Official Title:
Paradoxes in Probability

Title I wanted to use:
You are bad at Probability

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What is a Paradox in the context of Mathematics?

- Not particularly well defined, some things just happen to be named a **paradox**
- The term paradox comes from the ancient Greek terms for "against" or "beyond" ($\pi\alpha\rho\acute{\alpha}$) and "expectation" or "opinion" ($\delta\acute{o}\xi\alpha$)
- Typically you will see them come in one of two forms
  - A logical sequence or construction of an object which "breaks the rules" of Mathematics (**falsidical**)
  - A correct result which does not conform to our intuition (**veridical**)
- Historically, Mathematics was developed to solve real world problems, so many of the structures obey a sense of real world intuition

Some well known Paradoxes (there are a lot)

- The Banach-Tarski paradox
- Russel's paradox
- $\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$
- $\sum_{n=1}^{\infty} (-1)^{n-1} n = \frac{1}{4}$
- Gabriel's Horn
- Hilbert's Grand Hotel
- The friendship paradox
- The intransitive dice paradox
- The potato paradox
- The staircase paradox
- The string girdling Earth paradox
- Simpson's paradox
- Freedman's paradox
- The Will Rogers phenomenon
- Lindley's paradox
- The inspection paradox
- Braess's paradox
- Berkson's paradox
- The accuracy paradox
- Abelson's paradox
- The false positive paradox
- The Cramer-Euler paradox
- Grice's paradox
- The two envelope paradox
- Proebsting's paradox

The Banach-Tarski Paradox (veridical)

- **The Paradox:** A three dimensional ball may be decomposed into five disjoint sets such that after each is subjected to a rigid transformation, the resulting set is two copies of the original ball
- Our intuition tells us that the **volume** of a set in $\mathbb{R}^3$ must be preserved under rigid transformations
- Our intuition also tells us that the **volume** of a set in $\mathbb{R}^3$ is equal to the sum of **volumes** of disjoint subsets
- These are both correct statements, if we restrict the word **volume** to only apply to certain sets
- The Banach-Tarski "paradox" relies on decomposing the ball into **non-measurable sets**
- I encourage you to look up this decomposition, it is surprisingly easy to understand considering the result

$0 = 1$ (falsidical)

- Consider the following computation:

$$\begin{aligned}0 &= (1-1)\\ &= (1-1)+(1-1)\\ &= (1-1)+(1-1)+(1-1)+\cdots\\ &= 1+(-1+1)+(-1+1)+\cdots\\ &= 1\end{aligned}$$

- This relies on the **misuse** of infinite series
- In particular, they must be defined in terms of a limit of the partial sums, which in this example **do not converge**
- Also, "..." can be a very misleading symbol, but I don't have time to go into that

Nature of the examples discussed today

- Each result discussed here is a **veridical** type paradox
  - Except one, because I want to emphasize an important point
- The examples will each begin with a question that should be extremely easy to understand regardless of mathematical background
  - So you can take these to parties and impress all of your friends
- Each Paradox in Probability will be based on a question about the **probability** of an **event** or the **expected value** or **distribution** of a **random variable**
  - Distinct from paradoxes in Statistics, but those are also very interesting to consider
- Audience participation!!!!! When you see this symbol: **!!!** just shout out your gut feeling answer
  - But if you are familiar with the question, please let me have my fun (be quiet)
- I will certainly run out of time before presenting all of my examples
  - Come find me if you want to hear more!

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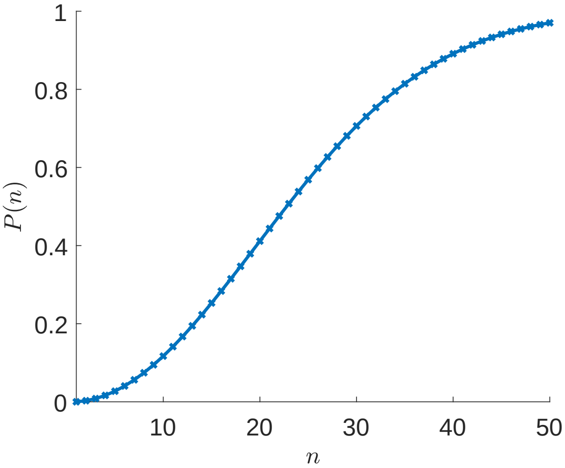
- What is the smallest value of n such that $P(n) > 0.5$? **!!!**

$$1 - P(n) = \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdots \frac{365 - (n-1)}{365}$$

The Birthday Paradox



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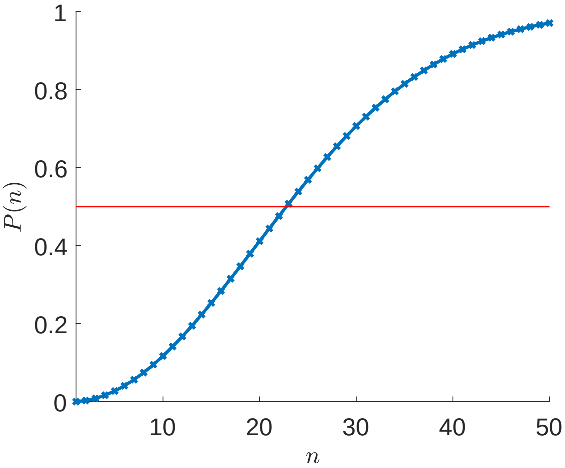

The Birthday Paradox

- $P(22) = 0.476$ and $P(23) = 0.507$
  - Thus, the answer is $n^* = 23$
- Why do most people find this counterintuitive?
- As an individual, the probability that one other person has your birthday is $1/365$
  - On average you would have to meet 365 people before meeting one with your birthday
- We naturally conflate this situation with the question of the Birthday Paradox, which is different!

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- What should your choice be? Does it make a difference? **!!!**

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- This can be explicitly computed

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		G	B
2^{nd} child	G		
	B		

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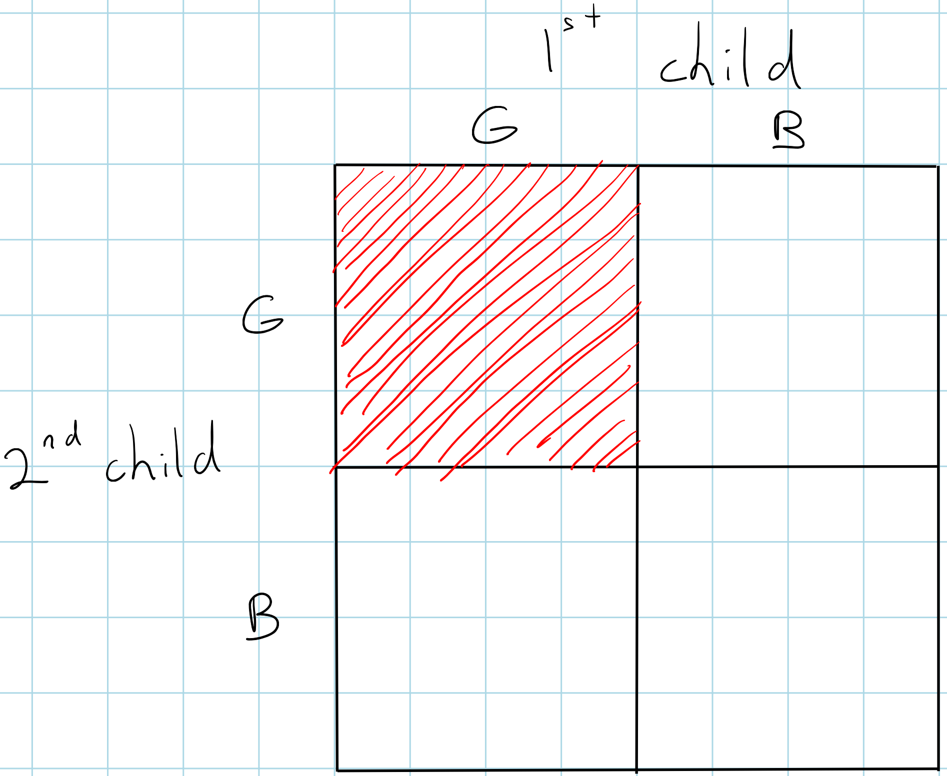
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- Conditional on at least one child being a **boy**, one of the above outcomes is **impossible**, leaving **three equally likely** outcomes
- Thus, the probability that there is a **girl**, **given** that there is **at least one boy**, is $\frac{2}{3}$!

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- The state space for one child has 14 points, and so the state space for both children has 196 points

		G							B						
		M	T	W	Th	F	S	Su	M	T	W	Th	F	S	Su
G	M														
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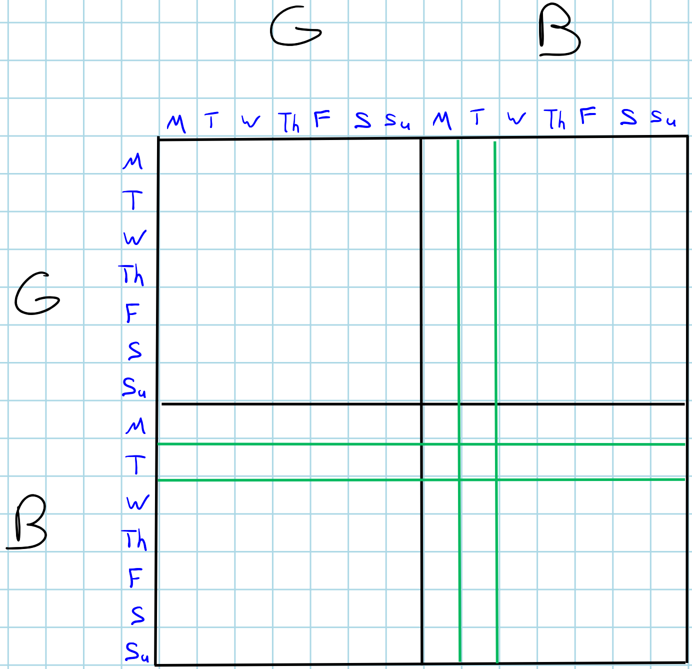
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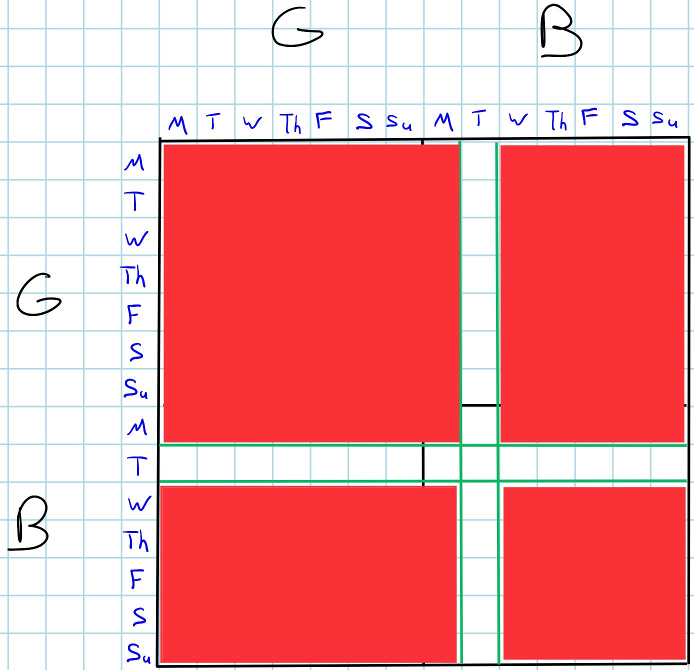
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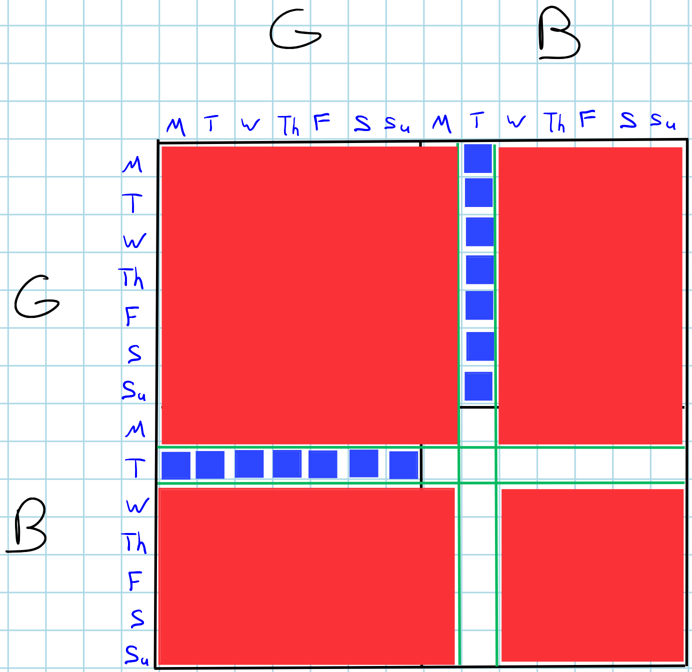
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- Use the same procedure as before, each child is assigned $\{B, G\}$ and $\{1, \ldots, 7\}$ uniformly and independently
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- Taking into account the given information, 169 points are **impossible**, leaving **27 equally likely** outcomes, 14 of which include a **single girl**

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- Thus, the probability that the other child is a **girl** is $\frac{14}{27} \approx 0.52$

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- **Conditioning** on no odd outcomes is **not the same** as **removing all odd outcomes**

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- NO! The conditional probabilities put **very low weight** on long sequences of rolls without stopping
- We can again explicitly compute this expected value

Die Against the Odds - General Solution

- For a general N sided die:
  - Let T be the random variable which is the number of rolls until the first 6
  - Let E be the event that all rolls are even

$$\mathbb{E}[T|E] = \sum_{k=1}^{\infty} k\mathbb{P}(T = k|E)$$
$$= \sum_{k=1}^{\infty} k\frac{\mathbb{P}(T = k \text{ and } E)}{\mathbb{P}(E)}$$

  - It is not difficult to compute $\mathbb{P}(T = k \text{ and } E)$ and $\mathbb{P}(E)$:

$$\mathbb{P}(T = k \text{ and } E) = \left(\frac{\frac{N}{2} - 1}{N}\right)^{k-1}\frac{1}{N}$$
$$\mathbb{P}(E) = \sum_{k=1}^{\infty}\left(\frac{\frac{N}{2} - 1}{N}\right)^{k-1}\frac{1}{N}$$

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- For the original 6-sided die, the result is 1.5

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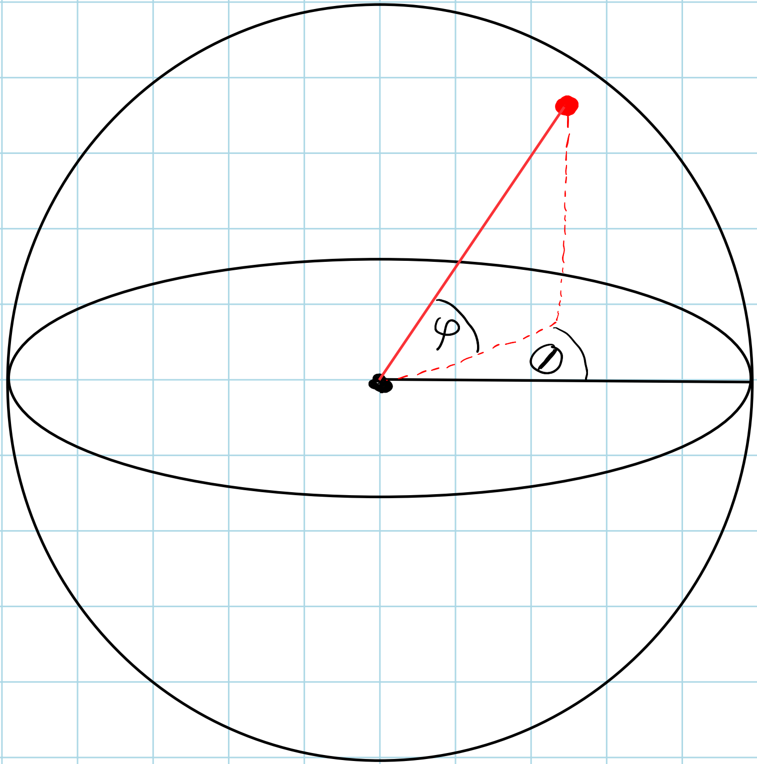

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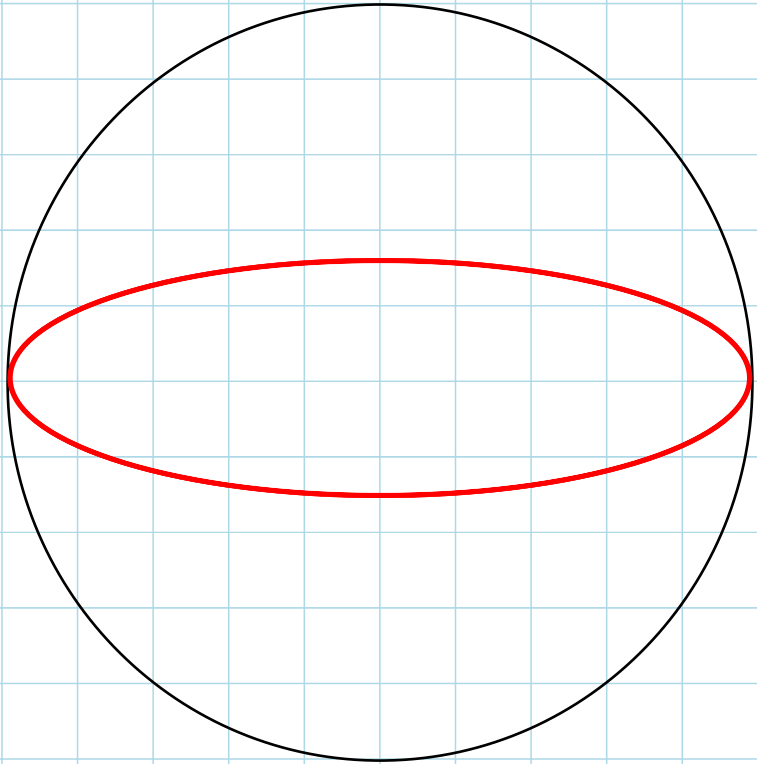
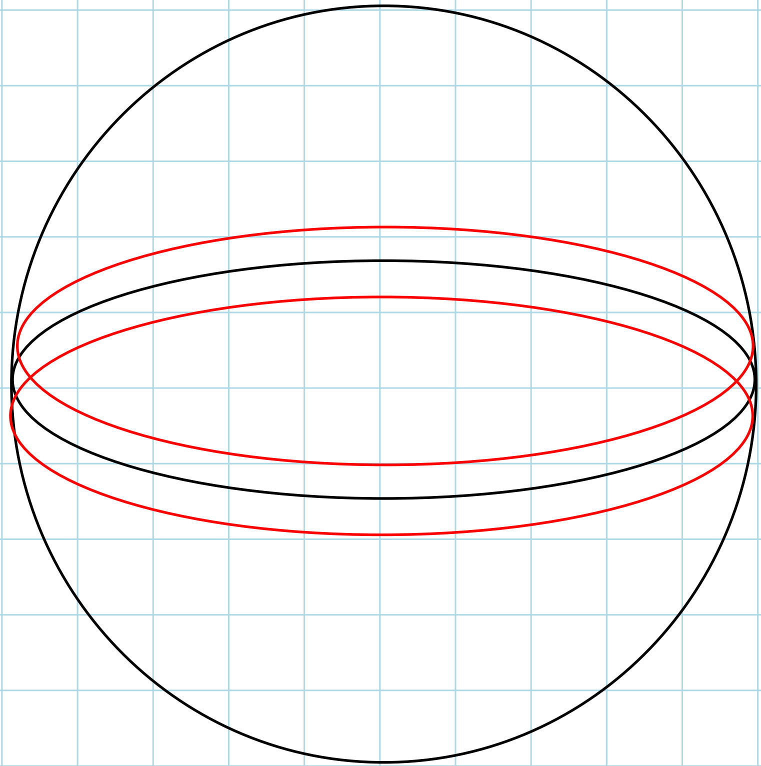
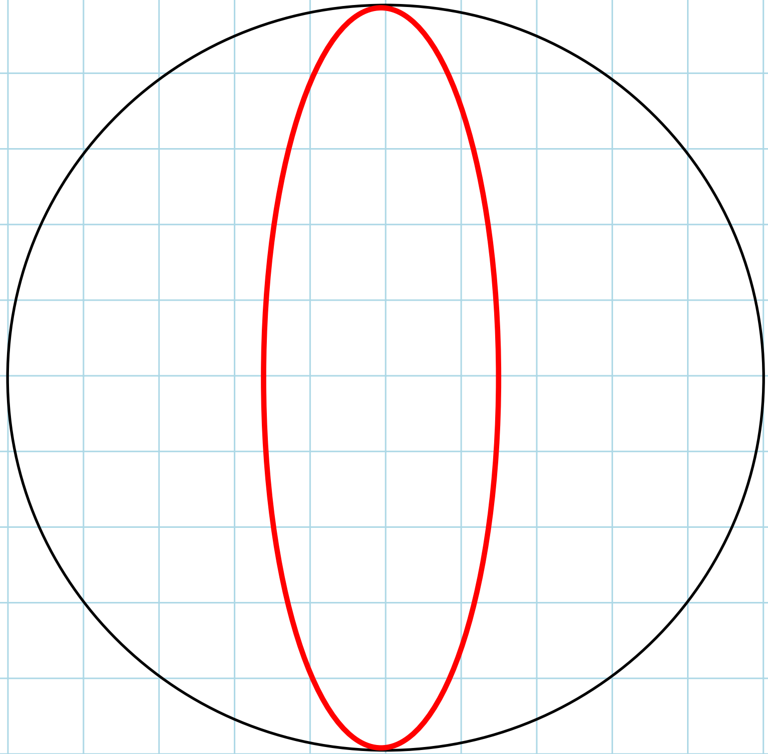
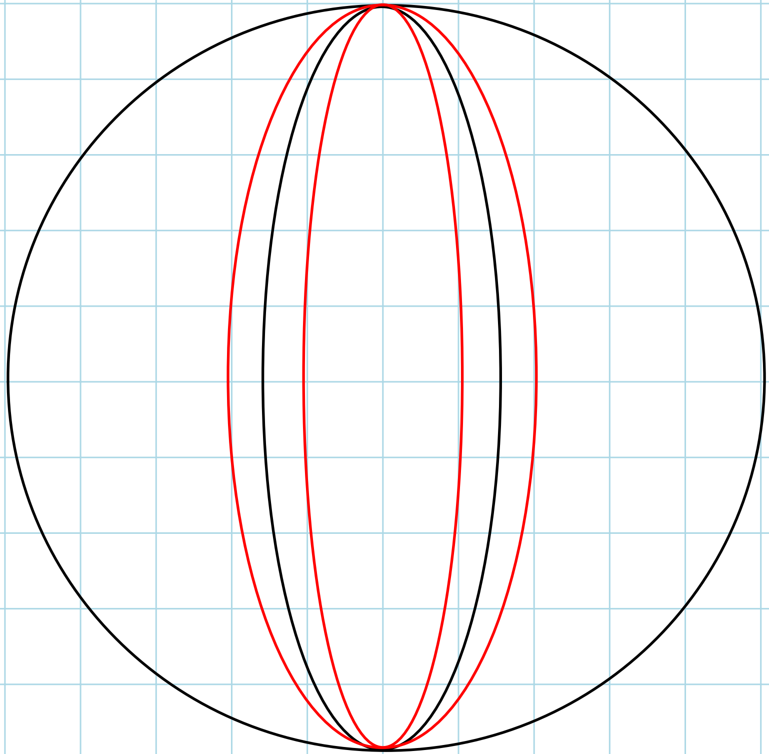
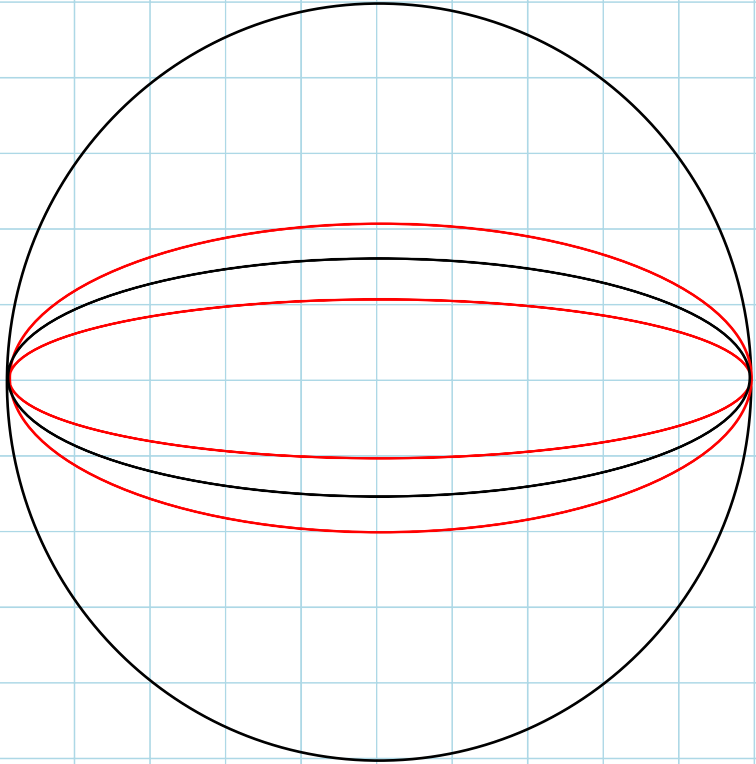
- **Conditional expectations** and **conditional probabilities** are **not defined** on events with probability zero!
  - **Conditional distributions** come from **conditional probabilities**
- **Conditional densities** are perfectly fine to deal with, but a density function **always appears inside an integral!**

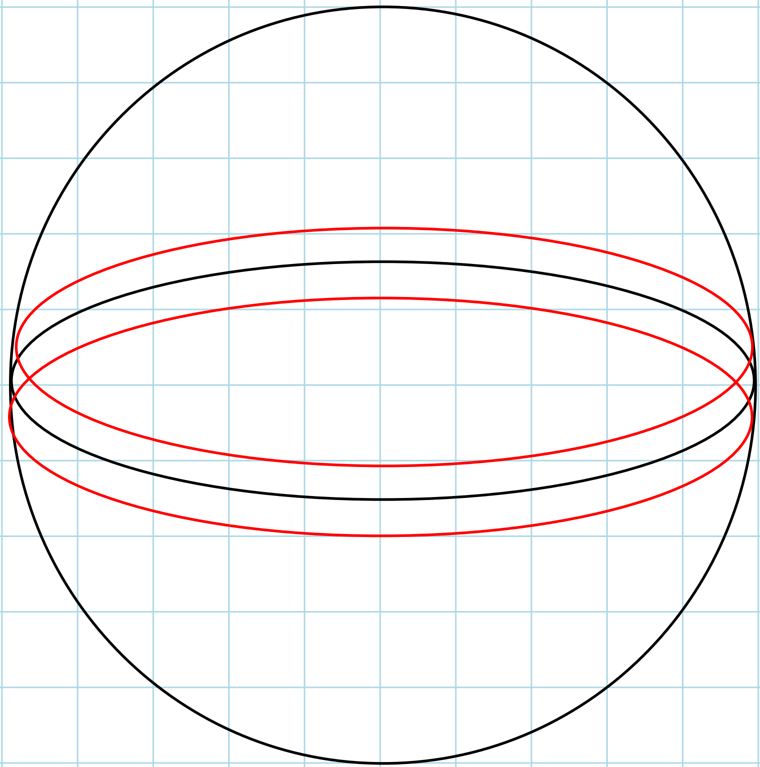












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- This makes sense because in order to roll three 6's in a row you need to roll at least three 6's

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- Even though you have to roll at least three in order to roll three in a row, the skewed probabilities push the likelihood of finishing with a run of three towards the beginning

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- **Alice** flips a coin repeatedly until seeing the sequence **HTH**

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  - **Alice** finishes on average with 10 rolls
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- Penny's Game is **non-transitive**:
  - For any sequence of three coin tosses, there is a different sequence that has greater than 0.5 probability of winning

What are the lessons here?

- Humans often **don't have good intuition** for probability
- This is especially true for **conditional probability**
  - Every example given involved conditional probabilities, except for the first (Birthday Paradox) and the last (Penny's Game)
- Every example was also **veridical**, except the Great Circle Paradox which is somewhere on the boundary between **veridical** and **falsidical**

Conditional Expectation (measure theoretic)

- This talk is not complete without the definition of **conditional expectation**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let X be an integrable random variable, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

Suppose a random variable Y is $\mathcal{G}$-measurable and satisfies

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}$$

for all $A \in \mathbb{G}$. Then we call Y the **conditional expectation** of X given $\mathcal{G}$ and write

$$Y = \mathbb{E}[X|\mathcal{G}]$$

From this we define **conditional probability** of and event E given $\mathcal{G}$ to be

$$\mathbb{P}(E|\mathcal{G}) = \mathbb{E}[1_E|\mathcal{G}]$$

Thanks for your attention!

Ryan Donnelly

ryan.f.donnelly@kcl.ac.uk