The Riemann Hypothesis and other unsolved problems

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What is your favourite conjecture or unsolved problem?

"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its future development?"

David Hilbert, *Mathematical Problems* (translation), 1902

David Hilbert

- Born 1862, died 1943. Worked in Königsberg and Göttingen
- Analysis, algebra, algebraic geometry, number theory, logic, mathematical physics
- Hilbert spaces, Hilbert basis theorem, Hilbert system, Hilbert axioms,...

Hilbert's Problems

- In 1900, David Hilbert published a list of 23 (at the time) unsolved problems
- Set as examples of problems whose solutions would further mathematics
- Hilbert presented ten of the problems at the Second International Congress of Mathematicians in Paris in 1900
- Problems in abstract algebra, analysis, differential equations, geometry, logic, number theory, physics

Hilbert's problems - some examples

Hilbert's first problem

There is no set whose cardinality is strictly between that of the integers and that of the real numbers.

Hilbert's second problem

Prove that the axioms of arithmetic are consistent.

Hilbert's sixth problem

Mathematical treatment of the axioms of physics.

Hilbert's tenth problem

Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.

Hilbert's twenty-third problem

Further development of the calculus of variations.

Hilbert's problems

MATHEMATICAL PROBLEMS.*

LECTURE DELIVERED BEFORE THE INTERNATIONAL CONGRESS OF MATHEMATICIANS AT PARIS IN 1900.

BY PROFESSOR DAVID HILBERT.

WHO of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled questions pass before our minds and look over the problems which the science of to-day sets and whose solution we expect from the future. To such a review of problems the present day, lying at the meeting of the centuries, seems to me well adapted. For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future.

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* Translated for the BULLETIN, with the author's permission, by Dr. MARY WINSTON NEWSON. The original appeared in the *Göttinger Nachrichten*, 1900, pp. 253–297, and in the *Archiv der Mathematik und Physik*, 3d ser., vol. 1 (1901), pp. 44–63 and 213–237.

- Some problems resolved (affirmatively or negatively)
- Some problems are partially resolved, for some there is no consensus
- Some problems unresolved, including the eighth problem - the Riemann Hypothesis
- Some problems are stated too vaguely to describe as 'solved/unsolved'

Millennium Prize Problems

- In 2000, to celebrate mathematics in the new millennium, The Clay Mathematics Institute published a list of seven unsolved problems
- Aimed to record the most difficult problems of the time, to raise awareness with the public, and to recognise achievement
- Each of the seven problems has a \$1 million dollar prize for a solution
- The prizes were announced at the Collège de France, Paris in 2000
- The Riemann Hypothesis is the only problem on both lists

Millennium Prize Problems

- Birch and Swinnerton-Dyer conjecture (number theory)
- Hodge conjecture (algebraic geometry)
- Navier-Stokes equation (physics/PDE)
- P vs NP (computer science)
- Poincaré conjecture (geometric topology)
- Riemann Hypothesis (number theory)
- Yang-Mills and the Mass Gap (physics)

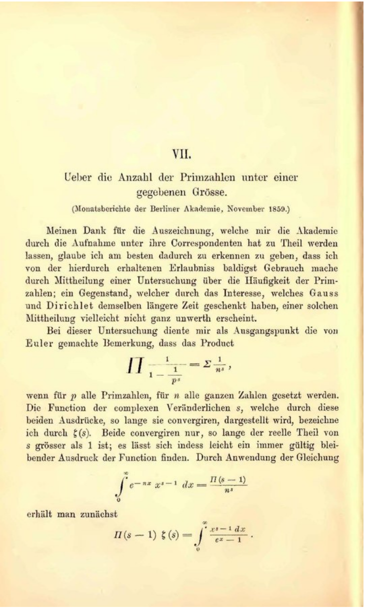
Millennium Prize Problems

- Only one problem solved so far - the Poincaré conjecture
- Perelman proved a more general result in 2002
- He turned down both the Millennium Prize in 2010, and the Fields Medal in 2006
- The prize instead funded the “Poincaré chair” at the Paris Institut Henri Poincaré

Bernhard Riemann

- Born 1826, died 1866. Worked in Göttingen.
- Geometry, analysis, number theory, physics
- Riemannian geometry, Riemann integral, Riemann surfaces, Riemann zeta function,...

Riemann's memoir



VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

- In 1859, Riemann published his only paper on number theory - the manuscript was six pages!
- Included several results and conjectures
- The only remaining unresolved conjecture from this memoir is the Riemann Hypothesis

The Riemann zeta function

Definition

Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then the **Riemann zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and in the whole complex plane by analytic continuation.

Riemann proved that $\zeta(s)$ extends meromorphically to $\mathbb{C}$ with only one pole, $s = 1$.

The functional equation and zeros

- Riemann also established the functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

  where $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ is the gamma function.

- **Trivial zeros:** $s = -2, -4, -6, -8\ldots$

- All non-trivial zeros must lie in the **critical strip**, $0 < \Re(s) < 1$.

The Riemann Hypothesis

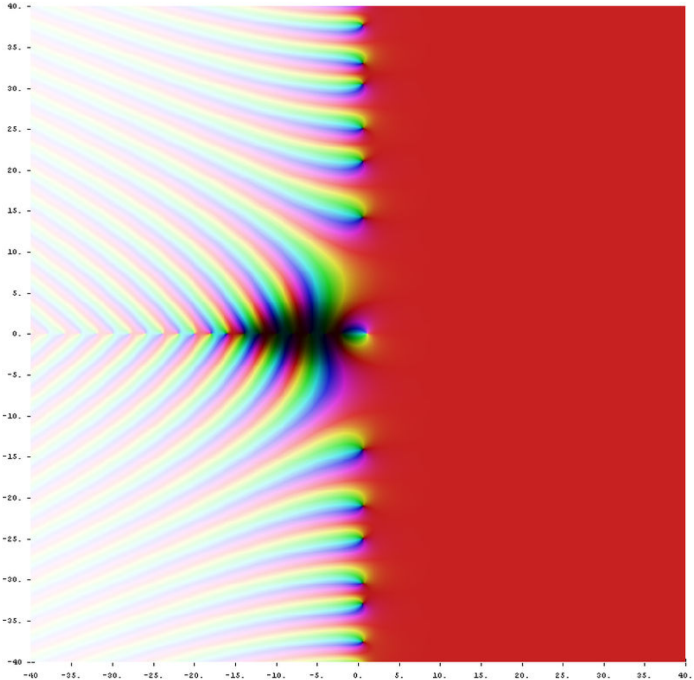
Riemann Hypothesis

The real part of all non-trivial zeros of the Riemann zeta function is $1/2$.

"Of course one would wish for a rigorous proof here; I have for the time being, after some fleeting vain attempts, provisionally put aside the search for this, as it appears dispensable for the immediate objective of my investigation."

Zeros

- Riemann calculated a few zeros; $\rho = 1/2 \pm it$, with $t = 14.13\ldots, 21.02\ldots, 25.01\ldots$
- Platt, Trudigan (2021) verified all non-trivial zeros up to height 3×10^{12} lie on the **critical line** $\Re(s) = 1/2$
- Hardy (1914) was first to show there are infinitely many zeros on the critical line
- Selberg (1942) proved that a positive proportion of zeros lie on the critical line
- Pratt, Robles, Zaharescu and Zeindler (2019) showed that 41.7% of zeros lie on the critical line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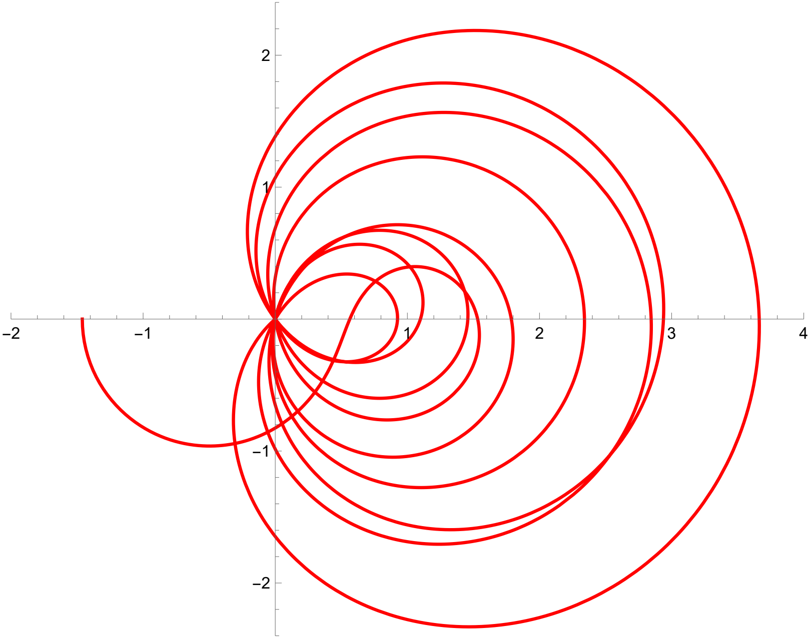
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Euler product and connecting to primes

- In 1748, Euler proved that for $\Re(s) > 1$,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

- Proof uses the Fundamental Theorem of Arithmetic
- Taking $s = 1$ gives an alternative proof that there are infinitely many primes
- (Left as exercises for the audience...)

Distribution of primes

Definition

For a given number X, we define the **prime counting function** to be

$$\pi(X) := \#\{p \leq X : p \text{ is prime}\}.$$

- Gauss (late 1700s) conjectured that as $X \to \infty$, we have

$$\pi(X) \sim \mathrm{li}(X) := \int_2^X \frac{dt}{\log(t)} \sim \frac{X}{\log(X)}.$$

- Legendre (1808) conjectured that

$$\pi(X) \sim \frac{X}{\log(X) - 1.08366}.$$

Distribution of primes

- Riemann conjectured there is an explicit formula for $\pi(x) - \mathrm{li}(x)$ involving a sum over the non-trivial zeros of zeta, proved by von Mangoldt in 1895.

- Finally, in 1896, Hadamard and de la Vallée Poussin independently proved:

Prime number theorem

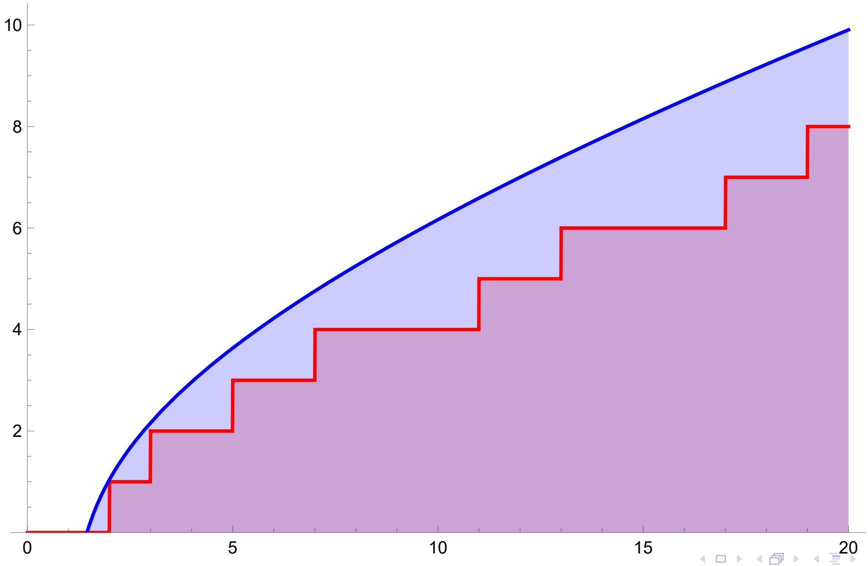
As $X \to \infty$, we have

$$\pi(X) \sim \mathrm{li}(X)$$

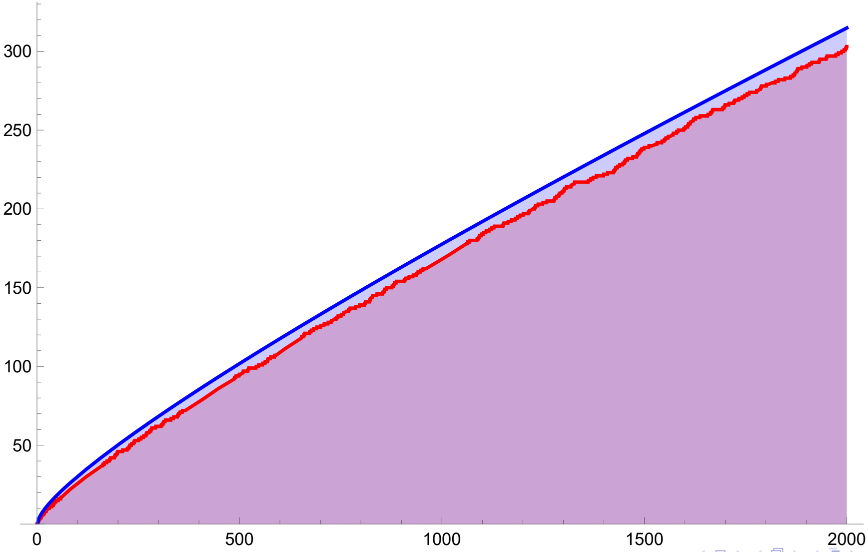
- Proof uses complex analysis and information about the zeros of ζ. Equivalent to no zeros with $\Re(s) = 1$

- RH is equivalent to $\pi(X) = \mathrm{li}(X) + O(\sqrt{X}\log(X))$

Prime Number Theorem



Prime Number Theorem



Prime Number Theorem