Entanglement Measures in Theoretical Physics

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Introduction

A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.

A typical example: a pair of opposite-spin electrons [Bell States]

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

With: $|00\rangle = |0\rangle \otimes |0\rangle$ and

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Electrons in Bell states are known to be "maximally entangled"

Maximally Entangled

Quantum Superposition

Unlike a state such as:

$$\frac{1}{2}(|01\rangle + |10\rangle + |11\rangle + |00\rangle) = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

entangled states cannot be factorised.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

The property of one particle (i.e. spin) is inextricably linked (upon measurement) to the property of the other particle, so that measuring the state of the first electron reveals everything about the state of the second electron without the need to measure again.

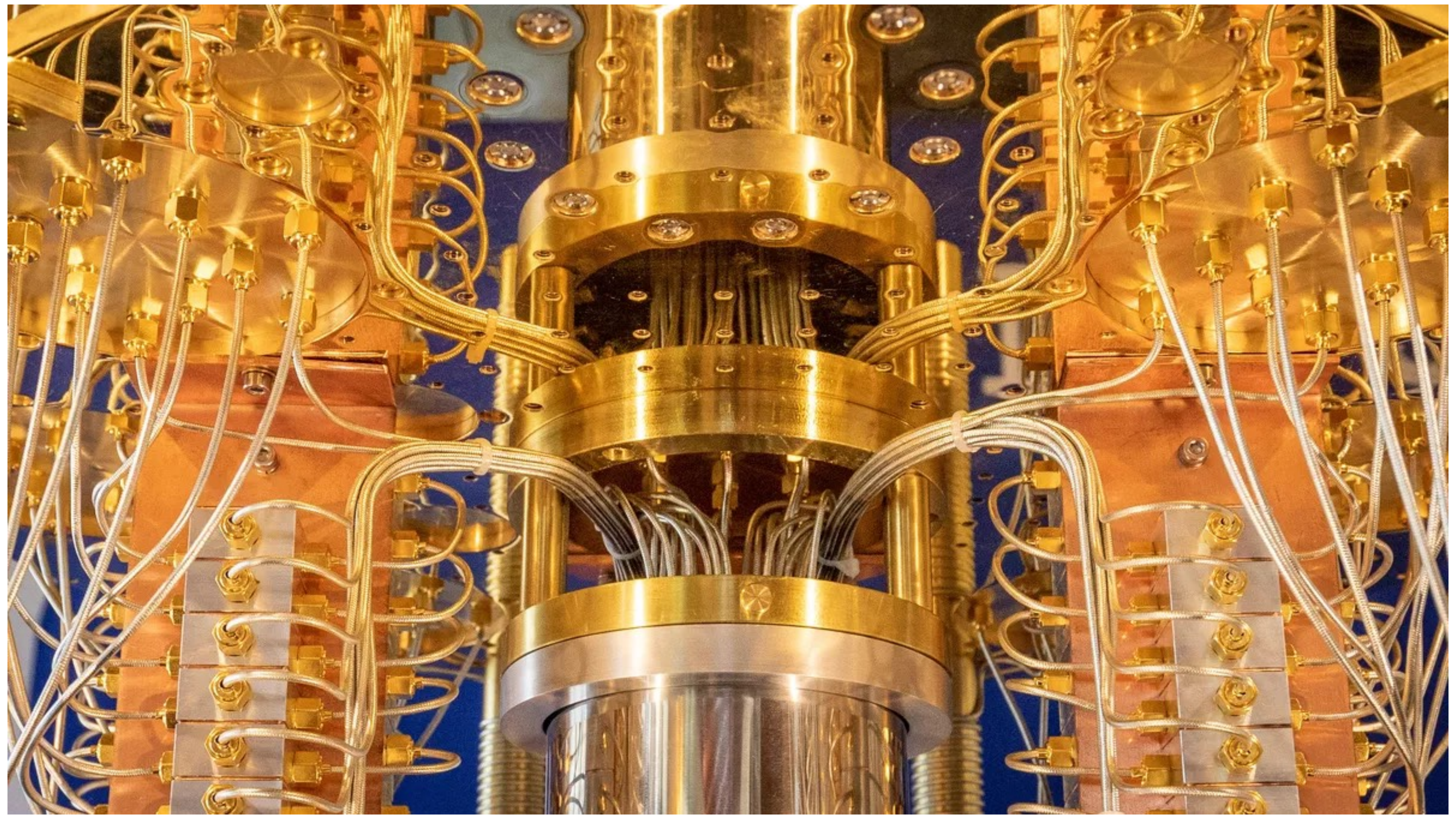
Spooky action at a distance! [EPR Paradox 1935]


MUAH!
RYAN
!
MUAH!
BRYAN

Quantum Information

- The properties of Quantum Entanglement and Quantum Superposition [the idea that a particle (qubit) can be in a superposition of states] can be used as resources for quantum computation protocols and this connects to the areas of Quantum Computing and Quantum Information.

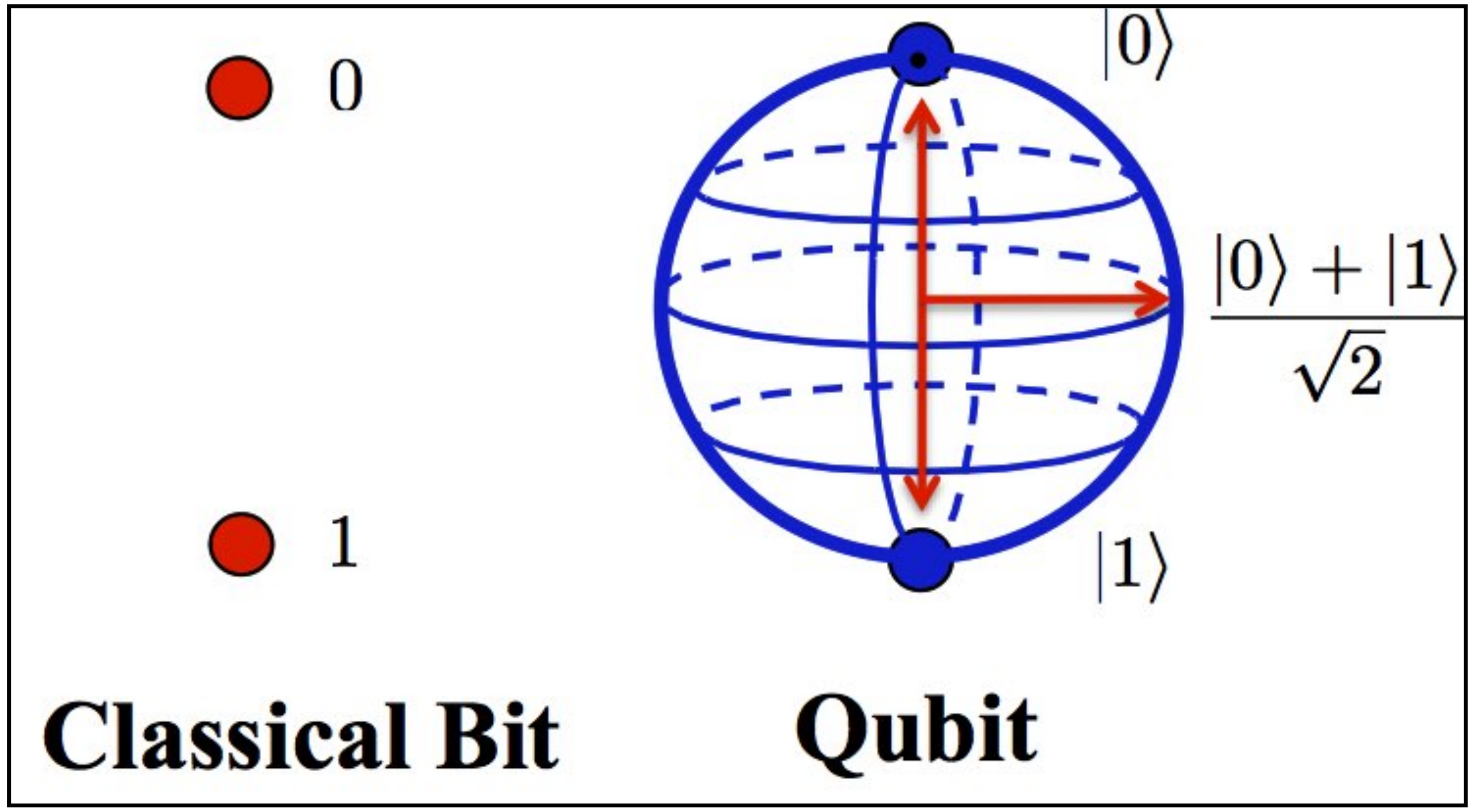
- There are several "classical" examples of problems that can be solved much more quickly or only by exploiting these properties. Let's see one example!



IBM Q (100 qubits)

Shor's algorithm for prime factorisation

Qubits & Quantum Computers



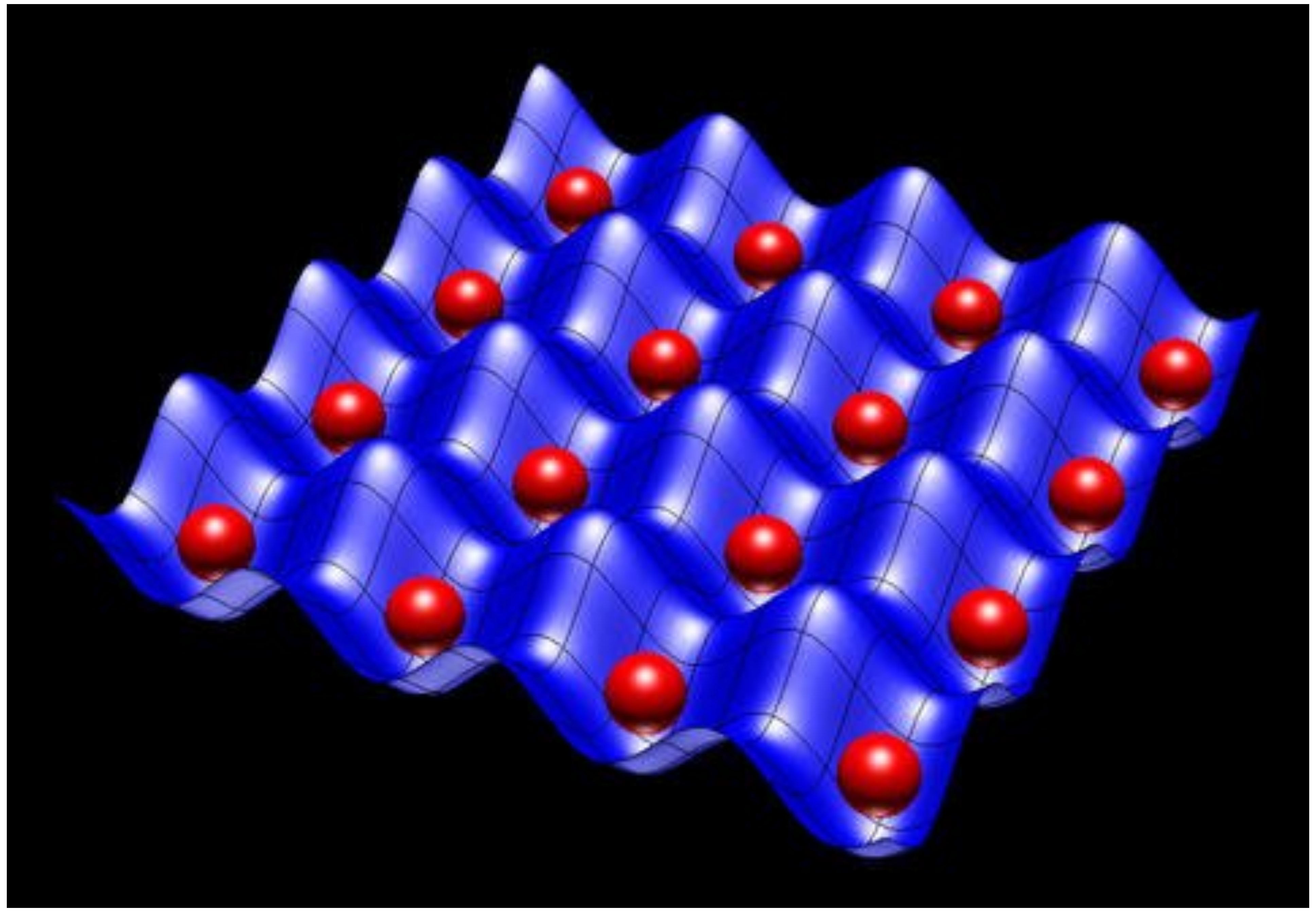
1 classical bit=one state

1 qubit= a superposition of 2 (or more) states

20 classical bits = 20 states

20 qubits = 2^{20}states

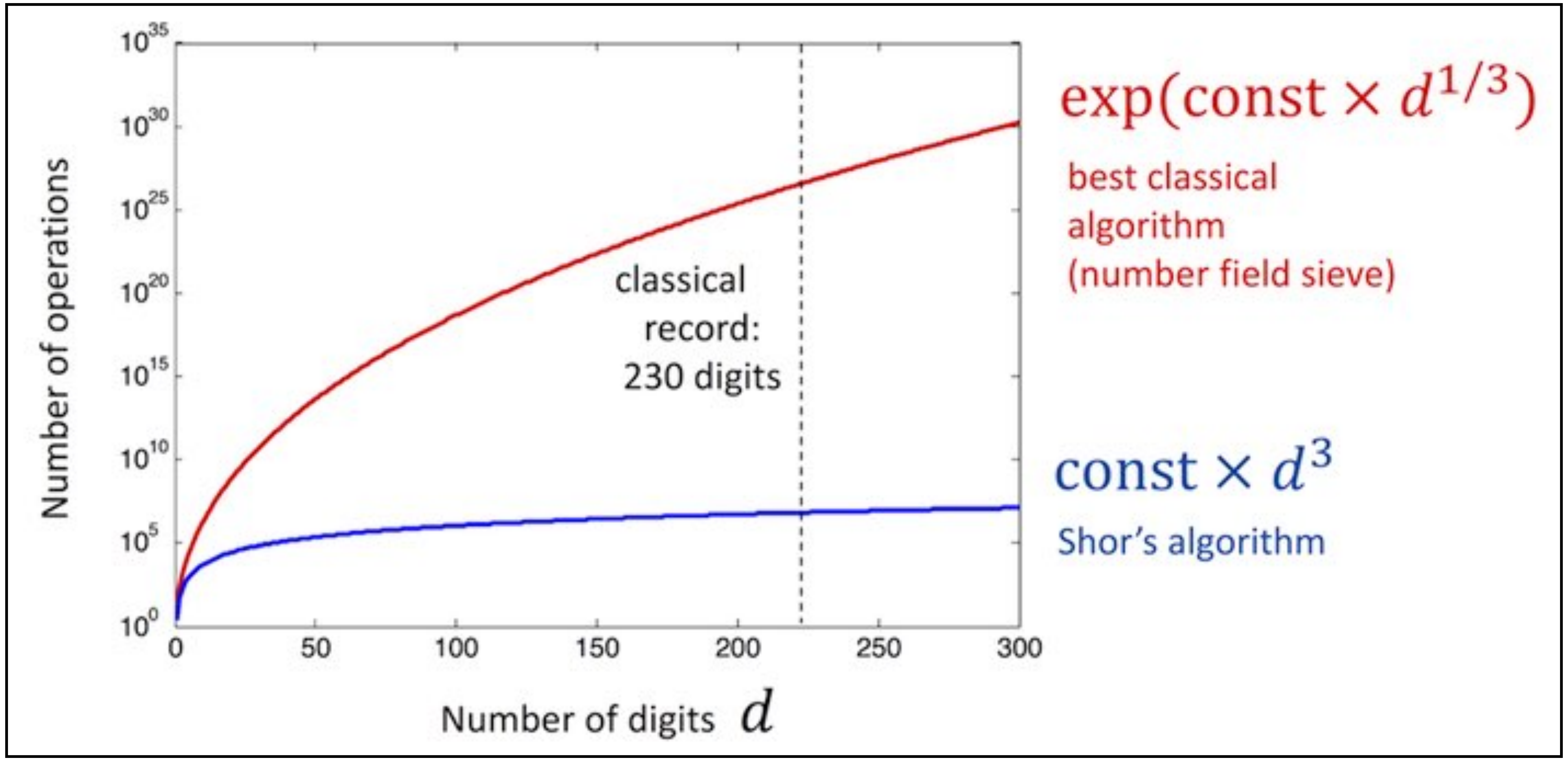
= 1 048 576 states



Optical lattices use lasers to separate rubidium atoms (red) for use as information bits in neutral-atom quantum processors.

Shor's Algorithm (1994)

- Problem: Given and integer N find its prime factors [e.g. 5055=5x3x337]
- Shor's algorithm can do this in polynomial time. This is exponentially faster than the most efficient classical algorithms.
- The efficiency of Shor's algorithm is due to the efficiency of the quantum Fourier transform, and modular exponentiation by repeated squarings.



Shor's Algorithm: Classical Part

1. Pick a random number $a < N$
2. Compute gcd(a, N). This may be done using the Euclidean algorithm.
3. If $gcd(a, N) \neq 1$, then there is a nontrivial factor of N, so we are done.
4. Otherwise, use a **period-finding subroutine** to find the period of: $f(x) = a^x \bmod N$, i.e. the smallest integer r for which $f(x+r) = f(x)$.
5. If $a^{x+r}(\mathrm{mod\,N}) = a^{x}(\mathrm{mod\,N})$ then $a^{r}(\mathrm{mod\,N}) = 1(\mathrm{mod\,N})$ or $a^{r} - 1\ (\mathrm{mod\,N}) = 0\,(\mathrm{mod\,N})$
6. If r is even then $a^{r} - 1\ (\mathrm{mod\,N}) = (a^{\frac{r}{2}} - 1)(a^{\frac{r}{2}} + 1)\,(\mathrm{mod\,N})$
7. Assuming N=p.q and that $(a^{\frac{r}{2}} \pm 1)\,(\mathrm{mod\,N}) \neq 0$ then p=$\mathrm{gcd}(a^{\frac{r}{2}} - 1, N)$ and q= $\mathrm{gcd}(a^{\frac{r}{2}} + 1, N)$

Shor's Algorithm: Quantum Part

- The bit of Shor's algorithm that uses quantum superposition and quantum entanglement is the finding of the period r of the function $f(x) = a^x \,(\mathrm{mod}\,\mathrm{N})$.
- Consider and example and a (very simplified) explanation of … take N=15 and a=7.
- Let us initialise our quantum computer using two "registers"

This operation can be done in a single step (the function acts on a superposition)

$$|0\rangle|0\rangle \mapsto (|1\rangle + |2\rangle + |3\rangle + \ldots)|0\rangle \mapsto |1\rangle|f(1)\rangle + |2\rangle|f(2)\rangle + |3\rangle|f(3)\rangle + \ldots$$

$$\mapsto |1\rangle|7\rangle + |2\rangle|4\rangle + |3\rangle|13\rangle + |4\rangle|1\rangle + |5\rangle|7\rangle + \cdots$$

This is generally an entangled

- Performing a measurement of the second register gives any result with equal probability. Suppose that we got the result $|7\rangle$. Then this means that the first state is $|1\rangle + |5\rangle + |9\rangle \cdots$ from which the period (r=4) can be read off.

Here is Shor himself speaking about his algorithm:

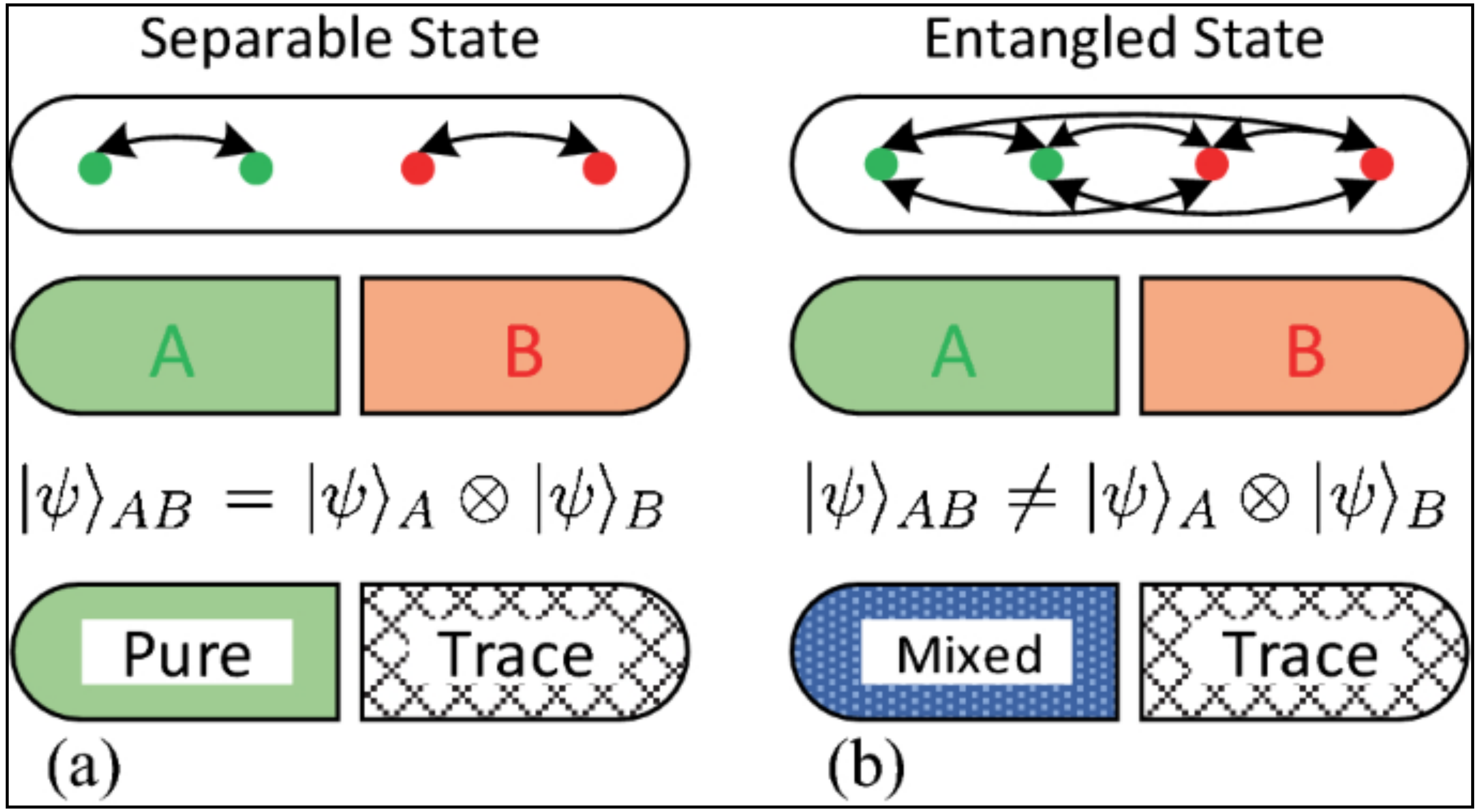
https://www.youtube.com/watch?v=hOlOY7NyMfs

and here is a slightly longer video of Shor as well:

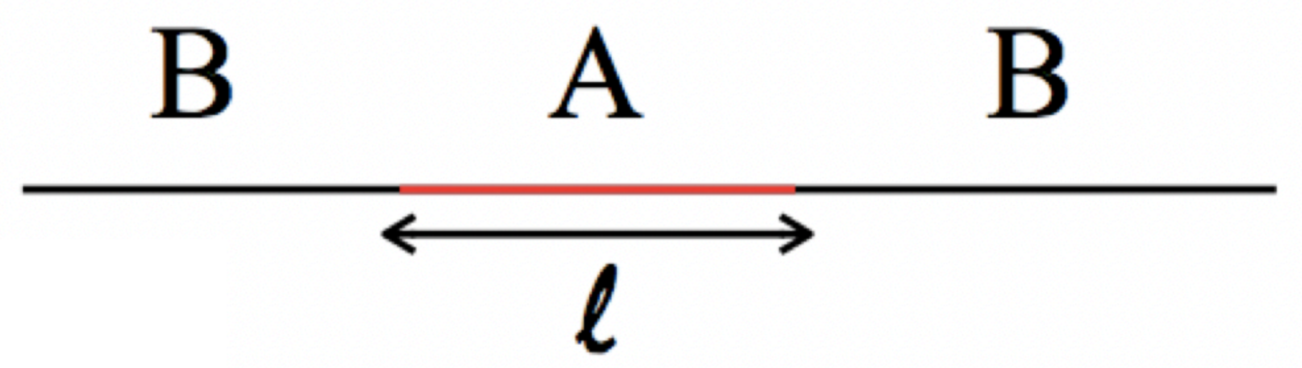
https://www.youtube.com/watch?v=6qD9XElTpCE

Theoretical Physics and Entanglement

- A lot of theoretical research on entanglement is instead interested on **what constitutes a good measure of entanglement** and **what that measure can tell** us about the properties of a quantum system.



- The first obvious question is how to define entanglement in a system consisting of more than two qubits (unlike Bell states). For instance, many people are interested in quantum spin chains or many-body quantum systems.



Bipartite Measures

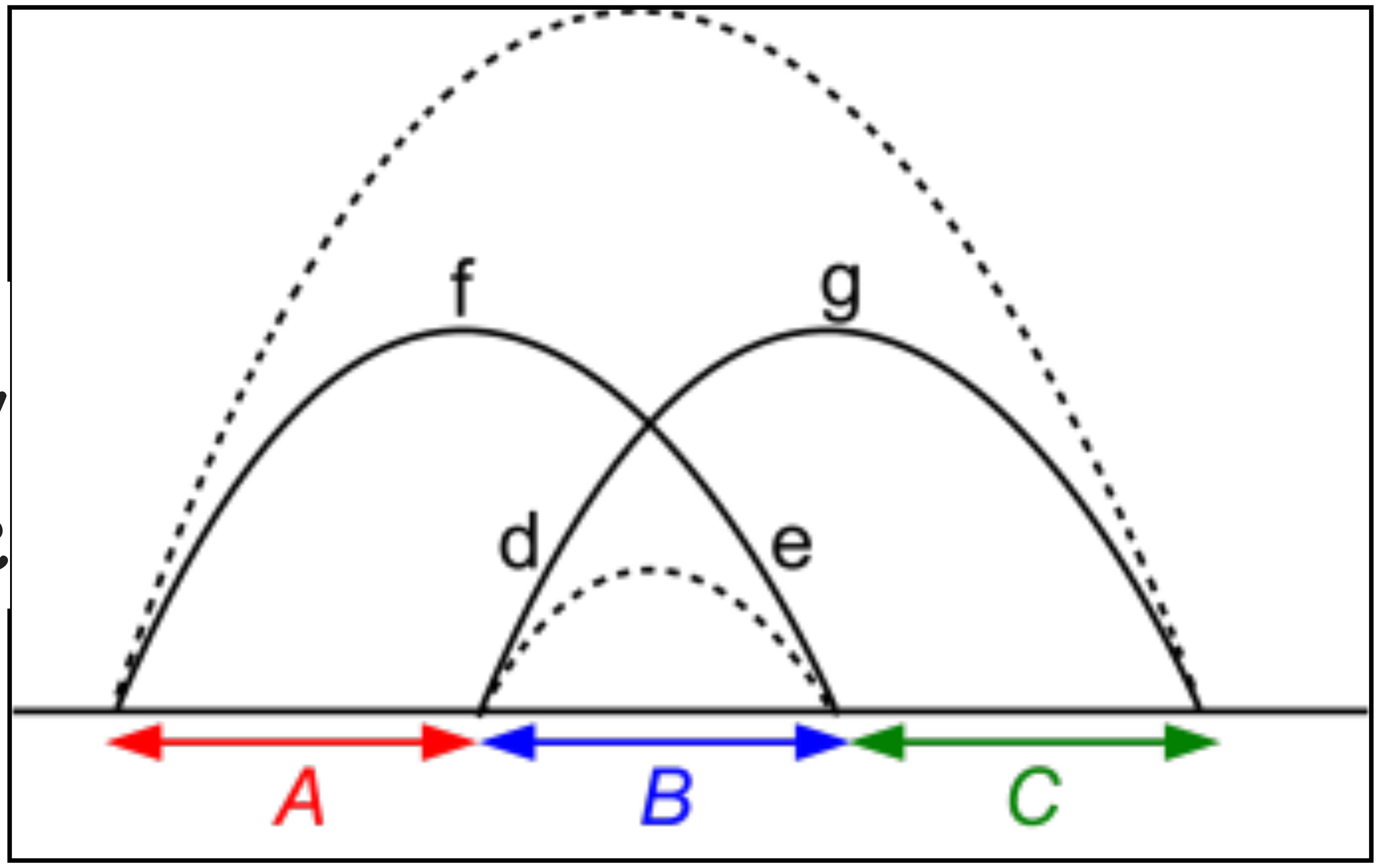
- The most popular measures are **bipartite**, that is based on dividing the system into two parts and looking at their mutual entanglement.
- Among these measures, the most famous is the **Von Neumann Entropy** or, simply, the **Entanglement Entropy**.
- The starting point is a bipartition of a quantum system into two complementary parts A and B. Suppose the system's state is described by a **pure state** $|\Psi\rangle$.
- We define a **reduced density matrix** $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$. This provides a measure of the correlations "seen" by subsystem A when we "forget" about subsystem B.
- Then, the Von Neumann Entropy is given by $S = -\mathrm{Tr}_A(\rho_A \log \rho_A)$.

Example: A Bell State

- Let us consider again a Bell state. If we identify the first spin as subsystem A and the second spin as subsystem B we have for $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:
- $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \frac{1}{2}\mathrm{Tr}_B(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|)$
  $= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{bmatrix} 10 \\ 01 \end{bmatrix}$
- So, $S = -\mathrm{Tr}(\rho_A \log \rho_A) = -\sum_{j=1}^{2} \eta_j \log \eta_j$ where η_j are the eigenvalues of the density matrix. In our case $\eta_1 = \eta_2 = \frac{1}{2}$ which gives $S = \log 2$

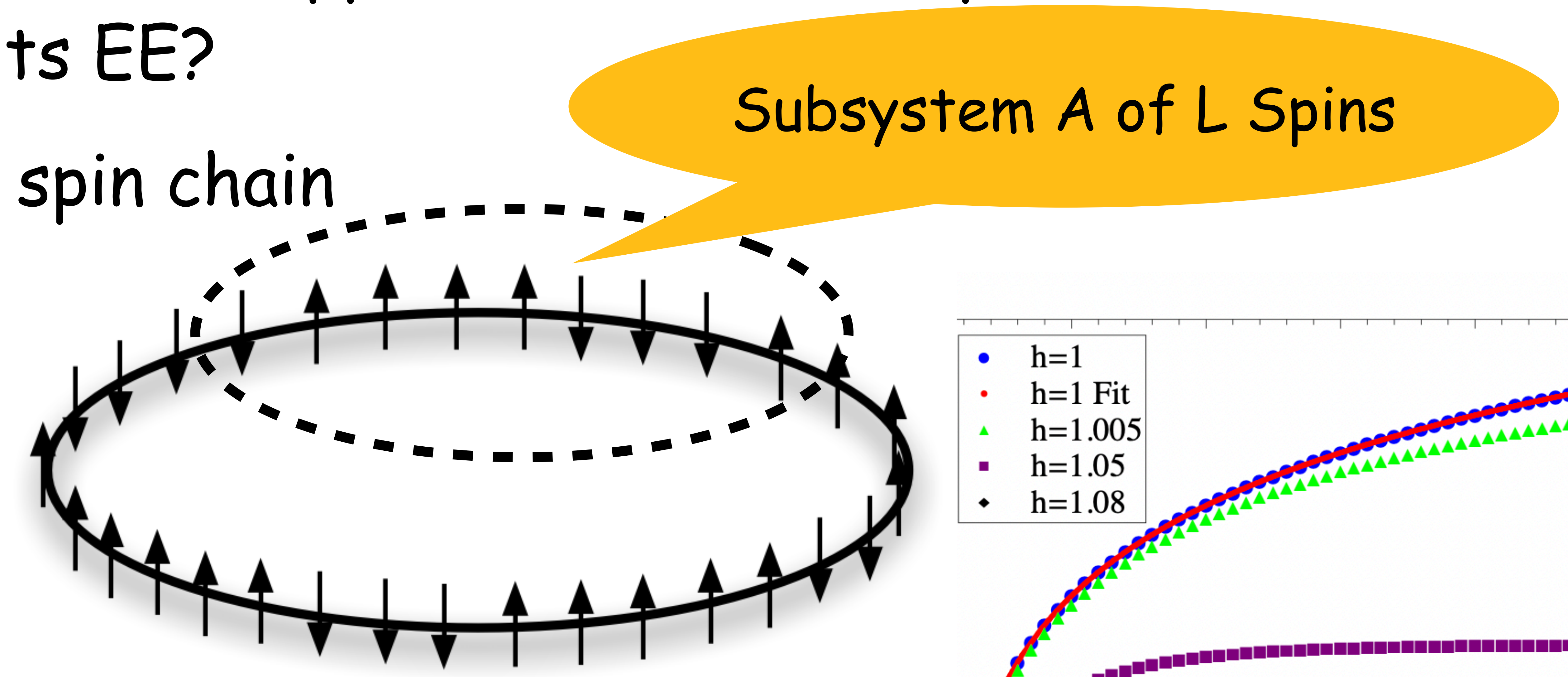
A Good Measure

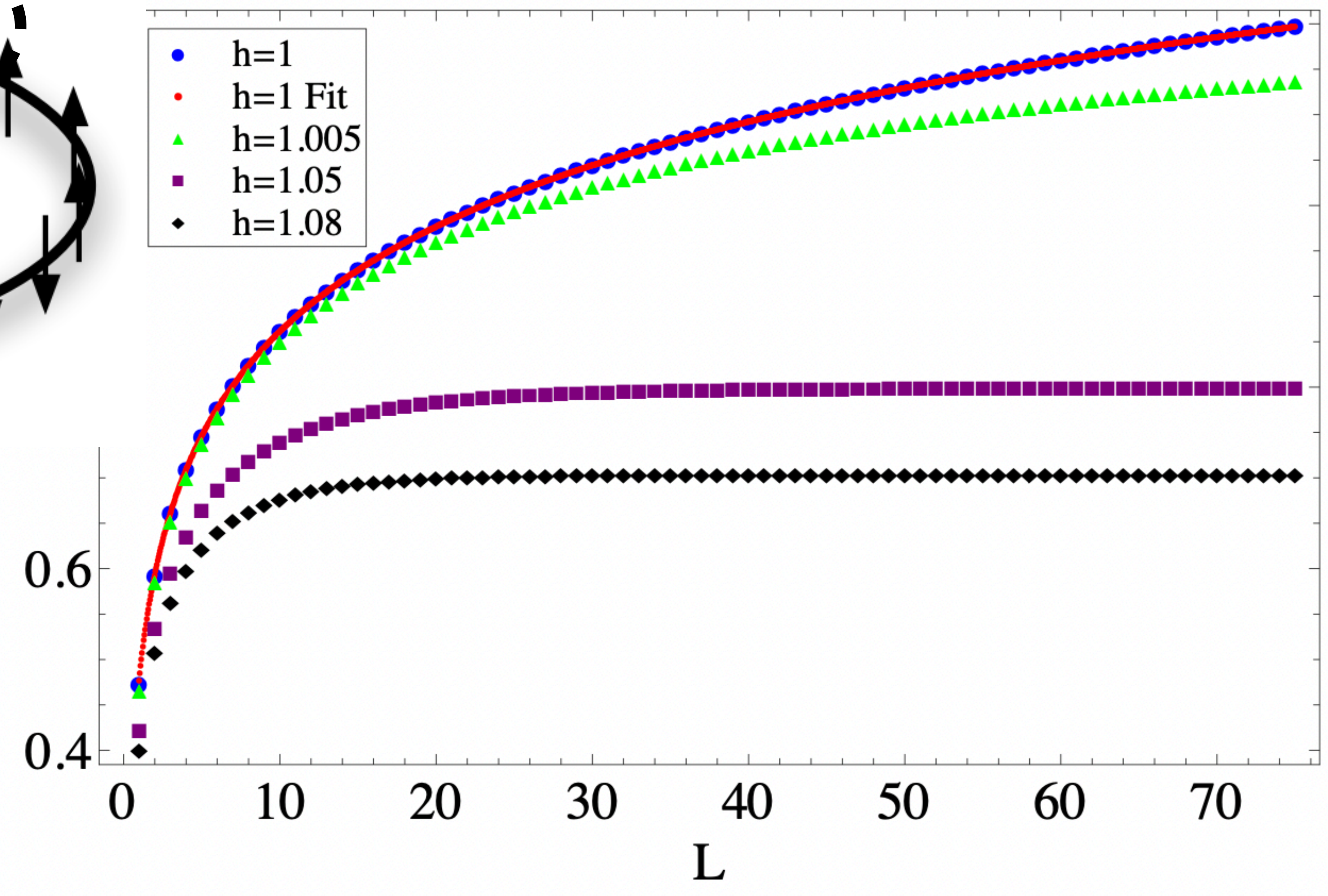
- S(ρ) is zero if and only if ρ represents a pure state.
- S(ρ) is maximal and equal to ln N for a maximally mixed state, N being the dimension of the Hilbert space.
- S(ρ) is invariant under changes in the basis of ρ, that is, S(ρ) = S(UρU†), with U a unitary transformation.
- S(ρ) is concave, which means, given $\eta_i > 0$ such that $\sum_i \eta_i = 1$ and density operators ρ_i we have that $S(\sum_i \eta_i \rho_i) \geq \sum_i \eta_i S(\rho_i)$
- S(ρ) is strongly subadditive. Given three systems A, B, C we have: $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$ where $\rho_B, \rho_{AB}, \rho_{BC}$ are reduced density matrices of ρ_{ABC}



Many-Body Quantum Systems

- Bell states are very simple, but what happens if we have a system made of a very large number of spins and compute its EE?
- A famous example is the Ising spin chain
- $$H = -\frac{J}{2}\sum_{j=1}^{N}[\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$$
- h=1 (critical in scaling limit)
- h>1 (near-critical QFT)
- $J \to \infty, a \to 0, Ja = v$



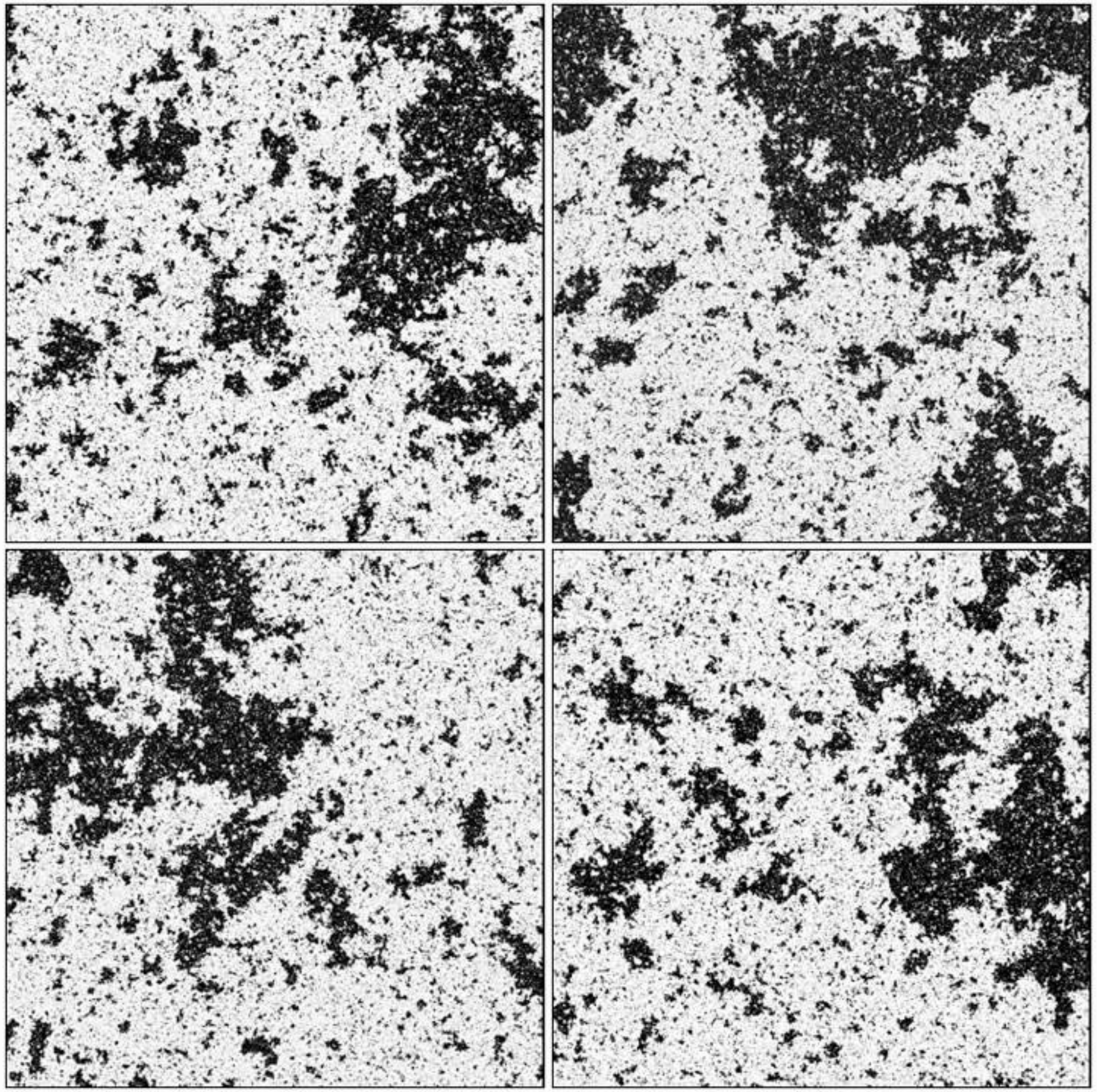


Logarithmic Scaling vs Saturation

- $$S(L) = \frac{1}{3}\log L + \text{constant}$$

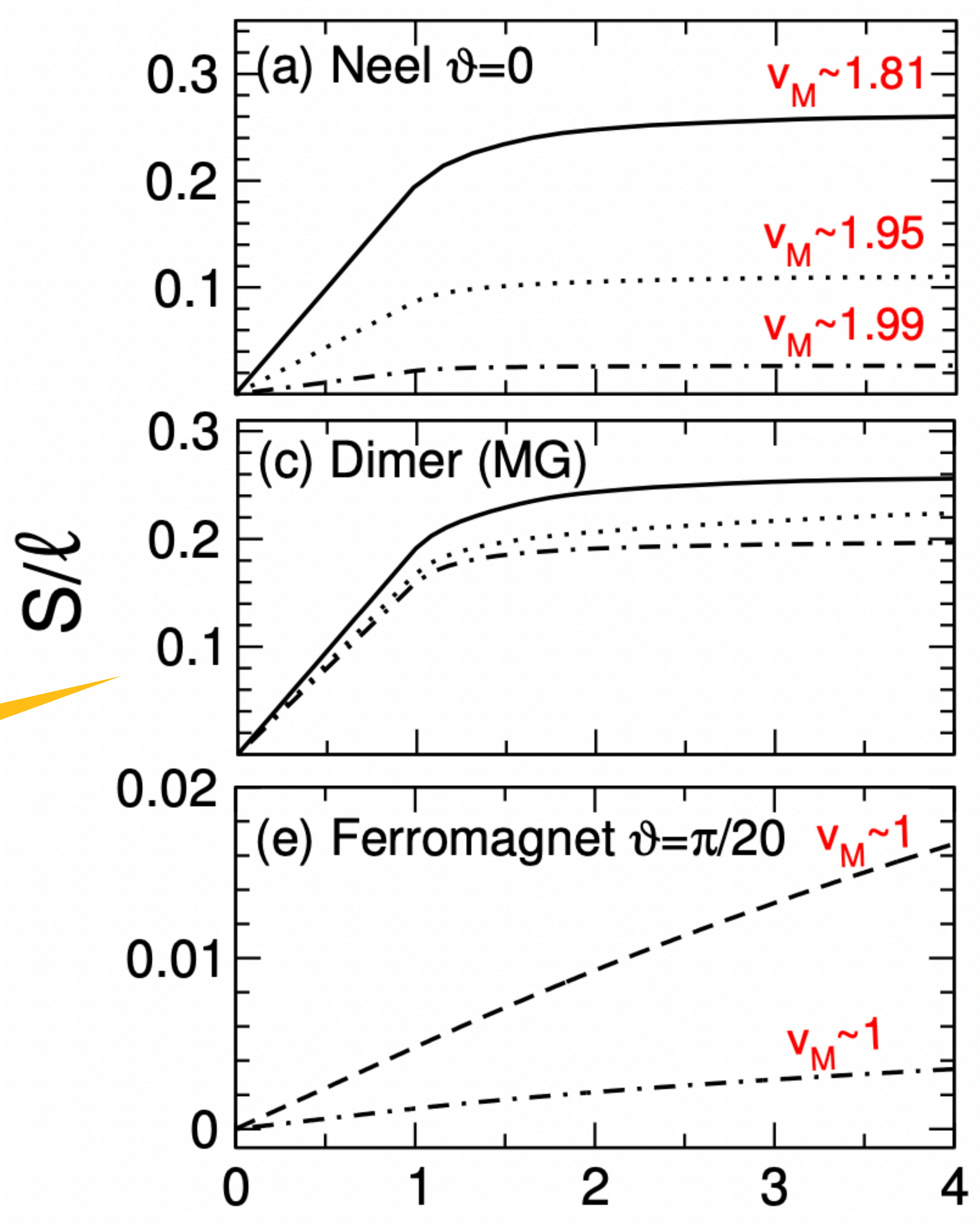
$$S(L) = \frac{1}{3}\log \xi + \text{constant} - \frac{1}{8}\sum_i K_0(2m_i L)$$

Scaling Limit



So, What is this Useful For?

- Entanglement measures display **Universal Behaviours** which means that they provide a natural way to classify quantum systems or "**quantum states of matter**"
- The value of the EE and other measures gives information about the state of a quantum system and about how feasible or not it is to **simulate that state in a classical computer** with classical algorithms (i.e. DMRG, ITEBD, MPS…)



Another Universal Feature: the EE density grows linearly in time and then saturates after a "quantum" quench

*density matrix renormalisation group , infinite time-evolving block decimation, matrix product states…

Conclusions

Entanglement Measures are of interest in many areas:

- Pure Mathematics
- Information Theory/Quantum Computation
- Theoretical Physics: Quantum Field Theory (including String theory) and Astrophysics/Gravity Theory (e.g. Black Holes)

They have interesting and sophisticated mathematical properties.

Can be computed numerically using advanced simulation techniques for many-body quantum systems

In some cases, analytical formulae can be found, usually showing universal trends in QFT.

What do People do Now?

- They come up with ever new and more sophisticated measures of entanglement for different kinds of states/geometries/dimensionalities.
- They study their mathematical properties and develop analytical methods to compute them.
- They develop numerical algorithms to simulate quantum systems in all their complexity.
- They develop quantum technology (quantum encryption, quantum computers…)
- They carry out experiments involving many qubit (cold atom experiments)
- They develop techniques to measure entanglement in the laboratory….

Some References

[EPR Paradox]

Einstein, A; B Podolsky; N Rosen (1935-05-15). "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" Physical Review. **47** (10): 777–780.

[Shor's Algorithm]

Shor, P.W. (1994). "Algorithms for quantum computation: discrete logarithms and factoring". Proceedings 35th Annual Symposium on Foundations of Computer Science. IEEE Comput. Soc. Press: 124–134. doi:10.1109/sfcs.1994.365700.

[Entanglement Entropy/Density Matrices]

Von Neumann, John (1932). Mathematische Grundlagen der Quantenmechanik. Berlin: Springer; Von Neumann, John (1955). Mathematical Foundations of Quantum Mechanics. Princeton University Press.

Landau, L. (1927). "Das Daempfungsproblem in der Wellenmechanik". Zeitschrift für Physik. **45** (5–6): 430–464.

[EE is a good measure of Entanglement]

C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).

[Entanglement Scaling in the Ising Model]

G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in Quantum Critical Phenomena, Phys. Rev. Lett. **90**, 227902 (2003).

[Entanglement Scaling in Critical Systems]

C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. **B424** (1994) 443-467.

P. Calabrese and J.L. Cardy, Entanglement Entropy in Quantum Field Theory, J. Stat. Mech. **0406**, P002 (2004).

[Entanglement in Gapped Systems]

J.L. Cardy, O. Castro-Alvaredo, and B. Doyon, J. Stat. Phys. **130**, 129-168 (2008)