# Farey neighbours and tilings of the Poincaré upper half plane

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Cumberland Lodge Retreat – 23 February 2025

# Motivation

The first time you were ever asked to add fractions, you probably thought:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}. \tag{1}$$

You were told by your teacher that this is wrong, and the correct way was:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}. \tag{2}$$

If this can be reassuring, you weren't completely wrong!

Expression (1) is called a Farey sum.

# Motivation

Let the denominator map be the map given by

$$\mathrm{D}_{\mathbb{Z}} : \mathbb{Q} \to \mathbb{Z}_{>0}, \ x \mapsto \mathrm{D}_{\mathbb{Z}}(x) := b,$$

where $x = \frac{a}{b}$, with $\gcd(a, b) = 1$ and $b > 0$.

Write the Farey sum using the notations $\oplus$ and $\mathrm{D}_{\mathbb{Z}}$, we get

$$x \oplus y = \frac{\mathrm{D}_{\mathbb{Z}}(x)x + \mathrm{D}_{\mathbb{Z}}(y)y}{\mathrm{D}_{\mathbb{Z}}(x) + \mathrm{D}_{\mathbb{Z}}(y)}.$$

Farey sums are just weighted averages! And have lots of nice properties!

# Farey sequence

The Farey sequence $\mathfrak{F}$ is defined using the following rules:

1. First two elements in $\mathfrak{F}$ are 0 and 1, and are Farey neighbours.
2. For $x, y \in \mathfrak{F}$, if $x$ and $y$ are Farey neighbours, add $x \oplus y$ to $\mathfrak{F}$, and declare that the Farey neighbours of $x \oplus y$ are $x$ and $y$.

$$
\begin{array}{lccccccccccccc}
\mathfrak{F}_1: & & & & & & 0 & & 1 & & & & & \\
\mathfrak{F}_2: & & & & & & 0 & \frac{1}{2} & 1 & & & & & \\
\mathfrak{F}_3: & & & & & 0 & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & 1 & & & & \\
\mathfrak{F}_4: & & & & 0 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & 1 & & & \\
\mathfrak{F}_5: & & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & 1 & \\
\mathfrak{F}_6: & 0 & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & 1
\end{array}
$$

**Facts.**

a. The Farey sequence $\mathfrak{F}$ is simply a reordering of the set of all rational numbers in the interval $[0, 1]$ by the size of their denominators.
b. As we keep repeating this procedure, we will eventually reach any rational number $0 < x < 1$.

# Farey neighbours

1. Let $\mathfrak{F}$ be the set of all rational numbers in the interval $[0, 1]$.
2. Recall that 0 and 1 are Farey neighbours.
3. For $x \in \mathfrak{F}$, with $0 < x < 1$, consider the set

$$\mathfrak{F}_x := \{y \in \mathfrak{F} : \mathrm{D}_{\mathbb{Z}}(y) < \mathrm{D}_{\mathbb{Z}}(x)\} .$$

Note that $\mathfrak{F}_x$ is finite (Exercise).

The Farey neighbours of $x$ are the two elements in $x_1, x_2 \in \mathfrak{F}_x$, with the largest denominators, such that $x_1 < x < x_2$.

# Farey tiling of the Poincaré upper half plane

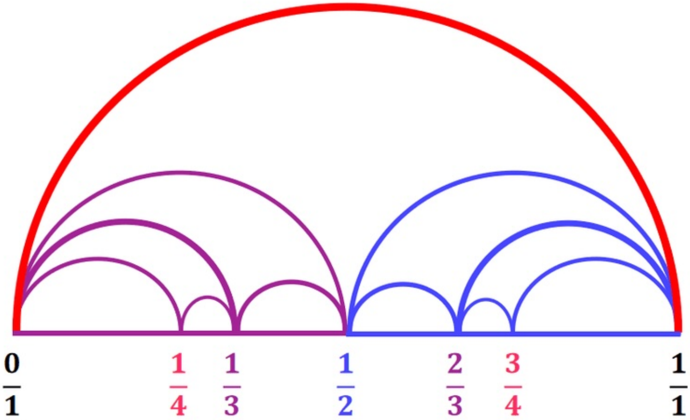
The Poincaré upper half plane is defined by

$$\mathfrak{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

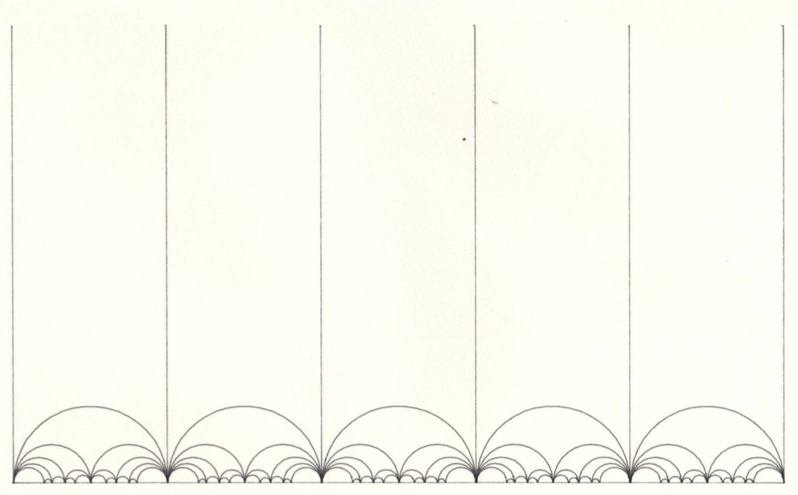
The Farey tiling or tessellation of $\mathfrak{H}$ is obtained as follows:

1. If $x, y \in \mathfrak{F}$ are Farey neighbours, join them by a semi-circle centred on the $x$-axis.
2. Draw vertical lines from $0, 1$ towards $\infty$.
3. For any integer $n \in \mathbb{Z}$, translate the above picture by $n$.

# Farey tiling of the Poincaré upper half plane



# Farey tiling of the Poincaré upper half plane



# Farey tiling of the Poincaré upper half plane

**Remark.**

1. Farey tiling is a very important 2-dimensional hyperbolic geometry.
2. In hyperbolic geometry, one can show that the (basic) hyperbolic triangles are all isometric.
3. In hyperbolic geometry, the shortest distance between any two points $z, z' \in \mathfrak{H}$ is the length of the arc on a semi-circle centred on the $x$-axis going through $z$ and $z'$.
4. The Farey titling uses those semi-circles that touch the $x$-axis at Farey neighbours.

# Möbius action on Poincaré upper half plane

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{H} \to \mathfrak{H}$$
$$(\gamma, z) \mapsto \gamma \cdot z := \frac{az+b}{cz+d}, \ \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is well-defined because:

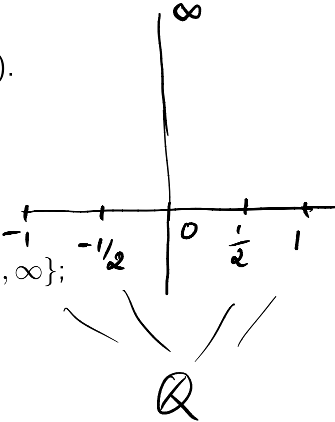
1. $cz + d \neq 0$, for all $c, d \in \mathbb{R}$ since $\mathrm{Im}(z) > 0$.
2. $\mathrm{Im}(\gamma z) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$, for all $z \in \mathfrak{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

**Theorem**

*For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the map $(\mathfrak{H} \to \mathfrak{H}, z \mapsto \gamma z)$ is bi-holomorphic. The inverse holomorphic map is given by $(\mathfrak{H} \to \mathfrak{H}, z \mapsto \gamma^{-1} z)$.*

# Extended Poincaré upper half plane

- Cusps: $\mathbf{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$.
- Extended upper half plane: $\mathfrak{H}^* := \mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q})$.




For a cusp $\sigma \in \mathbf{P}^1(\mathbb{Q})$, set

$$\gamma\sigma := \begin{cases} \infty, & \text{if } \sigma = -\frac{d}{c}; \\ \dfrac{a\sigma + b}{c\sigma + d}, & \text{if } \sigma \notin \{-\frac{d}{c}, \infty\}; \\ \dfrac{a}{c}, & \text{if } \sigma = \infty. \end{cases}$$

Get extended action:

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{H}^* \to \mathfrak{H}^*$$
$$(\gamma, z) \mapsto \gamma z := \frac{az + b}{cz + d}$$

# Distance to a cusp

The Siegel distance to a cusp is defined by

$$\begin{aligned} \mathbf{P}^1(\mathbb{Q}) \times \mathfrak{H} &\to \mathbb{R}_{>0} \\ (\sigma, z) &\mapsto d(\sigma, z), \ z := x + it, x \in \mathbb{R}, t \in \mathbb{R}_{>0}, \end{aligned}$$

$$d(\sigma, z) := \begin{cases} \dfrac{1}{t}, & \text{if } \sigma = \infty, \\ \mathrm{D}_{\mathbb{Z}}(\sigma) \dfrac{(x-\sigma)^2 + t^2}{t}, & \text{else.} \end{cases}$$

**Theorem**

*The distance map is $\mathrm{SL}_2(\mathbb{Z})$-invariant, i.e.*

$$d(\gamma\sigma, \gamma z) = d(\sigma, z), \ \textit{for all } z \in \mathfrak{H} \textit{ and } \sigma \in \mathbf{P}^1(\mathbb{Q}).$$

# Equidistant locus to two distinct cusps

The equidistant locus to two distinct cusps $\sigma, \tau \in \mathbf{P}^1(\mathbb{Q})$ is defined by

$$\Delta(\sigma, \tau) := \{z \in \mathfrak{H} : d(\sigma, z) = d(\tau, z)\}.$$

**Theorem**

*Let $\sigma, \tau \in \mathbf{P}^1(\mathbb{Q})$ be distinct cusp. Then, we have*

$$\Delta(\sigma, \tau) = \begin{cases} C(\sigma, \mathrm{D}_{\mathbb{Z}}(\sigma)^{-1}), & \text{if } \tau = \infty. \\ C(\sigma \ominus \tau, \varrho(\sigma, \tau)), & \text{if } \sigma, \tau \neq \infty, \ \mathrm{D}_{\mathbb{Z}}(\sigma) \neq \mathrm{D}_{\mathbb{Z}}(\tau), \\ \{z \in \mathfrak{H} : |z - \sigma| = |z - \tau|\}, & \text{else.} \end{cases}$$

*where $C(\sigma, \varrho)$ is the circle of centre $\sigma$ and radius $\varrho$,*

$$\varrho(\sigma, \tau) := \frac{\mathrm{D}_{\mathbb{Z}}(\sigma)\,\mathrm{D}_{\mathbb{Z}}(\tau)|\sigma - \tau|}{|\,\mathrm{D}_{\mathbb{Z}}(\sigma) - \mathrm{D}_{\mathbb{Z}}(\tau)|}.$$

# Farey neighbours and equidistant locus

## Theorem

*Let $\sigma \in \mathbf{P}^1(\mathbb{Q})$ be a finite cusp such that $0 < \sigma < 1$. Let $0 \leq \tau_1, \tau_2 \leq 1$ two finite cusps such that $\mathrm{D}_{\mathbb{Z}}(\tau_1), \mathrm{D}_{\mathbb{Z}}(\tau_2) < \mathrm{D}_{\mathbb{Z}}(\sigma)$. Then, $\tau_1$ and $\tau_2$ are the Farey neighbours of $\sigma$ if and only if $\varrho(\sigma, \tau_1)$ and $\varrho(\sigma, \tau_2)$ are the two smallest equidistant radii among such cusps. In that case, we have*

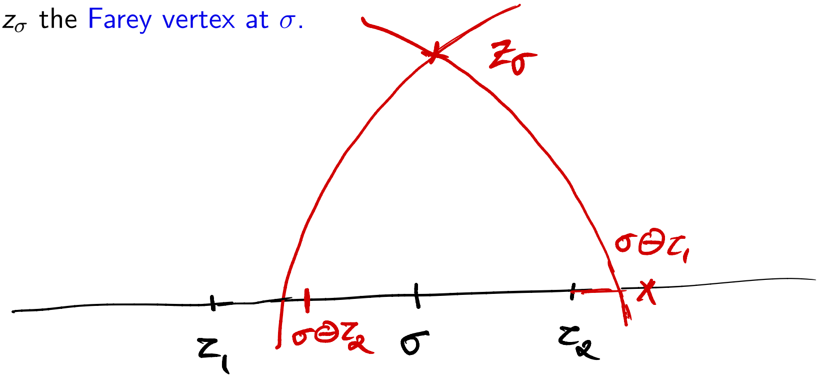
$$\varrho(\sigma, \tau_1) = \frac{1}{\mathrm{D}_{\mathbb{Z}}(\sigma \ominus \tau_1)},$$
$$\varrho(\sigma, \tau_2) = \frac{1}{\mathrm{D}_{\mathbb{Z}}(\sigma \ominus \tau_2)}.$$

# Farey graph

**Theorem**

*Let $\sigma \in \mathbf{P}^1(\mathbb{Q})$ be a finite cusp such that $0 < \sigma < 1$, and $\tau_1$ and $\tau_2$ be the Farey neighbours of $\sigma$. Then, the circles $C(\sigma \ominus \tau_1, \varrho(\sigma, \tau_1))$ and $C(\sigma \ominus \tau_2, \varrho(\sigma, \tau_2))$ intersect at a unique point $z_\sigma \in \mathfrak{H}$.*

We call $z_\sigma$ the Farey vertex at $\sigma$.

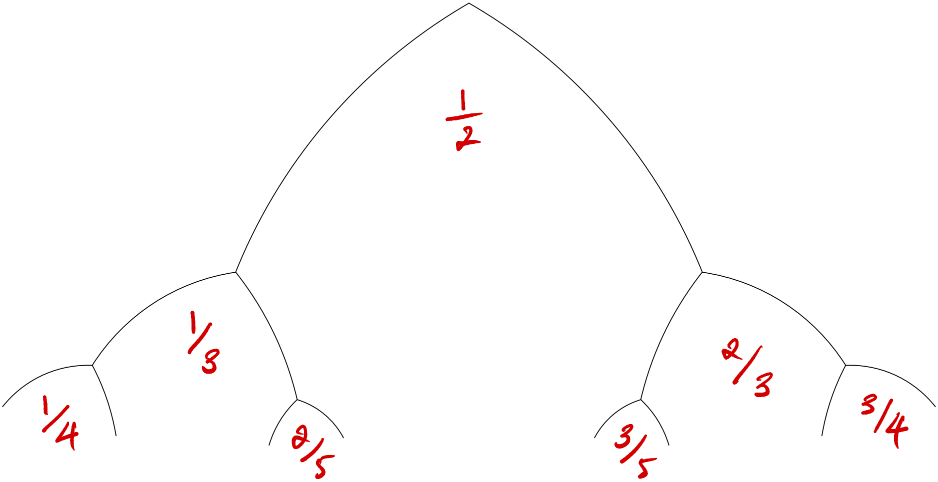


# Farey graph

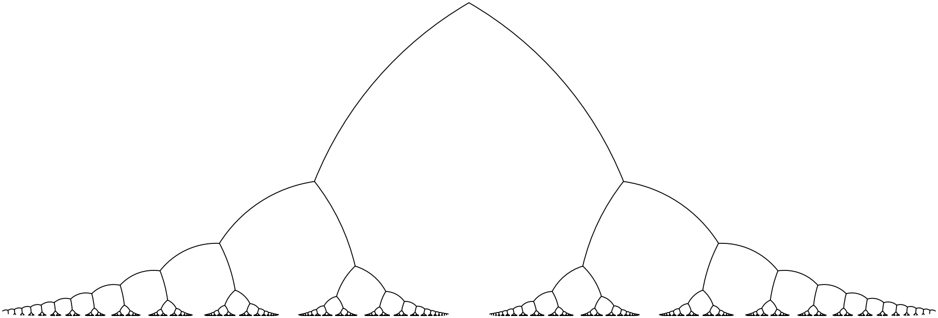
The geometric Farey graph $G = (V, E)$ is obtained as follows:

1. $V$ consists of all the Farey vertices associated to (finite) cusps.
2. Let $\sigma \in \mathbf{P}^1(\mathbb{Q})$ be a finite cusp such that $0 < \sigma < 1$; and let $\tau_1$ and $\tau_2$ be the Farey neighbours of $\sigma$. We connect $z_\sigma$ to $z_{\sigma \oplus \tau_1}$ and $z_{\sigma \oplus \tau_2}$ with the arc on the semi-circles $C(\sigma \ominus \tau_1, \varrho(\sigma, \tau_1))$ and $C(\sigma \ominus \tau_2, \varrho(\sigma, \tau_2))$.

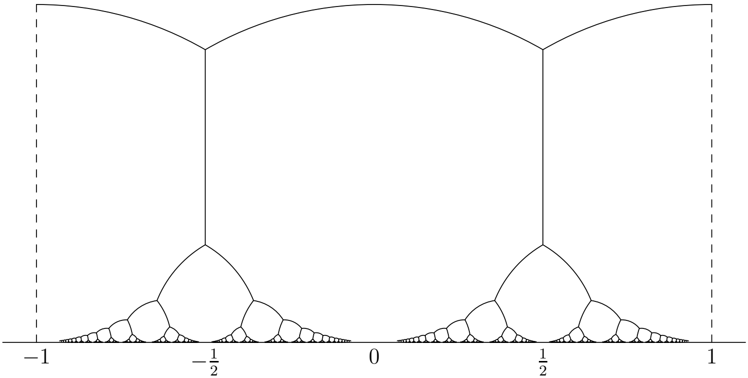
# Farey graph: Denominator at most 5



# Farey graph: Denominator at most 20



# Farey graph: Denominator at most 20



# Application

**Theorem**

*Every matrix* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *can be written as a finite product (word) in*

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thank you for your attention!