Games we play

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Zermelo's Theorem (1913)

In chess, **precisely one** of the following holds:

a. White has a winning strategy
b. Black has a winning strategy
c. Both the white and the black can force (at least) a draw

Most believed

Ruled out:

One of the sides can only force a draw and the other side cannot force a draw

Ernst Zermelo (1871 – 1951)

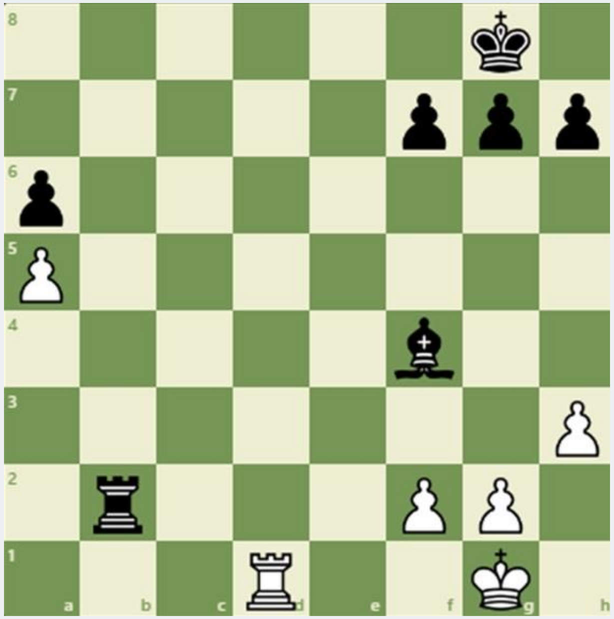
Proof of Zermelo's Theorem (incomplete)

- For simplicity modify rules. Loss if same position is repeated.
  E.g. White & Black move knight to initial position – loss for Black.

- Game is "finite" – bounded (but huge) number of moves.

- Auxiliary construction: Position **"Checkmate in k moves"**, $k \geq 1$.

- Checkmate in 1 move for White if:
  a) White move
  b) There exists a White move checkmates.



Proof of Zermelo's Theorem (cont.)
Auxiliary construction: checkmate in k moves

- Checkmate in k moves for White, $k \geq 2$ if:
  a) White move
  b) Exists White move s.t. every Black move results in "Checkmate in m moves", $m < k$.
  c) Minimal such k.

- E.g. "checkmate in 2 moves": move s.t. every Black move results in "checkmate in 1 move".

- E.g. "checkmate in 3 moves": move s.t. every Black move results in "checkmate in 2 moves" or "checkmate in 1 move".

Checkmate for white in
$k = 200$ moves
Draw under 50 moves rule

Proof of Zermelo's Theorem (cont.)

- 1. Assume initial position is "Checkmate in k moves", **some** k (White)

Winning strategy for White: Find the prescribed move that every Black move will bring to "Checkmate in m moves" $m < k$.
By induction, game will end at most k moves, **White wins**.

- 2. Initial position not in "Checkmate k moves", every k. **Negate.**

For every White move there exists a Black move that results in a position that is (still) not "Checkmate k moves", every k.

Winning strategy for Black: In response to White move, find move that results in "not "Checkmate k moves", every k".
Since game is finite, **Black wins**.

Proof of Zermelo's Theorem (cont.)

- Subtlety: What if black repeats position? What about castling?

- Negate "Initial position "Checkmate in k moves", **some** *k*".

For **every** White move there **exists** a Black move that results in a position that is (still) not "Checkmate k moves", **every** *k*.

- Negation of "Checkmate in k moves" doesn't quite do the trick, since Black might repeat position. Depends on move history (game tree).

- Solution (hint): Instead "Victory in k moves".

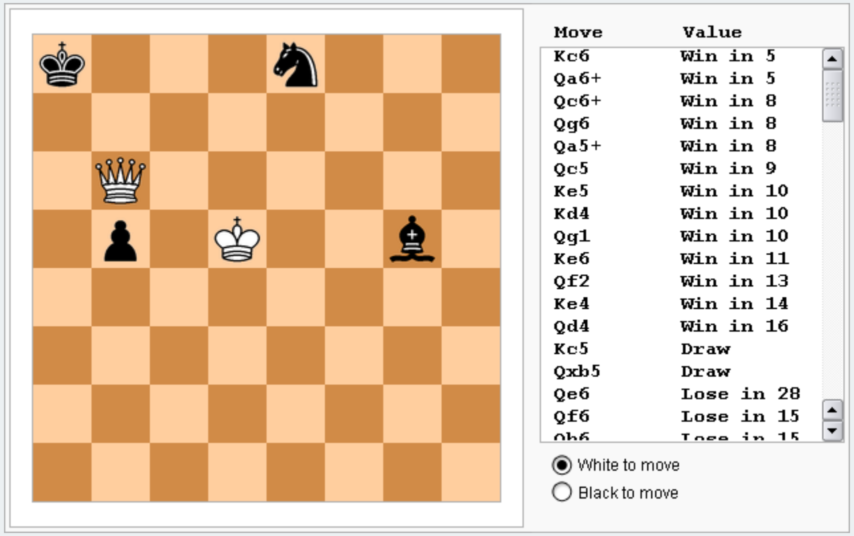
Allows "win by repetition".

Records all previous positions as part of induction.

Otherwise use the game tree.

Some insights into chess

- Chess is believed to be a "draw game"
  For one side to win, the other **must** make a mistake.
- '**First-move advantage**'???
- No meaning "advantage" (eg +1.5 pawns). Position "win/lose/draw".
- Precise meaning "mistake", even lost position.
- **Nalimov tablebases**
  Solve every position with ≤ 7 pieces.
  Huge amount of memory and computation. 8 pieces no pawns (2021).
  Closest to finding a solution, still very far.



Zermelo's Theorem (general)

Every **finite** [1] **deterministic** [2] **game** [3] of **two players** with **perfect information** [4] has a **solution** [5].

This theorem is applied to many games, not just chess!

Game Theory

Ernst Zermelo (1871 – 1951)

Solution of a game

Rudolph and Blitzen are given 100 carrots.

Rudolph can give X number of carrots to Blitzen out of the 100.

Blitzen can only **accept** or **decline** the offer.

If Blitzen agrees, she receives X number of carrots, and Rudolph receives the remaining.

If Blitzen declines, they both get 0 carrots.

Solution of a game (cont.)

What's the solution?
50 carrots each? No!

Rudolph decides to give 1 carrot to Blizten, and 99 carrots to himself.

Will Blitzen accept?

Assumes players **rational** – not always. Rational refuse low offers?

'Solution' might not be the most appropriate.

One-shot Vs. Repeated games. Force fair outcome.

People care (mainly) about their own benefit
Lifts in Covent Garden station

Zero vs nonzero sum games

- Zero sum game: one player's gain is the other player's loss.
  (Not: "Nobody gains from playing".)

- E.g. chess. 1M prize money 700K winner, 300K loser. Still zero sum.
- Both players can't gain. No common interest.

- Pure zero sum games are very rare. Even wars are non-zero sum. Prisoners exchange, rules of warfare, Geneva convention etc.

Zero vs nonzero sum games (cont.)

- Carrot dividing game – **not zero sum.**
- Strong common interest, smaller conflict of interest.

- Common life situation.

- Winning strategy for X: Present Y with two options A & B, where A is the most preferred for X, and A is better than B for Y (but <C).

Brexit negotiations

- Strong common interest (UK & EU): Make deal.
- Weaker contradictory interests.

EU: Retain the EU citizens in UK, maximize payout (£35B), punish UK*

UK: Minimize the payout (£35B), improve the deal.

- "Fair solution": EU citizens stay, no payout.

- **Why then did UK get such terrible, unfair, deal?**

- <u>Self-inflicted</u> "carrots game" declared that all EU citizens will be allowed to stay.
- Terrible strategy "fair towards unfair".
- Nobody is fair (even at low stake).

Battle of sexes / communication

- A married couple wants to spend their time together.
- Wife (W) wants to go to opera.
- Husband (H) wants to go to a football match.

- W prefers football with H over opera alone.
- H prefers opera with W over football alone.
- Winning strategy for H:

"Darling, I go to football, you are free to do whatever you please".

- Symmetric strategy for W.
- Rationality assumption reasonable?

- Message real estate agent: "You now have two options, stay at £…k or increase your offer, please do let me know what you would like to do?"

Repeated games, imperfect information

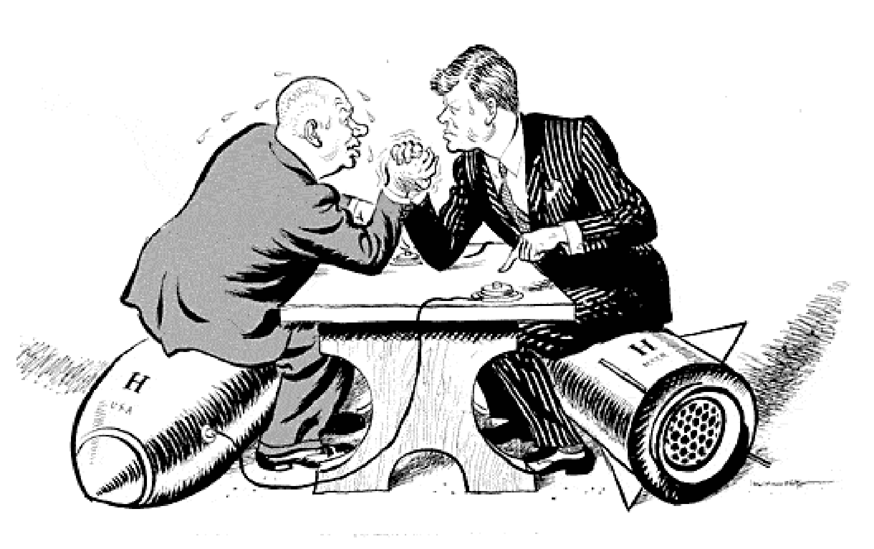
Nuclear disarmament negotiations: USA vs. USSR

- Imperfect information: neither side knew the nuclear arsenal of other.
- Repeated: negotiators meet every year.

Example: USA 200 bombs, USSR 100.

Destroying 100 bombs each:
good for the USA, bad for the USSR.

USA destroying 150, USSR destroying 50: good for USSR, bad for USA.



Kennedy versus Khrushchev: Cold War Political Cartoon

Disarmament negotiations

- Nuclear secrets might be (partially) revealed by way of negotiations.

- What information should the negotiator by given to optimize the result?

Robert J. Y. Aumann

- Mathematician, Nobel Prize in Economics Sciences 2005. Speech “War and piece”.

- Developed the theory of **repeated games.**

- Aumann’s theorem asserts: better send a negotiator who has **no information**.

Repeated games

- Solution very different than for single games.
- E.g. Blitzen can force Rudolph to make higher offers by repeatedly declining unfair deals.
- Assumes rationality. Is it rational to refuse low offers, thus punish unfair players?
- No impact on chess or zero sum games. The outcome will be the same every time (assuming correct strategy).
- Explains “Si vis pacem, para bellum”.

War is not irrational. To avoid war, dangerous to dismiss it.

Repeated games (cont.)

Aumann's theory applicable:

- Elections (opinion polls having an impact).

- **Brexit negotiations** – single or repeated?

- Single for UK, repeated for EU. Important to punish UK.
  UK stronger interest to make a deal?

- No theory "semi-repeated" games? "one shot" vs "long lived"? Idea PhD?

- Another example: Does a gym charge registration fee?

Repeated games (cont.)

- War strategies. Invest in defensive of attacking weapons?
- Moral conundrum: If enemy kills your civilians, to preserve own population, should order killing enemy's civilians?
- Would there be attack on nuclear facilities in Middle East?
- Warfare rules set by Geneva conventions and International Humanitarian Law etc.
- Rules are meaningless if violated by one of the sides without regulating mechanism. Only relevant because of "repeated" aspect.

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Concluding questions

Solve modified chess, e.g. two moves at a time?

Q1. Guarantees White win?
*No winning strategy for Black (prove!)

Another variant: 1 White move, then 2 each
Q2. Can one compare the difficulty?