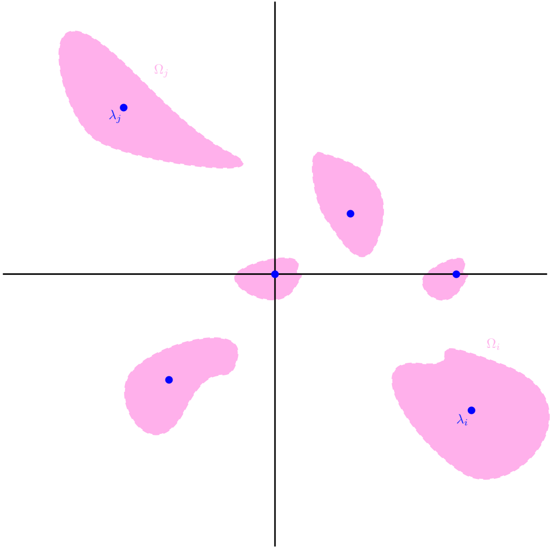
$\mathrm{spec}(M) = \{\lambda_1, \ldots, \lambda_N\}$.

Ω_j open subsets of $\mathbb{C}$.

$\Omega_j \cap \Omega_i = \varnothing$ for $i \neq j$.

$\lambda_j \in \Omega_j$.

$\Omega := \cup_{i=1}^{N} \Omega_j$.



Define $f_j : \Omega \to \mathbb{C}$:

$$f_j(\zeta) := \begin{cases} 1 & \zeta \in \Omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

$f_j^2 = f_j$, so $f_j(M)^2 = f_j(M)$, i.e., a projection.

$1 = \sum_{j=1}^{N} f_j(\zeta)$, so

$$v = \sum_{j=1}^{N} f_j(M)v, \quad Mv = \sum_{j=1}^{N} \lambda_j f_j(M)v \quad \text{and} \quad f(M)v = \sum_{j=1}^{N} f(\lambda_j) f_j(M)v.$$

where $f \in \mathrm{Hol}(\Omega)$.

Jordan decomposition of M!

Back to our evolution equation

Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a matrix. System of ODEs:

$$(\partial_t V)(t) = AV(t), \quad V(0) = v.$$

Solution in one stroke!

$$V(t) = \mathrm{e}^{-tA} v,$$

where $f(z) := \mathrm{e}^{-z}$ and $e^{-tA} = f(tA)$.

What's with matrices anyway?

$M = (m_{ij}) : \mathbb{C}^n \to \mathbb{C}^n$ matrix.

For $\alpha_i \in \mathbb{C}$, and $u_i \in \mathbb{C}^n$

$$M\left(\sum_{i=1}^{N} \alpha_i u_i\right) = \sum_{i=1}^{N} \alpha_i M u_i.$$

I.e., M is linear.

Other way?

I.e., given $L : \mathbb{C}^N \to \mathbb{C}^N$ linear $\implies$ there exists $M_L : \mathbb{C}^N \to \mathbb{C}^N$ matrix such that $L = M_L$?

Let $u = (u_1, \dots, u_N) \in \mathbb{C}^N$.

Then $u = \sum_{i=1}^{N} u_i e_i$ where $e_i = \underbrace{(0, \dots, 1, \dots, 0)}_{i\text{-th slot is } 1}$.

$$Lu = L\left(\sum_{i=1}^{N} u_i e_i\right) = \sum_{i=1}^{N} u_i L e_i.$$

Define

$$\begin{pmatrix} l_{i1} \\ \vdots \\ l_{iN} \end{pmatrix} := Le_i \quad \text{and} \quad M_L := \begin{pmatrix} l_{11} & \dots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{nn} \end{pmatrix}.$$

Then $Le_i = M_L e_i$ and $Lu = M_L u$.

☣ Linearity of a map is *intrinsic* property, matrix is a *representation* of such a map. ☣

Eigenvalues?

Linear map $L : \mathbb{C}^N \to \mathbb{C}^N$.

Consider $\zeta \in \mathbb{C}$ such that $(\zeta I - L)^{-1}$ exists.

This is the *resolvent set* denoted $\mathrm{res}(L)$ open set.

The *spectrum* is $\mathrm{spec}(L) = \mathbb{C} \setminus \mathrm{res}(L)$. These are the *eigenvalues* of any matrix representing L.

Open set $\Omega \subset \mathbb{C}$ with $\mathrm{spec}(L) \subset \Omega$ and $f : \Omega \to \mathbb{C}$ holomorphic

$$f(L)v := \frac{1}{2\pi \mathrm{i}} \oint_\gamma f(\zeta)(\zeta \mathrm{I} - L)^{-1} v \, d\zeta.$$

See that $f(L) = f(M_L)$.

Yellow brick road to infinite dimensions...

Set $\mathcal{V}$ is a *linear space* (or *vector space*) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ if $(\mathcal{V}, +)$ commutative group with $\cdot : \mathbb{K} \times \mathcal{V} \to \mathcal{V}$ (scalar multiplication):

$$\alpha(\beta u) = (\alpha\beta)u, \quad 1u = u, \quad \alpha(u+v) = \alpha u + \alpha v, \quad (\alpha+\beta)u = \alpha u + \beta u$$

for all $\alpha, \beta \in \mathbb{K}$ and $u, v \in \mathcal{V}$.

Examples

1) $\mathcal{V} = \mathbb{C}^N$ or $\mathcal{V} = \mathbb{R}^n$;
2) $\mathcal{V} = \mathrm{C}^0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous}\}$;
3) $\mathcal{V} = \mathrm{L}^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty \right\}$.

Linear operators on vector spaces

$L : \mathbb{C}^N \to \mathbb{C}^N$ linear is automatically continuous. I.e.

$$u_n \to u \implies Lu_n \to Lu \quad \iff \quad |u_n - u|_{\mathbb{C}^N} \to 0 \implies |Lu_n - Lu|_{\mathbb{C}^N} \to 0.$$

In general, $L : \mathcal{V} \to \mathcal{V}$ is a linear operator if

$$L(\alpha u + \beta v) = \alpha L u + \beta L v$$

for all $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathcal{V}$.

☣ Need a notion of norm (size of a vector). ☣

Norms!

Norm $\|\cdot\|:\mathcal{V}\to[0,\infty)$ satisfying:

$$\|\alpha u\|=|\alpha|\|u\|,\quad \|u+v\|\le\|u\|+\|v\|,\quad \|u\|=0\implies u=0.$$

Examples

1) $\mathcal{V}=\mathbb{C}^N$ or $\mathcal{V}=\mathbb{R}^n$, any two norms are equivalent;
2) $B_R\subset\mathbb{R}^n$ an R-ball , $\mathcal{V}=\mathrm{C}^0(B_R)$, $\|f\|_{\mathrm{C}^0}=\max_{x\in B_R}|f(x)|_{\mathbb{C}^N}$;
3) $\mathcal{V}=\mathrm{L}^p(\mathbb{R}^n)$, $\|f\|_{\mathrm{L}^p}:=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{\frac{1}{p}}$. On $\mathbb{C}^N$, $u\cdot u=|u|^2_{\mathbb{C}^N}$. For $\mathrm{L}^2(\mathbb{R}^n)$, *inner product* $\langle u,v\rangle=\int_{\mathbb{R}^n}u(x)\overline{v(x)}\,dx$, satisfies

$$\|u\|^2_{\mathrm{L}^2}=\int_{\mathbb{R}^n}|u(x)|^2\,dx=\int_{\mathbb{R}^n}u(x)\overline{u(x)}\,dx=\langle u,u\rangle\,.$$

Continuity

$$L : \mathcal{V} \to \mathcal{V} \text{ linear and continuous} \iff \|u_n - u\| \to 0 \implies \|Lu_n - Lu\| \to 0$$
$$\iff \exists C < \infty \, \forall u \in \mathcal{V} \qquad \|Lu\| \le C\|u\|.$$

Spectral theory:

$$\mathrm{res}(L) = \{\zeta \in \mathbb{C} : (\zeta \mathrm{I} - L)^{-1} \text{ exists and continuous } \}$$
$$\mathrm{spec}(L) = \mathbb{C} \setminus \mathrm{res}(L).$$

Note: L need not be continuous or invertible.

$\mathrm{res}(L)$ open $\iff$ $\mathrm{spec}(L)$ closed.

Functional calculus for continuous operators

Theorem: $\mathcal{V}$ with norm $\|\cdot\|$, $L : \mathcal{V} \to \mathcal{V}$ continuous linear operator $\implies$ $\exists R > 0 \quad \mathrm{spec}(L) \subset B_R$.

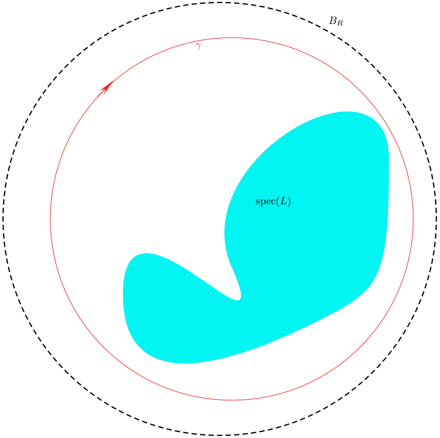
Proceed as before: Ω open set $\mathrm{spec}(L) \subset \Omega$
$f : \Omega \to \mathbb{C}$ holomorphic.

γ smooth curve parametrised anti-clockwise enveloping $\mathrm{spec}(L)$ inside Ω,

$$f(L)u := \frac{1}{2\pi \mathrm{i}} \oint_\gamma f(\zeta)(\zeta \mathrm{I} - L)^{-1} u \, d\zeta.$$

Satisfies:

$1(L) = \mathrm{I}, \quad \mathrm{id}(L) = L, \quad (\zeta\mathrm{I}-\mathrm{id})^{-1}(L) = (\zeta\mathrm{I}-L)^{-1}$



Generalisation of symmetric matrices

Symmetric matrix: $A = A^T \iff Au \cdot v = u \cdot Av$.

$\mathcal{V}$ vector space $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ *inner product*.

$L : \mathcal{V} \to \mathcal{V}$ *self-adjoint*: $\langle Lu, v \rangle = \langle u, Lv \rangle \implies \mathrm{spec}(L) \subset \mathbb{R}$.

Suppose $\mathrm{spec}(L) = \{\lambda_i\}_{i \in \mathbb{Z}}$ discrete with accumulation point at 0

For $Le_k = \lambda_k e_k$ have $\langle e_i, e_j \rangle = 0$ if $Le_k = \lambda_k e_k$ and for $u \in \mathcal{V}$, have $u = \sum_i u_i e_i$.

Take $\Omega = \cup_i \Omega_i$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$and $f_i : \Omega \to \mathbb{C}$, $f_i(z) = \begin{cases} 1 & z \in \Omega_i \\ 0 & z \notin \Omega_i \end{cases}$

$\implies f_i(L)^2 = f_i(L)$ projection and $f(L)u = \sum_i f(\lambda_i) u_i e_i$.

Beyond bounded operators - Fourier series

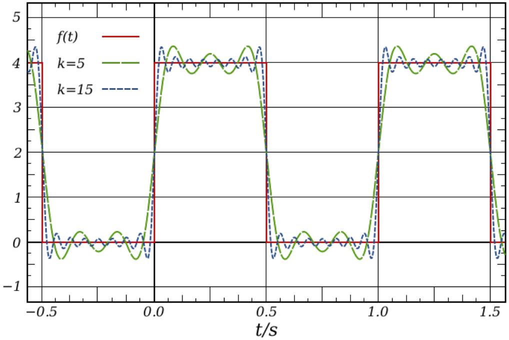
$u \in C^{\infty}(\mathbb{S}^1), \exists a_n \in \mathbb{C}$:

$$u(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Consider $\mathbb{S}^1 = {}^{[0,1]}\!/\!_{\sim}$ with $0 \sim 1$.

Laplacian: $\Delta_{\mathbb{S}^1} = -\frac{d^2}{d\theta^2}$ with periodic boundary conditions,

$$(\Delta_{\mathbb{S}^1} u)(\theta) = \sum_{n=-\infty}^{\infty} a_n n^2 e^{in\theta}.$$



Eigenvalues: $\left\{\lambda_n = n^2\right\}_{n=0}^{\infty}$, and corresponding eigenfunctions: $\left\{e^{in\theta}, e^{-in\theta}\right\}_{n=0}^{\infty}$.

Symbol - appropriate function f:

$$(f(\Delta_{\mathbb{S}_1})u)(\theta) := \sum_{n=-\infty}^{\infty} f(n^2) a_n e^{in\theta}.$$

Heat equation:

$$(\partial_t U)(t,\theta) = -\Delta_{\mathbb{S}^1} U(t,\theta), \quad \lim_{t\to 0} U(t,\theta) = u(\theta).$$

Unique solution:

$$u(t,\theta) = \left(e^{-t\Delta_{\mathbb{S}^1}} u\right)(\theta) = \sum_{n=-\infty}^{\infty} e^{-tn^2} a_n e^{-in\theta}.$$

More than one operator?

Form $f(L_1, L_2, \ldots, L_n)$ given $L_1, \ldots, L_n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$?

Let $\{e_0, e_1, \ldots, e_n\}$ standard basis for $\mathbb{R} \times \mathbb{R}^n$. Define algebra:

$$\begin{aligned} e_0 &= 1 \\ e_j^2 &= -e_0 \quad 1 \le j \le n \\ e_i e_j &= -e_j e_i \quad 1 \le i < j \le n \end{aligned}$$

$S = \{j_1 < j_2 < \cdots < j_k\}$ define $e_S = e_{j_1} e_{j_2} \ldots e_{j_k}$, $e_\varnothing = e_0$.

$\mathbb{R}_{(n)}$ *Clifford algebra* defined

$$\mathbb{R}_{(n)} = \operatorname{span}\{e_S : S \subset \{1, \ldots, n\}\}.$$

Examples $\mathbb{R} = \mathbb{R}_{(0)}$, $\mathbb{C} = \mathbb{R}_{(1)}$, $\mathbb{R}_{(2)}$ - *quaternions*.

Clifford conjugate for e_S is $\overline{e}_S$ satisfying $\overline{e}_S e_S = 1 = e_S \overline{e}_S$.

$\mathbb{R}^{n+1} \subset \mathbb{R}_{(n)}$ via $(x_0, \ldots, x_n) \mapsto \displaystyle\sum_{j=0}^{n} x_j e_j$.

Kelvin inverse of $x \in \mathbb{R}_{(n)}$

$$x^{-1} = \frac{\overline{x}}{|x|^2} = \frac{x_0 - \sum_{j=1}^{n} x_j e_j}{\sum_{j=0}^{n} x_j^2}.$$

Notion generalising holomorphic: $f : \mathbb{R}_{(n)} \to \mathbb{R}_{(n)}$ *monogenic*.

Generalised Cauchy integral formula:

$$f(x) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n+1})} \int_{\partial\Omega} \frac{\overline{x}}{|x-\omega|^{n+1}} \nu(\omega) f(\omega)\ d\mu_{\partial\Omega}(\omega).$$

Need: notion of *joint* spectrum for $L_1, \ldots, L_n$.

Leads to $f(L_1, \ldots, L_n)$ via generalised Cauchy integral formula.