Emergence:
From tornados to Hilbert spaces

$$\mathbb{P}: \quad \mathcal{H} \to \mathcal{Q} \subset \mathcal{H}$$

$$a \to \mathbb{P}a$$

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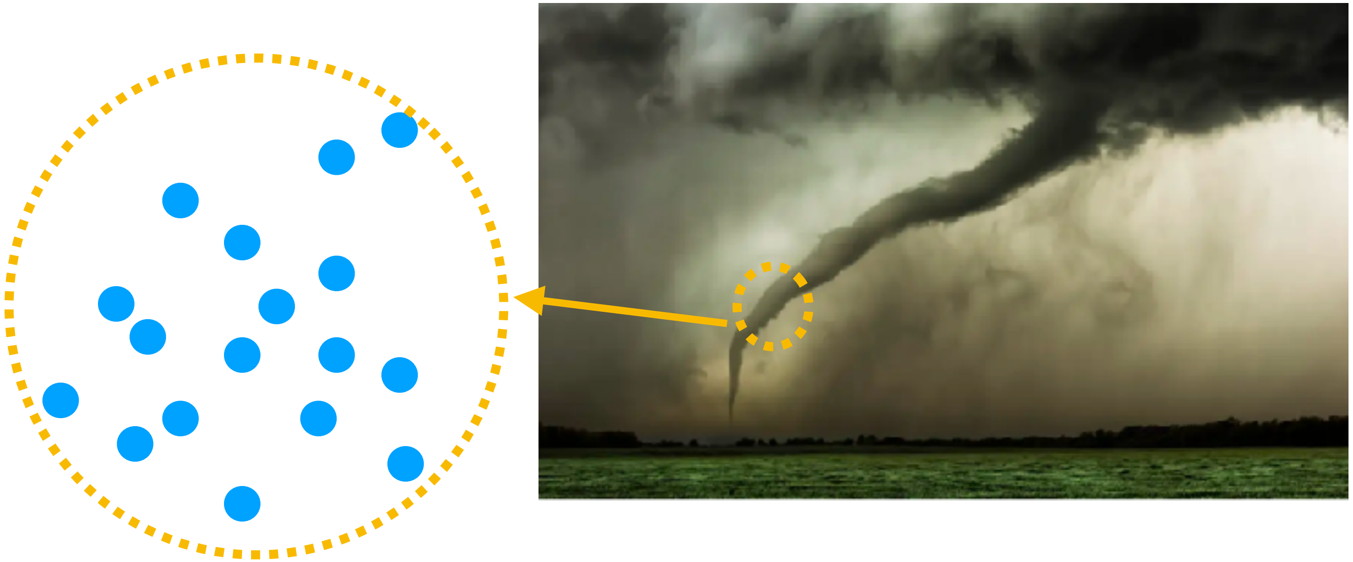
Emergence

- What is a tornado, fundamentally?
- Is it many, many molecules with trajectories $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t), \ldots, \vec{x}_{10^{26}}(t)$, flying around and interacting a little bit?

$$m\frac{d^2\vec{x}_i(t)}{dt^2} = \sum_j \vec{F}(\vec{x}_i(t) - \vec{x}_j(t)) \quad \text{(Newton's equations)}$$

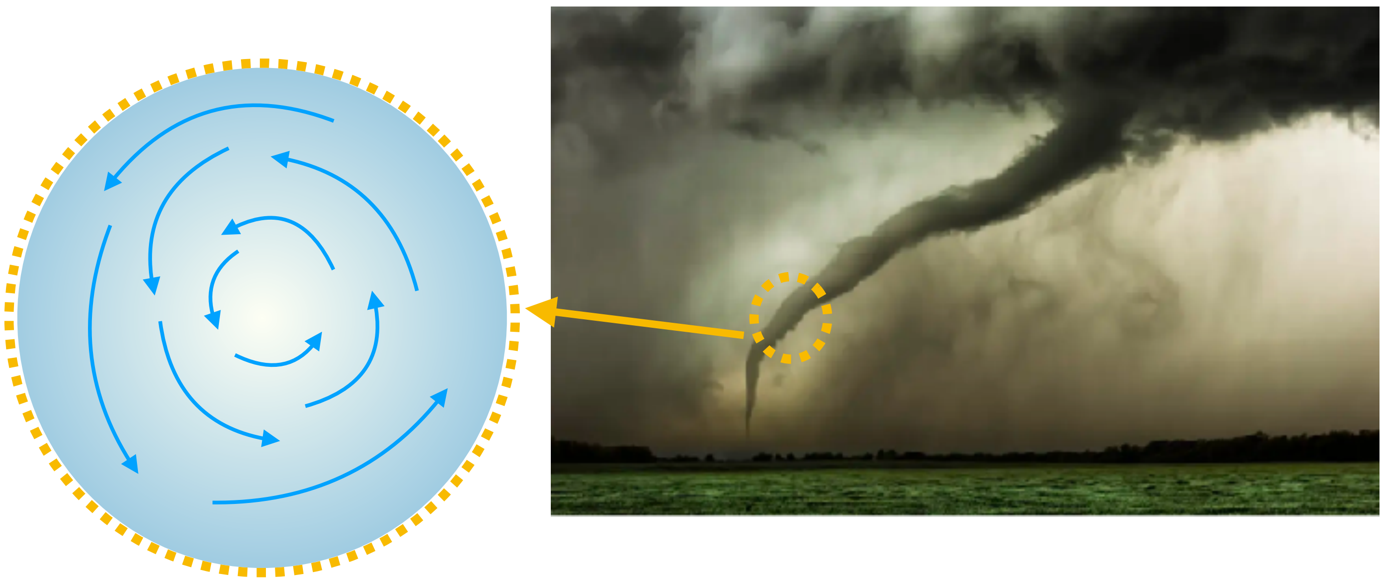


Emergence

- Or is it a gas or fluid, with some density $\rho(\vec{x}, t)$ and velocity field $\vec{v}(\vec{x}, t)$ flowing around smoothly into a nice whirling structure?

$$\frac{d\rho}{dt} + \vec{\nabla} \cdot (\vec{v}\rho) = 0$$
$$\frac{d\vec{v}}{dt} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} w(\rho) + \vec{g}$$

(Euler's equations)



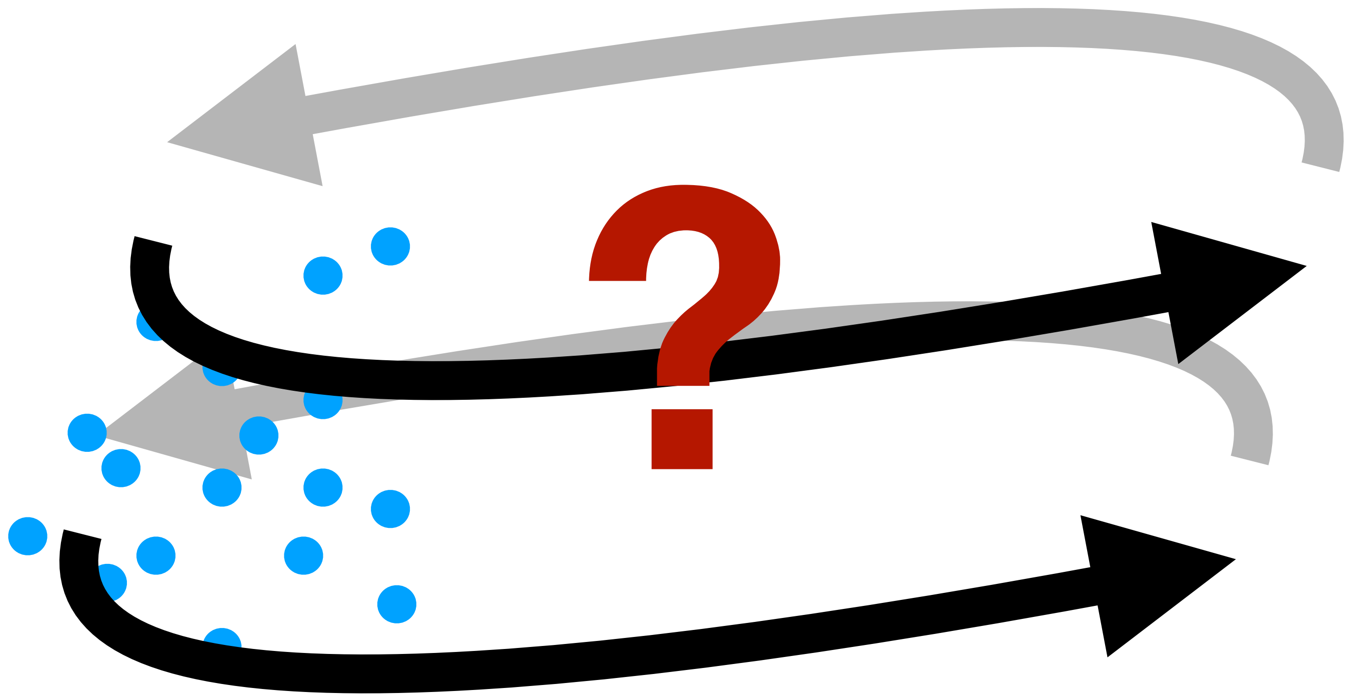
Emergence

- This is the dichotomy between the **discrete** and the **continuum**
- It dates back to ancient Greece: atomism by Leucippus and Democritus (c. 5th BC), then emphasised by Boyle and Newton (c. 1700); and theory of “substance” by Aristotle (c. 4 BC), popular in medieval Europe (for philosophical / theological reasons, not very scientific...).
- In fact **both are true** – in an appropriate sense...

Emergence

- But how can all these 10^{26} molecules, following basically their own little straight trajectories until the next quick interaction, organise themselves to make this structure at large scales?

Emergence of structures



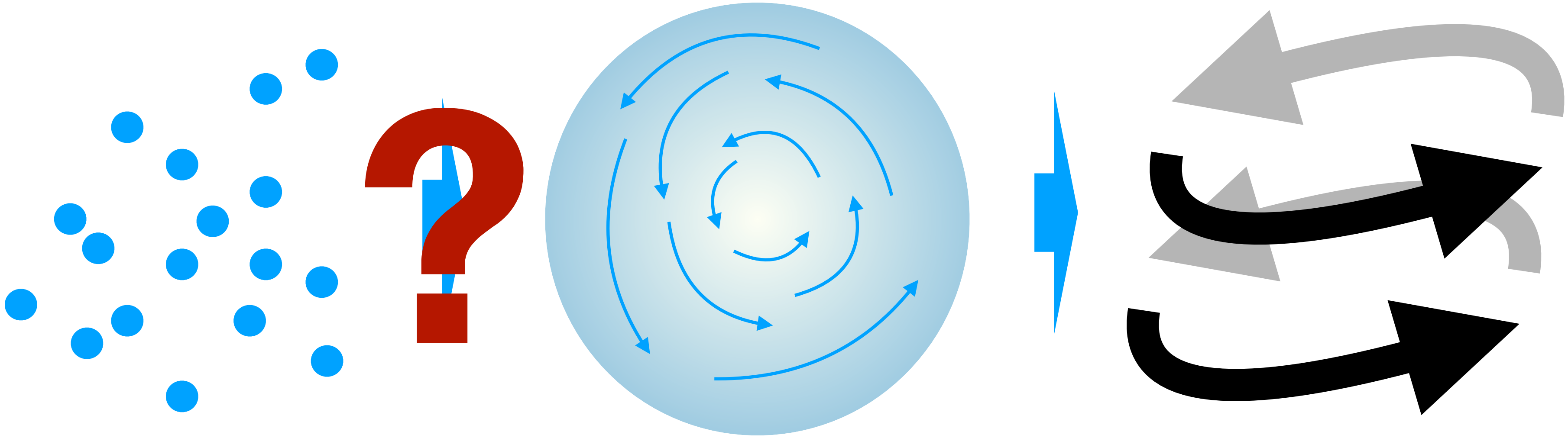
Emergence

- Or, why is it possible for us to describe this emergent structure using much less equations than the 10^{26} Newton equations, just the few Euler equations?

Reduction of the apparent number of degrees of freedom

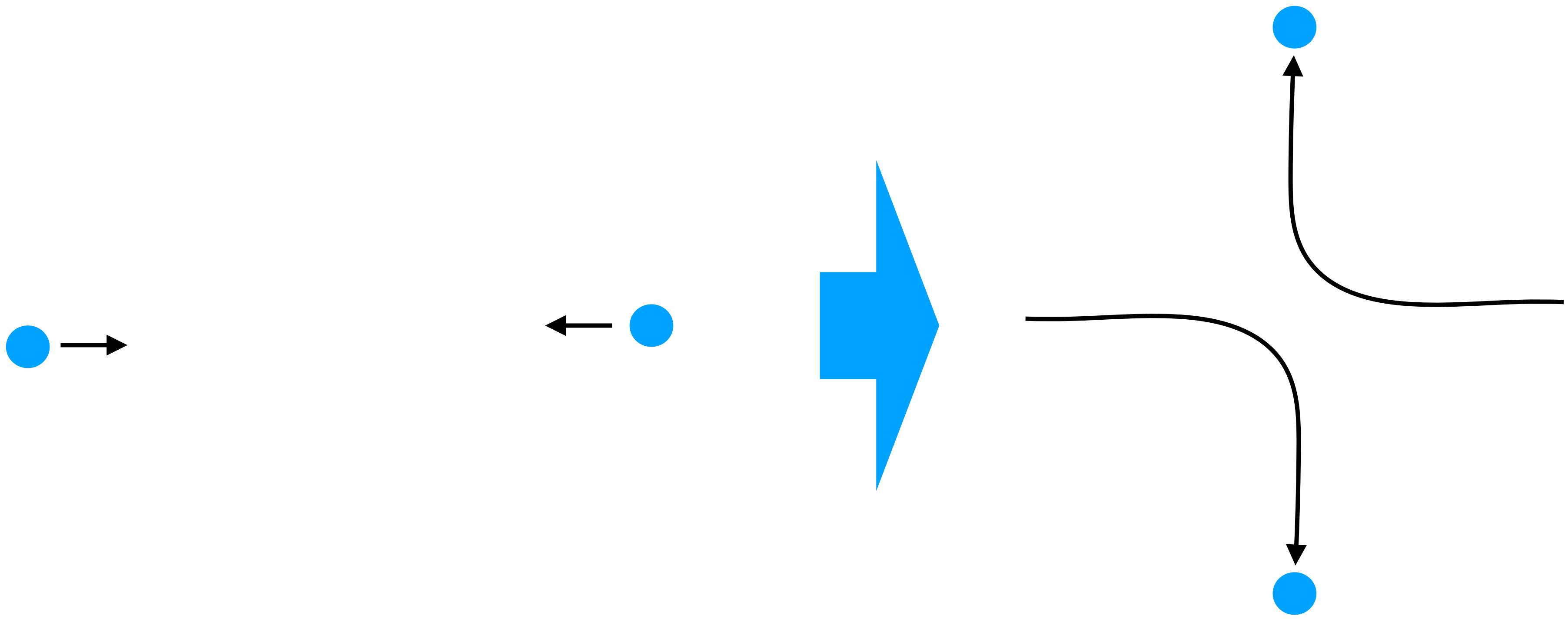
and emergence of new physical laws.

Microscopic particles, what is "really" going on, Newton's laws of motion	**Emergent physical laws for what we "see" at large scales, Euler's fluid equations**	**Structures can be explained by these emergent laws**



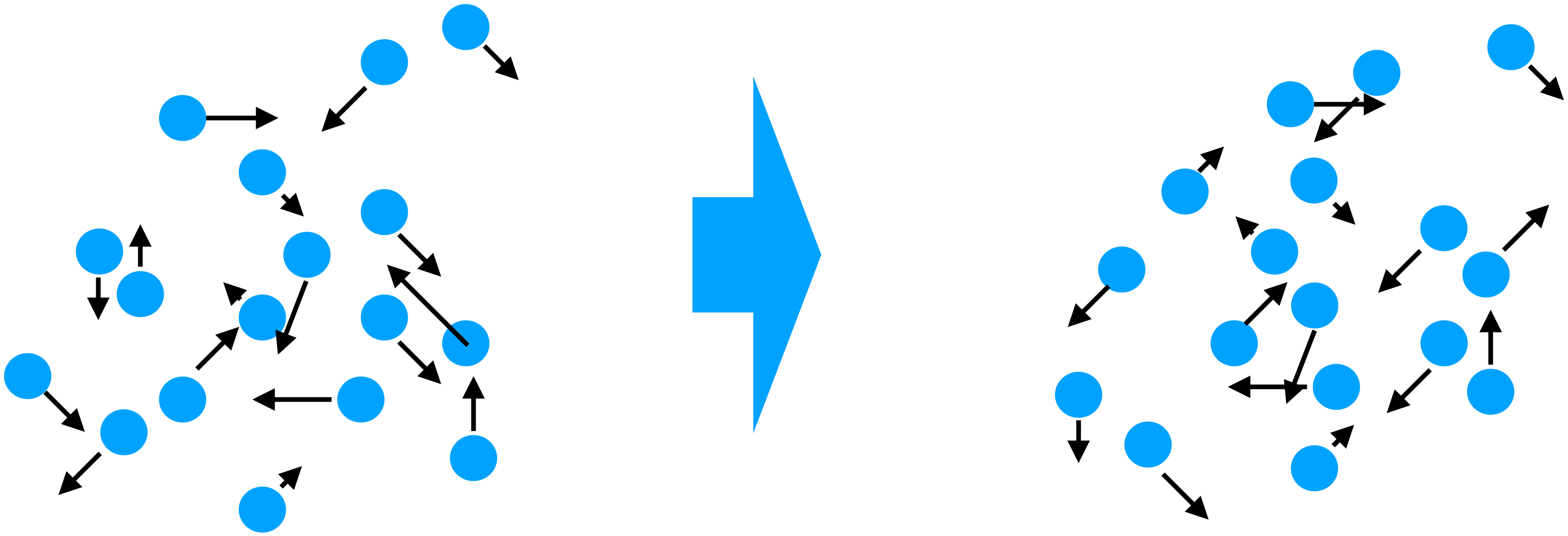
Small-scale randomness: loosing the plot

- The idea to understand emergence is to consider some kind of **randomness**.
- Trajectories are not random: they follow the very deterministic Newton's laws of motion. Once you know the initial positions and velocities of all particles, all trajectories are determined, forever!!!



Small-scale randomness: loosing the plot

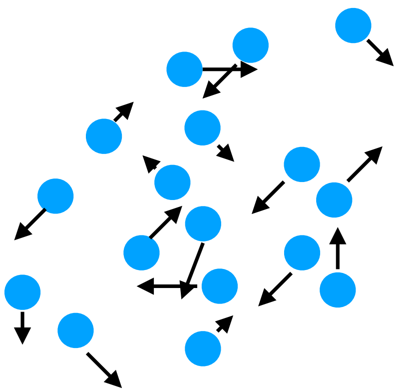
- But when there are so many of particles, all of them going in different directions and at different velocities, and interacting at pair collisions from time to time, overall the whole will quickly look somewhat random, and more or less the same, statistically, over time.



Small-scale randomness: loosing the plot

- Maxwell, Boltzmann and others realised, end of 1800's / beginning of 1900's, that **most likely, the particles will distribute more or less uniformly in space, with more or less uniformly distributed directions, and with some more or less Gaussian distribution of speeds**

$$\text{Probability}(\vec{v}) \propto \exp\left[-\frac{m|\vec{v}|^2}{2T}\right]$$



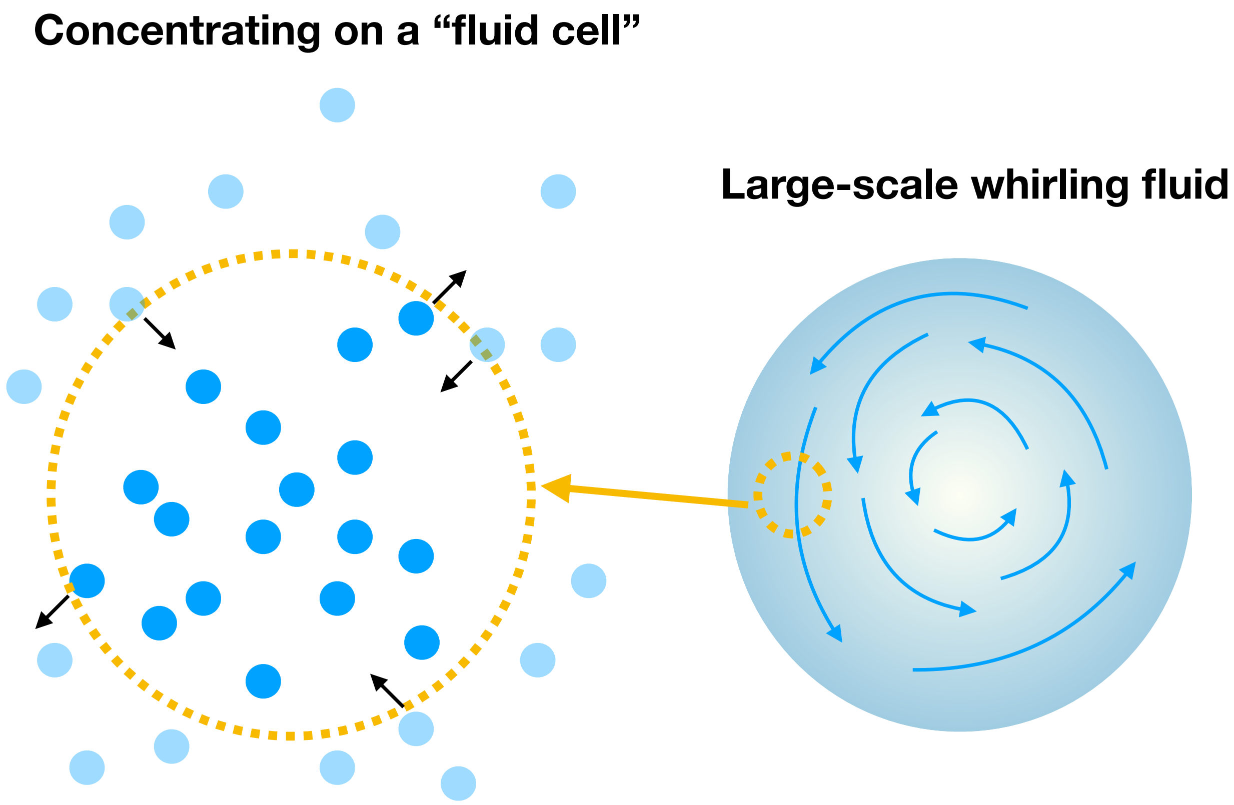
- It is certainly possible that, as a consequence of (blindly!) following Newton's laws, at some point in time all particles in the gas suddenly go almost all parallel to each other in some direction, but this is extremely, extremely unlikely!

Large-scale order: finding your way

- We have lost almost all information about trajectories! But what **does** remain?

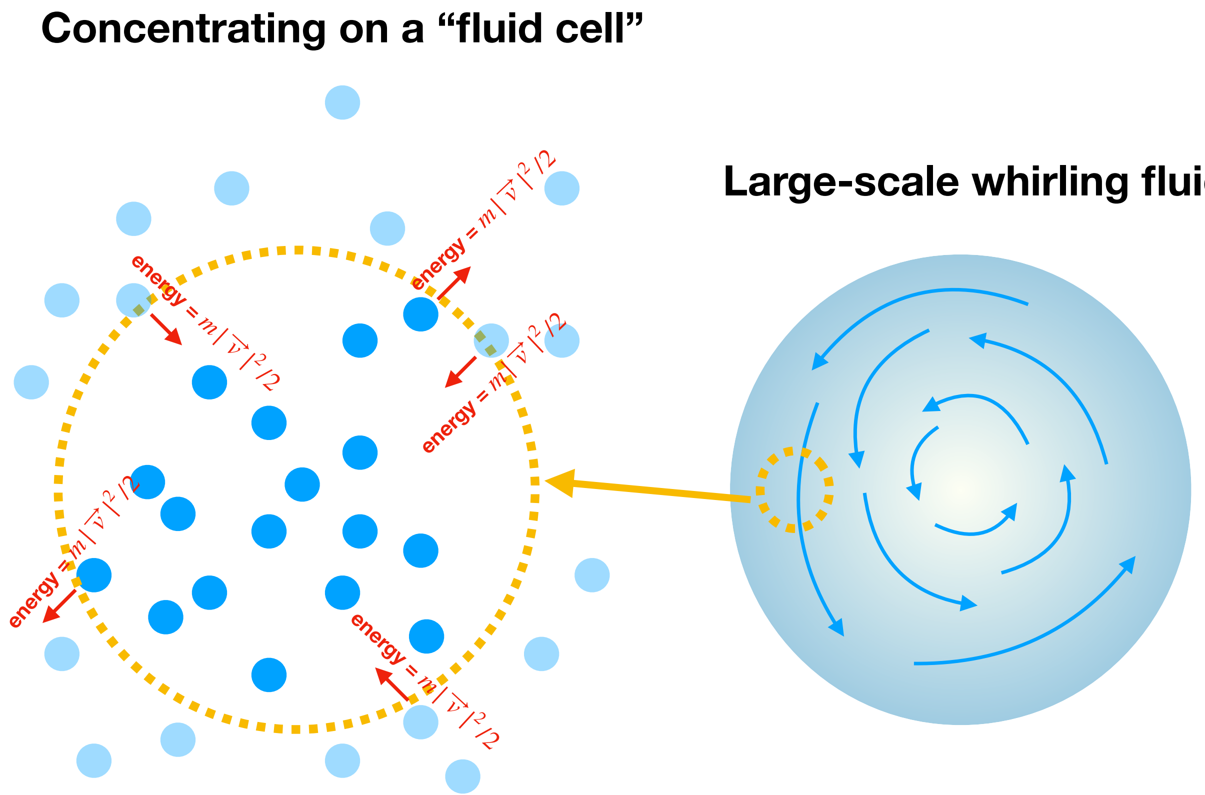
Large-scale order: finding your way

- No matter how particles collide with each other, their **number doesn't change**. Looking at some small region, say 1cm^3, the number can only change by **"surface effects": particles coming in and going out of the region**. Because **surface** $\ll$ **volume**, this is much slower than how particles move and collide. This is an **emergent degree of freedom**, which changes at a completely different scale than the microscopic particles!



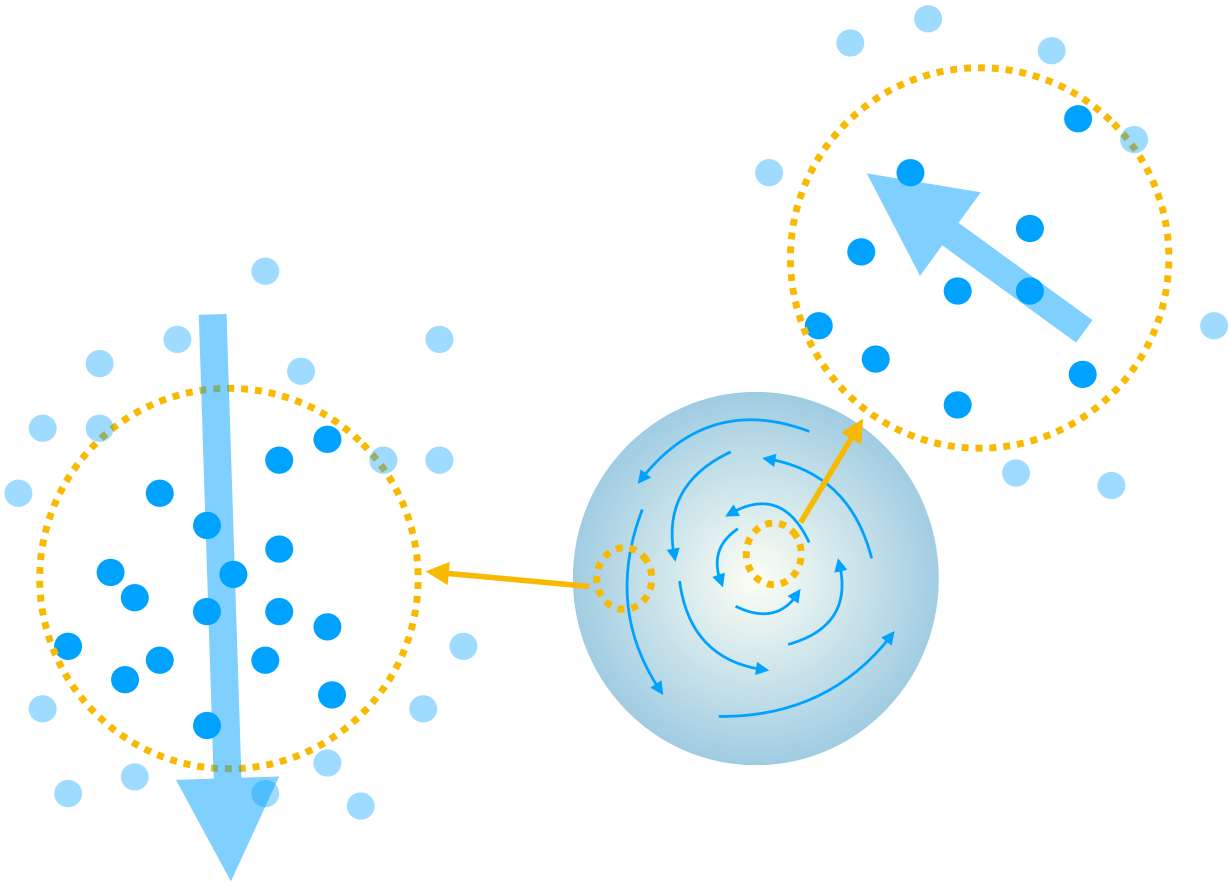
Large-scale order: finding your way

- The same goes for anything that is **conserved by Newton's laws**: the total energy, the total velocity (or center-of-mass velocity). These are other **emergent degrees of freedom**, which also change at a completely different scale than the microscopic particles.



Large-scale order: finding your way

- In this way we have **fluid cells** in space-time, and **fields** $\rho(\vec{x}, t), \ \vec{v}(\vec{x}, t)$ that give the number of particles and velocity (and also energy) of each cell.

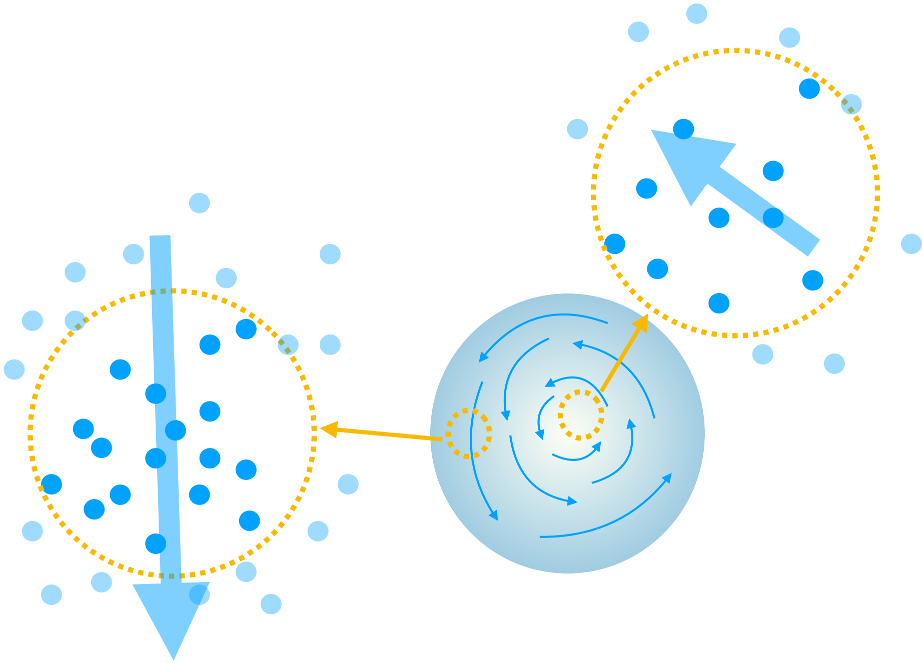


Large-scale order: finding your way

- Newton's equations then translate into **equations for these fields**, the Euler equations, just by an analysis of particles coming in and out by the surface of the cell. These are the emergent equations that describe the tornado.

Newton's equations + surface effects for slowly changing quantities

= Euler's equations



The math: from small scales to large scales by Hilbert spaces and projections

- But how do we know that we got **all the slowly-moving emergent degrees of freedom**?
- We have to revert to a mathematically accurate theory of emergence...

The math: from small scales to large scales by Hilbert spaces and projections

- A good way is to form **a very big vector space $\mathcal{V}$ of all things we can observe in a fluid cell**

$$a_f = \sum_i f(\vec{x}_i, \vec{v}_i), \quad c_1 a_f + c_2 a_g = a_{c_1 f + c_2 g}$$

for instance the density is $\rho(\vec{x}) = a_{\delta(\cdot - x)} = \sum_i \delta(\vec{x}_i - \vec{x})$. [With appropriate bells and whistles, this forms a C^* algebra...]

- These things fluctuate rapidly as particles move and interact. Averaging over the fast motion [this needs the notion of statistical ensembles...], we get a **state**: a continuous linear functional on $\mathcal{V}$

$$\omega(a_f) = \mathbb{E}[a_f]$$

- Newton's equations describe how the state changes in time, $t \mapsto \omega_t$, or equivalently how observables change $t \mapsto a_f(t)$, and become a **big linear operator U_t on $\mathcal{V}$**

Newton's evolution: $\omega_t(a_f) = \omega(a_f(t)) = \omega(U_t(a_f))$

The math: from small scales to large scales by Hilbert spaces and projections

- We define an ($\vec{x}$- and state-dependent) "inner product" by looking at the **statistical covariances under the rapid fluctuations inside fluid cells** [using statistical ensembles representing local fluid cells; this needs an operation of separation of scales...]

$$\langle a_f, a_g \rangle = \omega_{\vec{x}}(a_f a_g) - \omega_{\vec{x}}(a_f)\omega_{\vec{x}}(a_g), \quad \omega_{\vec{x}} = \omega_{\text{fluid cell at } \vec{x}}$$

  We get a Hilbert space $\mathcal{H}$, which has a lot of nice mathematical properties!

- We look at the **space of elements that have constant covariances** (they don't fluctuate rapidly)

$$\mathcal{Q} = \{q : \langle a_f, \frac{dq(t)}{dt} \rangle = 0 \, \forall \, a_f\}$$

- They still change slowly as elements of $\mathcal{V}$, because of particles coming in and going out of fluid cells, so they satisfy a **continuity equation**:

$$\frac{dq}{dt} + \vec{\nabla} \cdot \vec{j} = 0$$

The math: from small scales to large scales by Hilbert spaces and projections

- Now any a_f, as element of $\mathcal{H}$, can be seen as a vector field on the (very very big) space of $\omega' s$, with value $a_f(\omega) = \omega(a_f)$... Then the arguments of randomness mean that, when restricting to local observables, the Newton evolution essentially restricts to a **small manifold of what is observed at large scales**, and its tangent space is exactly $\mathcal{Q}$:

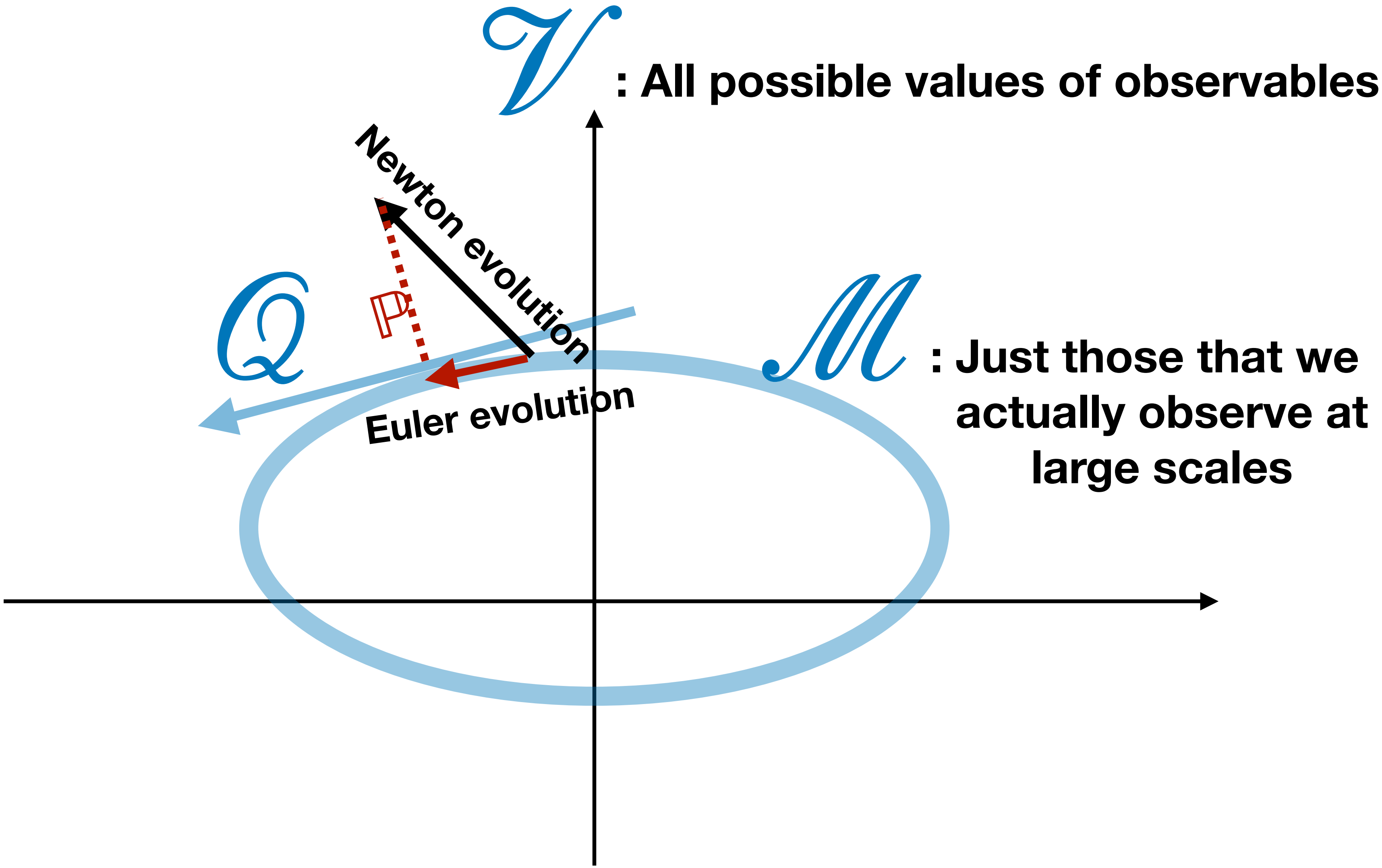
$$\mathcal{M} \subset \mathcal{V}, \quad T\mathcal{M} = \mathcal{Q}$$

- Euler's equations are obtained from Newton's equation by projecting $\mathbb{P} : \mathcal{H} \to \mathcal{Q}$ onto this tangent space after every infinitesimal time evolution:

$$\text{Euler's equations:} \quad \left(\frac{d}{dt}\right)_{\text{Euler}} \omega(a_f) = \dot{\omega}(\mathbb{P}a_f), \quad \dot{\omega}(q) = (\vec{\nabla}\omega) \cdot (\mathbb{P}\vec{j})$$

The math: $\dot{\omega}$ is a co-vector field on the space of local states, and thus can be contracted with elements of $\mathcal{H}$ to get a number. $\mathbb{P}a_f$ is the projection onto $\mathcal{Q}$ of the vector field $a_f \in \mathcal{H}$. The first equation says that to get the evolution of the state, we only need its projection onto $\mathcal{Q}$. Then the second equation says that this projection on $\mathcal{Q}$ is obtained by overlapping the other co-vector fields $\vec{\nabla}\omega$, the spatial state variations, onto the projection of the current $\vec{j}$ onto $\mathcal{Q}$.

The math: from small scales to large scales by Hilbert spaces and projections

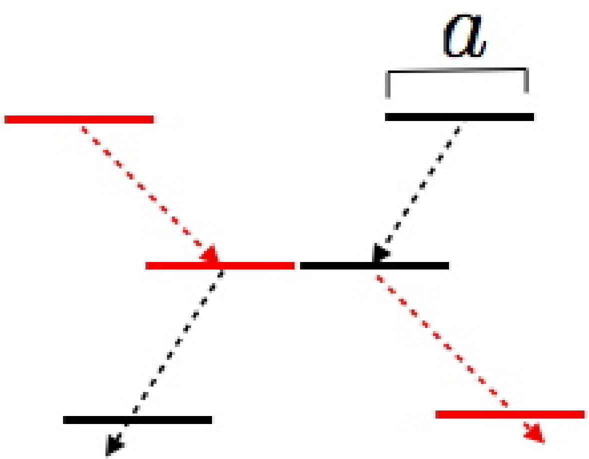


Emergence beyond the tornado...

- For instance, consider a simple system of rods hitting each other elastically

- At each collision they just exchange their velocities, so we have many more conserved quantities

$$a_n = \sum_i v_i^n \quad \text{for any } n$$



Emergence beyond the tornado...

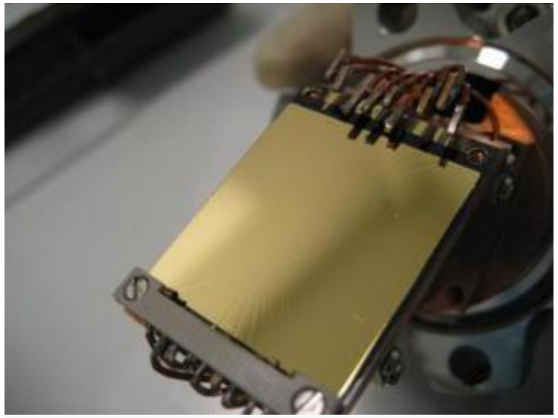
- The "Euler" equations we obtain are completely different

$$\frac{d}{dt}\rho(v,x,t) + \frac{d}{dx}(v^{\rm eff}(v,x,t)\rho(v,x,t)) = 0$$

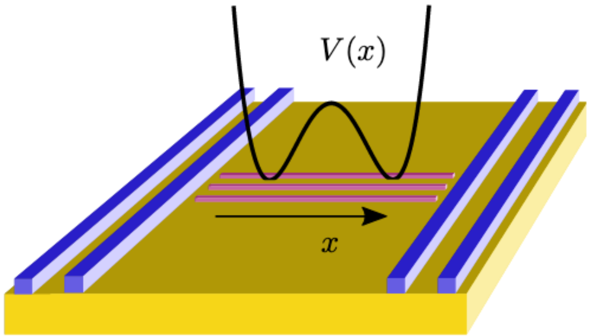
where $v^{\rm eff}(v,x,t)$ is a complicated functional of $\rho(\cdot,x,t)$....

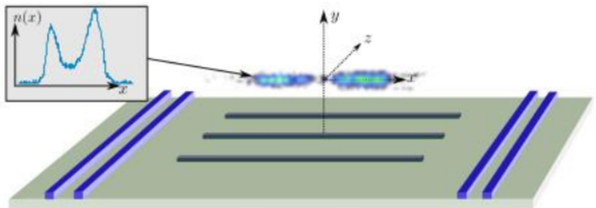
Emergence beyond the tornado...

- Constrain Rubidium atoms at very low temperatures to essentially one dimension using magnetic fields, and you get a similar system, but quantum....



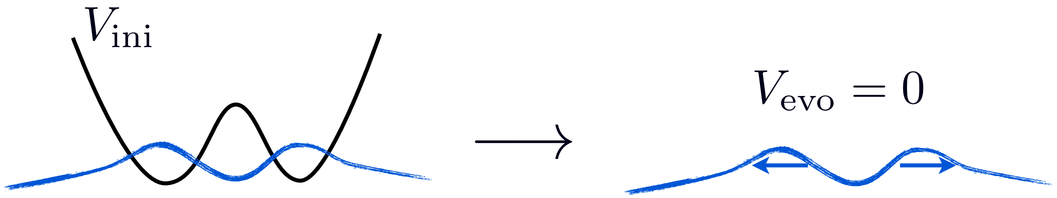
atom chip (Institut d'Optique – I. Bouchoule)

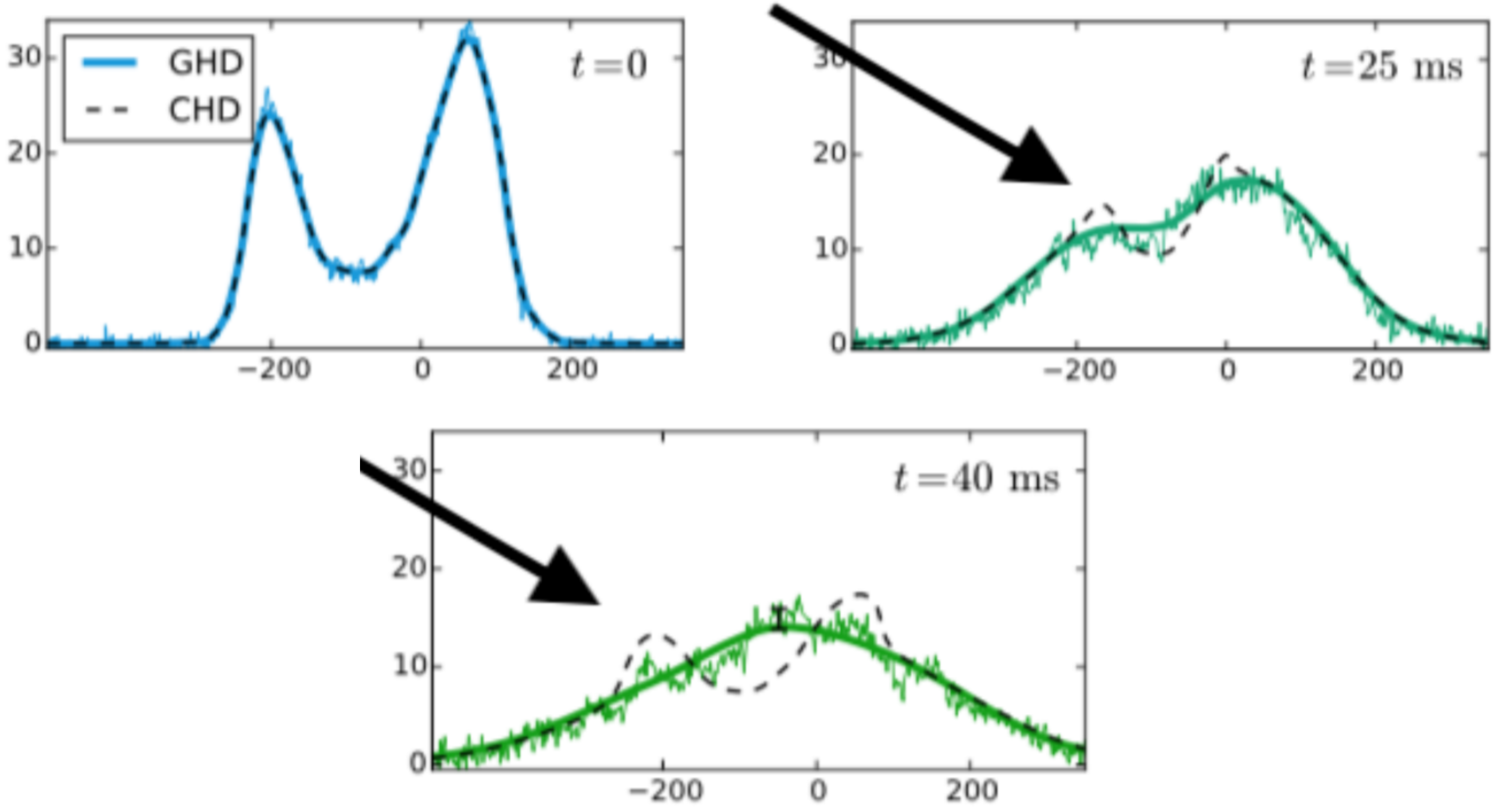




Emergence beyond the tornado...

Dotted curves: conventional hydrodynamics. Full curves: the new theory of GHD
[Schemmer, Bouchoule, BD, Dubail PRL 2018]





Outlook: a general theory for emergence

- There is no need to restrict ourselves to Newton's equations and particles.
- We may use Schrödinger's equations instead (quantum mechanics), Einstein's equations (relativistic particles), or even weirder systems such as lattices of quantum spins, cellular automata, gases of solitons which can be done experimentally, or even flocks of birds or markets of investors!
- The same principles of emergence apply, "hydrodynamic equations" emerge, and this mathematical formulation in terms of projections on Hilbert spaces tells us how to do this no matter the system.
- One can go much further: fundamental particles are emergent "waves" in quantum fields; classical mechanics is a theory emerging from measurements in quantum mechanics; is there some emerging motion at the scale of galaxies which we are not yet aware of?....

And thanks go to my funder on this research!

EPSRC

Engineering and Physical Sciences Research Council

Emergence of hydrodynamics in many-body systems:

new rigorous avenues from functional analysis

(EP/W000458/1)