Towards Depth-Bounded Natural Deduction for Classical First-Order Logic

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Abstract

In this paper we lay the foundations of a new proof-theory for classical first-order logic that allows for a natural characterization of a notion of *inferential depth*. The approach we propose here aims towards extending the proof-theoretical framework presented in [6] by combining it with some ideas inspired by Hintikka's work [18]. Unlike standard natural deduction, in this framework the inference rules that fix the meaning of the logical operators are symmetrical with respect to assent and dissent and do not involve the discharge of formulas. The only discharge rule is a classical dilemma rule whose nested applications provide a sensible measure of inferential depth. The result is a hierarchy of decidable depth-bounded approximations of classical first-order logic that expands the hierarchy of tractable approximations of Boolean logic investigated in [11, 10, 7].

1 Introduction

In the Manifesto of the Vienna Circle, written in 1929, Rudolph Carnap, Hans Hahn and Otto Neurath claimed that logic is analytic and tautological:

> Logical investigation leads to the result that all thought and inference consists of nothing but a transition from statements to other statements that contain nothing that was not already in the former (tautological transformation). [...] The scientific world-conception knows only empirical statements about things of all kinds, and analytic statements of logic and mathematics. [5, pp. 308, 311]

In a series of papers collected in the volume *Logic, Language-Games and Information* published in 1973, Hintikka attacked the logical empiricists' thesis. Starting from the Church-Turing result that classical first-order logic is undecidable (1936), Hintikka argues that there is a class of polyadic first-order logical truths that are synthetic and informative. He formulates the theory of distributive normal forms for classical first-order logic, on the basis of which he defines two objective and non-psychological notions of information content. The former, which he calls "depth information", is equivalent to Bar-Hillel and Carnap's semantic information and might not be increased by deductive reasoning, thus justifying the traditional claim that logic is tautological. The latter, which he calls "surface information", might be increased by deductive reasoning and is calculable in practice, thus vindicating the idea that logic is informative.

The non-trivial deductive reasoning that does increase surface information is, according to Hintikka, to be regarded as synthetic. But, of course, the terms "analytic" and "synthetic" are given a meaning that is different from the one used in the Vienna Circle and that is founded on an original interpretation of the distinction put forward by Kant in this first *Critique*. According to Hintikka, a derivation is synthetic if at least one of its steps introduces *new individuals*; a derivation is analytic if all its steps merely discuss the individuals which we have already introduced. Inferences can be synthetic at any degree k and whether or not a sentence follows from a given set of premises by means of a k-degree synthetic inference is *decidable* for every fixed k.[1]

Despite Hintikka's rejection of the idea that all logical inferences are analytic, his approach still classifies as analytic a wide class of inferences that includes not only many valid inferences of polyadic predicate logic, but also the entire set of valid inferences of propositional logic and monadic predicate logic. As a result, his work might be charged of being only a *partial* vindication of the idea that logical deduction is informative. These doubts find an important confirmation in the theory of computational complexity: if the decision problem for Boolean logic is (most

[1] For a thorough discussion of Hintikka's view and its comparison with Kant's analytic/synthetic distinction, see [22].

probably) intractable[2], that is to say, undecidable in practice, how is it possible to maintain that propositional logic is analytic and uninformative?

This observation was at the origin of the approach of Depth-Bounded Boolean Logics (DBBLs) [11, 10, 7]. In this approach the standard semantics of the Boolean operators is replaced by a weaker "informational semantics" whereby the meaning of a logical operator $\star$ is fixed by specifying the sufficient and the necessary conditions for an agent a to *actually possess* the information that a $\star$-sentence φ is true (respectively false), and be therefore in the disposition to *assent to* (respectively *dissent from*) it, solely in terms of the information that a actually possesses about the immediate components of φ. An inference is analytic if its conclusion can be established in terms of the actual information that is implicitly contained in its premises according to this weaker explanation of the logical operators. Synthetic inference are those that essentially require the introduction of "virtual information", i.e., information that we do not actually possess, but must be temporarily assumed in order to reach the conclusion (as, for example, in the discharge rules of natural deduction). In this approach a propositional inference can be synthetic at any given degree k (the "depth" of the inference), depending on the nested use of virtual information; moreover, whether or not a sentence follows from a given set of premises by means of a k-degree synthetic inference is *tractable* for every fixed k.

Our main purpose in this paper is to lay the foundations for a unified treatment of classical first order logic that brings together the main insights of the two approaches outlined above. More specifically, our main aim is to extend to the standard quantifiers the informational semantics for the Boolean operators in order to obtain a general view of the analytic/synthetic distinction and of the classification of inferences in terms of their depth.

The main contributions of this paper are: i) a natural characterization of the intuitive "surface" meaning of quantifiers along the same lines as the characterization given for the Boolean operators in DBBLs; (ii) the definition of a suitable first-order extension of the propositional natural deduction system of [7][3] and of the associated notion of inferential depth, in such a way that k-depth inference is decidable. Typical technical results such as soundness, completeness and subformula property are stated but their proofs are omitted.

[2] See, for example, [14].

[3] See also [6] for a thorough proof-theoretical investigation.

2 Is logical inference "tautological"?

What do "analytic" and "tautological" mean? The affinity between the notion of analyticity and the discipline of logic is not something natural and atemporal, but rather the result of a precise historical development. This development, which concerns both the analytic-synthetic distinction and the conception of logic, might be summarized in three turning points. First, according to Kant, a judgment is analytic if and only if the concept of the predicate is (covertly) contained in that of the subject [20, A6-7/B10-11]. However, although Kant uses traditional logic as an instrument to define the notions, such as that of containment, that lie at the basis of his definition of analyticity, in his first *Critique* (1781, 1787^2) he is not interested in determining whether logic itself is analytic. It is with Frege's *Foundations of Arithmetic* (1884), that represents our second step, that the relationship between logic and analyticity becomes stronger. Frege holds that a truth is analytic if and only if it can be proved with help of logical laws from definitions only [13, §3, p. 4]; as a result, logical truths, being provable through logical laws, are analytic. Interestingly enough, however, Frege explicitly rejected the idea that truths or inferences that are analytic in this sense are uninformative. With the logical empiricist movement, we reach the third step: logical truths are assumed to be analytic and they are used in order to catch the rest of analytic truths and inferences. Even W.V.O. Quine, despite his thorough criticism of the sharp analytic/synthetic distinction made by the Vienna Circle, in his *Two dogmas of empiricism* (1951), maintained that logical truths can after all be safely classified as "analytic" [24, p. 23].[4]

Now, if we assume, as the logical empiricists do, that logical deduction is analytic — and thus its conclusions result from some kind of analysis that unfolds the meaning of the logical operators —, then we seem to be obliged to conclude that it must be trivial, that is, uninformative and tautological, at least on the basis of the standard theory of semantic information [9]. This appears to be the side effect of the paradox of analysis [21, p. 323], which states that analysis cannot be sound and informative at the same time: for if it is sound, the analyzed and the analyzandum are equivalent and analysis cannot be augmentative; and if it is informative, then the analyzed and the analyzandum are not equivalent and the analysis is incorrect. The logical empiricists were bold enough to accept the "triviality" of deductive reasoning as a consequence of their commitment to the principle of analyticity of logic.

[4]On this point see also [8], Section 3.

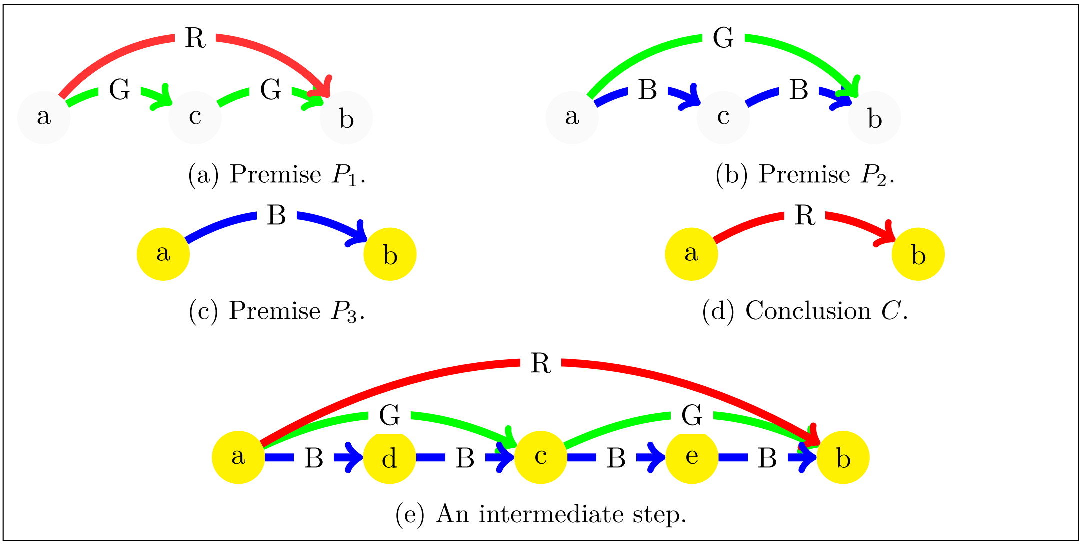

Figure 1: Configurations of individuals involved in the argument from premises P_1, P_2 and P_3 to conclusion C.

3 Hintikka on "synthetic" logical inference

In order to convey the main idea underlying Hintikka's approach, while avoiding technicalities, consider the following example, which is a simplified version of the case first presented in [4] and then discussed in [17, p. 86 ff.], that illustrates a kind of reasoning that is synthetic according to Hintikka's sense of the term. Consider the argument from the premises P_1, P_2 and P_3 to the conclusion C:

$$
\begin{aligned}
&P_1 : \forall x \forall y (Rxy \to \exists z (Gxz \land Gzy))\\
&P_2 : \forall x \forall y (Gxy \to \exists z (Bxz \land Bzy))\\
&P_3 : \forall x \forall y ((Bxy \land Cx) \to Cy)\\
&C : \forall x \forall y ((Rxy \land Cx) \to Cy).
\end{aligned}
$$

As Figure 1 suggests, P_1 says that whenever two points are connected through a red arrow, then there exists a third point, which is interpolated through green arrows. Similarly, P_2 says that whenever two points are connected through a green arrow, then there exists a third point, which is interpolated through blue arrows. P_3 says that whenever two points are connected through a blue arrow and the former is colored, then also the latter is colored. Similarly, C says that whenever two points are connected through a red arrow and the former is colored, then also the latter is colored.

What is the reasoning that leads us from the premises to the conclusion? We could start from premise P_1 and say that whenever two points, a and b, are connected through a red arrow, then there exists a third point, call it c, which is interpolated through green arrows. Then, we could use premise P_2 and reason as follows. Since a and c are connected through a green arrow, then there is another individual d, which is linked to a and c through blue arrows; similarly, since also c and b are connected through a green arrow, then there is a fifth point e, which is linked to c and b by blue arrows. Then, given the premise P_3, if a is colored, then also d is colored (because they are connected through a blue arrow); for the same reason and given that d is colored, then also c is colored; again, since c is colored then also e is colored; last, we get that b is also colored. In this way and since we didn't assume anything about the instantiating individuals, we reach the general conclusion that the colored marker ink spreads along red arrows too.

According to Hintikka's theory, this argument is synthetic, because some of its intermediate steps introduce new individuals into the argument. In particular, the intermediate step depicted in Figure 1e makes use of individuals d and e, which do not enter the configurations of the premises and conclusion of the argument, but, at the same time, are in certain relations with the other individuals.

As mentioned in the Introduction, this approach still classifies as analytic and informationally trivial all the inferences of propositional logic and of the monadic predicate calculus. This has been widely regarded as unsatisfactory especially in light of the development of the theory of NP-completeness according to which the decision problem for propositional logic is most likely to be intractable. The tension between the (probable) intractability of Boolean logic and its alleged informational triviality seems very similar to the tension that motivated Hintikka in arguing that the undecidability of first-oder logic is at odds with the philosophical claim that its inferences are analytic and tautological.

4 Depth-Bounded Boolean Logics

Standard formalizations of classical logic cannot capture the essential difference between these two inferences:

$$\frac{\begin{array}{l} P \vee Q \\ Q \to R \\ \neg P \end{array}}{R} \qquad \frac{\begin{array}{l} P \vee Q \\ P \to Q \end{array}}{Q}$$

The argument to establish the soundness of the first inference is the following:

$$\begin{array}{ll} 1 & P \lor Q \\ 2 & Q \to R \\ 3 & \neg P \\ \hline 4 & Q \text{ (from 1 and 3)} \\ 5 & R \text{ (from 2 and 4)} \end{array}$$

Notice that, here, at each step we are using information that we actually possess. On the other hand, a typical argument for the second example would run as follows:

$$\begin{array}{llll} & & 1 & P \lor Q \\ & & 2 & P \to Q \\ \hline 3.1 & \textit{Suppose that } P & 3.2 & \textit{Suppose that } \neg P \\ & Q \text{ (from 2 and 3.1)} & & Q \text{ (from 1 and 3.2)} \\ \hline \multicolumn{4}{c}{Q} \end{array}$$

The sense in which the conclusion of the first argument is "implicitly contained" in the premises is different from the sense in which the conclusion of the second argument is. In the latter we make essential use of information that we do not *actually possess* and is not even implicitly contained in the information that we actually possess. This is what we call "virtual information". We simulate information states that are richer than the actual one and consider the two possible outcomes of the process of acquiring such information.

In Gentzen's Natural Deduction the use of virtual information is associated with a technical device, known as "discharging of assumptions":

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ P \lor Q\end{array} \quad \begin{array}{c}\Delta, [P]\\ \vdots\\ R\end{array} \quad \begin{array}{c}\Lambda, [Q]\\ \vdots\\ R\end{array}}{R} \qquad \dfrac{\begin{array}{c}\Gamma, [P]\\ \vdots\\ Q\end{array}}{P \to Q} \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ (\exists x)P(x)\end{array} \quad \begin{array}{c}\Delta, [P(a)]\\ \vdots\\ R\end{array}}{R}$$

with the usual restrictions on a. In the propositional rules, the sentences in square brackets represent *virtual* information that may not be (and typically is not) contained in the information that is actually "given" in the premises. In the existential quantifier rule, the sentence in the square brackets may represent information on an individual that is not actually "given" in the set of individuals associated with the quantifiers that occur in the premises.

After making this fundamental distinction between inferences that make use only of actual information and those that require the use of virtual information, we can ask ourselves the following question: can we fix the meaning of the logical operators in terms of the *information that is actually possessed* by an agent, that is, without appealing to virtual information?

The *informational semantics* of the logical operators is based on the following principle:

> The meaning of an n-ary logical operator $\star$ is determined by specifying the sufficient (necessary) conditions for an agent x to actually hold the information that a sentence of the form $\star(P_1, \ldots, P_n)$ is true, respectively false, in terms of the information that x actually holds about the truth or falsity of $P_1, \ldots, P_n$.

Here by saying that x *actually holds* the information that P is true (respectively false) we mean that this is information *practically available* to x and with which x can operate (e.g., in decision-making).

In [10, 8, 7, 6] a suitable set of introduction and elimination (*intelim*) rules for the Boolean operators were presented that comply with the basic principle of informational semantics. These rules characterize a subsystem of classical propositional logic that is a logic in Tarski's sense and is *tractable*. Interestingly enough, this logical system is sound and complete w.r.t. to a *non-deterministic matrix*, in the sense of [1], that complies with the basic principle of informational semantics and was first proposed by W.V.O. Quine in [25] to capture the "primitive" meaning of the logical operators.[5] The full deductive power of classical propositional logic is retrieved by adding a single discharge rule that governs the use of virtual information and consists in a form of *classical dilemma* rule. The maximum number of nested application of this single discharge rule that are needed to obtain a conclusion from a given set of premises provides a natural measure of the *propositional depth* of the associated inference. For each given k, k-depth validity can also be decided in polynomial time, so providing an infinite hierarchy of *tractable* approximations to classical propositional logic.[6] In the next section we shall propose a way of extending these rules to first-order logic, to provide a similar measure of the quantificational depth of an inference and a hierarchy of *decidable* approximations to full classical logic.

To summarize, we have examined the main ideas of two theories that reject the logical empiricists' tenet that logic is analytic and tautological. Hintikka focuses on the the tension between this tenet and the undecidability of first-order logic. In his conceptual framework, an inference is analytic if it does not introduce new individuals into the argument beyond those that one needs to consider in order to grasp the premises and the conclusion. The approach of DBBLs focuses on the similar tension between the (probable) intractability of Boolean logic and the claim, shared

[5] See [8] for a discussion.

[6] Recently in [2] the DBBL approach has been adopted as the logical foundation of a depth-bounded approach to belief functions.

by Hintikka and the logical empiricists, that it is informationally trivial. According to this perspective, an inference is analytic if and only if the informational meaning of the logical operators is sufficient to derive the conclusion from the premises; an inference is synthetic (at different *degrees*) when virtual information (up to a certain depth) is needed to derive the conclusion from the premises.

Although each of these two theories suggests a compelling reason for which logic is informative, neither of them, if taken in isolation, is sufficient to provide a *complete* vindication of the thesis that first-order logical inferences are synthetic and informative. On the one hand, Hintikka's work classifies as analytic all propositional and monadic inferences. On the other, DBBLs are restricted to propositional logic and do not capture the dimension of quantificational depth that in Hintikka's work is related to the introduction of new individuals that were not "given" in a surface understanding of the premises.

The main contribution of this paper consists in merging the two approaches by introducing a new family of logical systems, that we call *Depth-Bounded First-Order Logics* (DBFOLs), which extends DBBLs to first-order languages by exploiting Hintikka's insight (in particular, appropriate rules for quantifiers are added to the introduction and elimination rules for DBBLs). The structure of DBFOLs, which resembles that of DBBLs, is given by an infinite hierarchy of logics representing increasing levels of syntheticity or informativeness of classical first-order logic. The logic $\vdash_0$, which is the basic element of the hierarchy, validates only analytic inferences; for every $k > 0$, the logic $\vdash_k$ validates synthetic inferences in such a way that the greater k is, the more synthetic and informative are the inferences that are valid in it. Here, the terms "analytic" and "synthetic" are given a new meaning that conciliates the intuitions of Hintikka's work and of the DBBL-approach.

5 An intuitive informational semantics for quantifiers

In order to define inference rules that comply with the basic ideas of informational semantics outlined in the previous section, we need to ask ourselves a fundamental question. What do we mean when we say that we hold the information that a sentence of the form $\forall xF$ or $\exists xF$ is true, respectively false?

Let's start with the notion of *actually possessing* the information that $\forall xF$ is true. The answer cannot be that we actually possess the information that $F[x/a]$ is true for *all* the infinitely many individuals that may be denoted by a. A more feasible answer is the following: we are in the disposition to assent to *any* sentence of the form $F[x/a]$. A typical analogy widely used in this context is that we have an urn W whose composition is unknown to us, and if we draw an individual at random from this urn, assign to it the name a (or a is its given name, in case it already has one), then we are in the disposition to assent to $F[x/a]$. We may also imagine that, once an individual has been drawn from the W urn, we move it into a box D that represents the known domain of discourse. The composition of the box D, unlike that of W, is fully known and always consists of a finite number of individuals at each stage of the reasoning process. How many draws from W are needed in association with the given universal quantifier in order to grasp the meaning of the sentence? It makes sense to say that, in order to grasp the meaning of $\forall x F$, envisaging a situation in which an agent is in the disposition to assent to $F[x/a]$ for *any single* random draw of an individual a from W *as well as* for all individuals a contained in D is a minimal sufficient meaning condition. We can call this explanation the *surface meaning* of $\forall$.

Similar considerations can be made for the case of the falsity of $\exists x F$. That an agent actually possesses the information that $\exists x F$ is false, or equivalently that $\neg\exists x F$ is true, means in essence that for any possible draw of an individual a from the W urn and for all the individuals a in D, the agent is in the disposition to dissent from $F[x/a]$, i.e., to assent to $\neg F[x/a]$, and this explanation is sufficient to grasp the meaning of $\neg\exists x F$.

What about the notion of actually possessing the information that $\exists x F$ is true? A natural answer is that one is informed that a search for an individual that fits the description given by the open sentence F (assuming that F contains x as a free variable) will eventually be successful. The search involves both the urn W and the box D, meaning that the sought individual might be unknown or already known. Similar considerations hold for the notion of actually possessing the information that the sentence $\forall x F$ is false, or equivalently that $\neg\forall x F$ is true. We are guaranteed the the search for an individual that fits the description given by the open sentence $\neg F$ will eventually succeed. We can call this explanation the *surface meaning* of $\exists$.

6 Perfect PNF and analytic rules for quantifiers

Our aim is to put forward a set of introduction and elimination rules for quantifiers that are in accordance with their surface meaning as fixed by the intuitive informational semantics outlined in the previous section. Their application will therefore be *analytic* as well as informationally trivial.

In order to keep technicalities to minimum and focus on the conceptual analysis, we shall assume that all *premises* of an inference are in prenex normal form. It is well-known that this involves no loss of generality, for it is computationally easy to transform every formula in a formula in prenex normal form such that the matrix has exactly the same Boolean structure as the original formula — namely it results from the original formula by removing all quantifiers. We shall not impose, however, that any formula other than the premises is in prenex normal form.

In what follows we shall assume also that the quantified variables in the *set of premises* Γ of an inference are renamed in such a way that (i) the number of existentially quantified variables occurring in Γ is maximal and (ii) the number of universally quantified variables occurring in Γ is minimal. When such a renaming has been performed we shall say that Γ is in *perfect normal form.* This kind of transformation of the premises, albeit non essential, makes for easier formulation of suitable analytic rules for quantifiers. The transformation can be avoided at the price of more contrived restrictions on the quantifier rules or of a new kind of format for proofs, other than the standard Gentzen-style or Fitch-style format. For the purposes of this preliminary investigation we shall therefore adopt this simplification and leave other options for future research.

Recall that a formula is in *prenex normal form* (PNF) if it has the form

$$Q_{x_1} \cdots Q_{x_n} F[x_1, \ldots, x_n]$$

where each Q_i is either an occurrence of $\forall$ or an occurrence of $\exists$, F is quantifier-free and all variables in F are bound by some quantifier in the prefix. A formula is in *minimal PNF* (*min-PNF*) if there is no logically equivalent formula with the same matrix and a lower number of occurrences of quantifiers in the prefix.

Examples 1. *$\forall x \exists y \forall z (Rxy \wedge Ryz)$ is in min-PNF. $\forall x \exists y \forall z \forall w (Rxy \wedge Ryz)$is not, for the last occurence of $\forall$ is redundant. $\forall x \forall y (Px \wedge Py)$ is not, for it is equivalent to $\forall x (Px \wedge Qx)$.*

A *set* Γ of formulae is in *perfect normal form* (PPNF) if:

- Every formula in Γ is in min-PNF;
- All occurrences of existential quantifiers in Γ bind variables that are different from each other and from all the universally quantified variables;
- The number of distinct universally quantified variables occurring in Γ is minimal.

Every set Γ of formulae in PNF can be easily transformed into a set Γ' in PPNF, by renaming of variables, in such a way that every formula A in Γ is transformed into a logically equivalent formula A' in Γ' such that the matrix of A' is the same as the matrix of A modulo renaming of variables.

Examples 2. *The set*

$$\{\forall x \exists y \forall z (Rxy \wedge Rxz), \forall x \exists y \forall w (Sxy \wedge Sxw)\}$$

is not in PPNF. A possible transformation of this set into PPNF is the following:

$$\{\forall x \exists y \forall z (Rxy \wedge Rxz), \forall x \exists w \forall z (Sxw \wedge Sxz)\}.$$

The main motivation behind the requirement that the set of premises of an inference is in PPNF is given by the following:

Proposition 1. *Let* Γ *be a set of formulae in PPNF. Then, the number of distinct bound variables in* Γ *is the same as the number of distinct bound variables in any min-PNF of* $\bigwedge \Gamma$*.*

So, we can argue that when a set of premises is in PPNF the number of distinct individuals that are considered in these premises and are required to grasp their surface meaning is mirrored by the number of distinct bound variables. Moreover, the situation does not change if we take any min-PNF of the conjunction of the premises.

The preliminary transformation of any set of premises into PPNF makes the explanation of the inference rules quite transparent. All inferences that can be justified by means of these rules will be "analytic" in the following (informal) sense:

> No more individuals need to be considered in proving the conclusion than those that were already considered in grasping the surface meaning of the premises. (PQA)

PQA stands for "Principle of Quantificational Analyticity". This sense of analyticity is not strictly equivalent to any of the senses discussed by Hintikka in [19], except perhaps the sense IIIc (p. 181).

Recall that when a set of premises is in PPNF, all occurrences of the existential quantifier bind different variables, while different occurrences of the universal quantifier may bind the same variable. When a set of premises is in PPNF, every distinct universally quantified *variable* involves the consideration of a distinct "arbitrary" individual, and every distinct existentially quantified variable involves the consideration of a distinct "specific" individual. These are all the individuals that may be regarded as been "thought of" in the premises. To take a simplest example, in order to grasp the surface meaning of the set of premises $\{\exists x \forall y Rxy, \forall y \exists z Syz\}$, we need to consider three distinct individuals. The first is a specific one that results from the search, associated with the existentially quantified variable x, of an individual that fits the description given by the open formula $\forall y Rxy$. The second is an unknown one, say b, which is drawn at random from the W urn and is associated with the universally quantified variable y. The same "drawn" individual b can be taken to instantiate the universally quantified variable y in the second formula. Finally we need to consider a third specific individual, say c, which results from the search associated with the existentially quantified z and fits the description given by the open formula Sbz.

Note that, according to our explanation of the informational meaning of $\forall$, holding the information that $\forall x\exists yRxy$ means that we are in the disposition to assent to $\exists yRay$ for *any* single unknown individual drawn from the W urn, as well as for *all* already known individuals taken from the D box (when it is not empty). Being in the disposition to assent to $\exists yRay$ means that we hold the information that the search for a suitable individual that fits $\exists yRay$ will eventually be successful. This does not immediately imply that we hold information about a specific individual that does fit the description. However, when $\forall x\exists yRxy$ is used as premise of an inference, we may consider as part of the surface meaning of $\forall x\exists yRxy$ that the result of this search can somehow be "given" to us. This allows us to choose a new name for this individual, say b and infer Rab. These are the two individuals that are thought of in grasping the meaning of the premises, as witnessed by the fact that any "concrete" (e.g. graphical) explanation of what counts as a model of this sentence would need to involve two individuals and no more. In essence, what $\forall x\exists yRxy$ says is: "let a be an arbitrary unknown individual drawn from W and let b any result of the search for an individual that fits the description $\exists yRay$, we are in the disposition to assent to Rab".

Now, it makes sense to claim that the meaning of $\forall$ implies that we are also in the disposition to assent to $\exists yRby$. However, it would not be equally natural to assume that the new search for an individual that fits this description and the result of this search has already been thought of in the premise $\forall x\exists yRxy$ and therefore required to grasp its surface meaning.

For the sake of simplicity we assume that our first-order language contains *no constants* and is equipped with a *set of parameters* (as in [28]) $a, b, c, \ldots$ possibly with subscripts, that may occur in the proof, but neither in the premises, nor in the conclusion. Given that formulae are in PPNF, let the *Q-complexity* of a finite set Γ of premises be the number of distinct variables that occur in Γ (we assume all variables are bound by some quantifier in the prefix). Then our notion of analytic proof in the sense of (PQA) above, that is in a sense that is restricted to the informational meaning of the quantifiers, can be simply rephrased as follows:

> A proof is analytic *only if* the number of distinct parameters that occur in it never exceeds the Q-complexity of its initial premises. (PQA*)

The use of "only if" stems from the fact that a proof may be analytic in the sense of (PQA*) above and yet be synthetic in that it may still make use of virtual information at the propositional level. In the sequel we shall see how the structural DBBL rule that governs the introduction of virtual information may be used to mark the transition to the next degree of depth both at the propositional and the quantificational level, so as to provide a unified approach to the notion of analytic proof for full first-order logic.

Our next problem is then: can we define a set of intelim rules for quantifiers that comply with the surface informational meaning of the quantifiers outlined in Section 5 and deliver *only* analytic inferences in the sense of (PQA*)? Can we construe the transition from analytic to synthetic inferences and from one degree of depth to the next *only* in terms of the nested use of virtual information?

As far as no virtual information is used, we shall display our proofs in the format of sequences of *signed formulae*, of the form $\mathsf{T}\,A$ or $\mathsf{F}\,A$. In accordance with the informational approach to classical logic our interpretation of signed formulae will be non-standard. We take "$\mathsf{T}\,A$" to mean "we actually possess the information that A is true" and "$\mathsf{F}\,A$" to mean "we actually possess the information that A is false". So the signs T and F do not refer to classical truth and falsity (as in Smullyan's semantic tableaux [28]), but to "informational truth" and "informational falsity". As mentioned above, a natural way of thinking of these notions is in terms of an agent's disposition to *assent* to a sentence or *dissent* from it depending on the available information. A straightforward consequence of this epistemic interpretation of the signs is that one cannot assume, in general, that for every sentence A an agent is either in the disposition of assenting to A or in the disposition of dissenting from A. When neither is the case, the agent may abstain for lack of sufficient information.

Although the use of signed formulae is appropriate for conceptual clarity, it is by no means essential in our approach. For all practical purposes one can always revert to standard formulae simply by removing all the T signs and replacing all the F signs with the negation operator.

In the DBBL approach, for each logical operator, there are intelim rules for a signed formula containing it as main operator as well as for its conjugate (the conjugate of "$\mathsf{T}\,A$" is "$\mathsf{F}\,A$" and viceversa) . This feature is shared by the tableau method (which, however, is restricted to refutations of sets of formulae via elimination rules only) and other bilateral systems of deduction, such as Bendall's [3] or Rumfit's [26]. The first-order version of the propositional DBBL rules, that allows for the presence of parameters in formulae, is given in Figures 2 and 3. In the sequel we shall make use of the following notation:

- F^x_a denotes the result of replacing every occurrence of the variable x in F with

$$\dfrac{\begin{array}{c}\mathsf{T}\,A \to B\\ \mathsf{T}\,A\end{array}}{\mathsf{T}\,B}\,\mathsf{T} \to E_1 \qquad \dfrac{\begin{array}{c}\mathsf{T}\,A \to B\\ \mathsf{F}\,B\end{array}}{\mathsf{F}\,A}\,\mathsf{T} \to E_2$$

$$\dfrac{\mathsf{F}\,A \to B}{\mathsf{T}\,A}\,\mathsf{F} \to E_1 \qquad \dfrac{\mathsf{F}\,A \to B}{\mathsf{F}\,B}\,\mathsf{F} \to E_2$$

$$\dfrac{\begin{array}{c}\mathsf{T}\,A \vee B\\ \mathsf{F}\,A\end{array}}{\mathsf{T}\,B}\,\mathsf{T} \vee E_1 \qquad \dfrac{\begin{array}{c}\mathsf{T}\,A \vee B\\ \mathsf{F}\,B\end{array}}{\mathsf{T}\,A}\,\mathsf{T} \vee E_2$$

$$\dfrac{\mathsf{F}\,A \vee B}{\mathsf{F}\,A}\,\mathsf{F} \vee E_1 \qquad \dfrac{\mathsf{F}\,A \vee B}{\mathsf{F}\,B}\,\mathsf{F} \vee E_2$$

$$\dfrac{\mathsf{T}\,A \wedge B}{\mathsf{T}\,A}\,\mathsf{T} \wedge E_1 \qquad \dfrac{\mathsf{T}\,A \wedge B}{\mathsf{T}\,B}\,\mathsf{T} \wedge E_2$$

$$\dfrac{\begin{array}{c}\mathsf{F}\,A \wedge B\\ \mathsf{T}\,A\end{array}}{\mathsf{F}\,B}\,\mathsf{F} \wedge E_1 \qquad \dfrac{\begin{array}{c}\mathsf{F}\,A \wedge B\\ \mathsf{T}\,B\end{array}}{\mathsf{F}\,A}\,\mathsf{F} \wedge E_2$$

$$\dfrac{\mathsf{F}\,\neg A}{\mathsf{T}\,A}\,\mathsf{F}\neg E] \qquad \dfrac{\mathsf{T}\,\neg A}{\mathsf{F}\,A}\,\mathsf{T}\neg E$$

Figure 2: Elimination rules for the propositional operators.

the parameter a;

- F^a_x denotes the result of replacing every occurrence of the parameter a with the variable x;
- $F[a/x]$ denotes the result of replacing some or all occurrences of a with x.

$\mathsf{T}\,\forall$-elimination and $\mathsf{F}\,\exists$-elimination.

$$\dfrac{\mathsf{T}\,\forall x F}{\mathsf{T}\,F^x_a}\;\mathsf{T}\,\forall E \qquad\qquad \dfrac{\mathsf{F}\,\exists x F}{\mathsf{F}\,F^x_a}\;\mathsf{F}\,\exists E$$

where a is any parameter that already occurs above in the proof; or else a is a *new* parameter, provided that no other parameter has been already introduced above by an application of the same rule to a formula of the form $\mathsf{T}\,\forall x G$, respectively $\mathsf{F}\,\exists x G$.

$$\frac{\mathsf{F}\,A}{\mathsf{T}\,A \rightarrow B}\mathsf{T}\rightarrow I_1 \qquad \frac{\mathsf{T}\,A}{\mathsf{T}\,A \vee B}\mathsf{T}\vee I_1 \qquad \frac{\mathsf{F}\,A}{\mathsf{F}\,A \wedge B}\mathsf{F}\wedge I_1$$

provided that every parameter occurring in B already occurs above.

$$\frac{\mathsf{T}\,B}{\mathsf{T}\,A \rightarrow B}\mathsf{T}\rightarrow I_2 \qquad \frac{\mathsf{T}\,B}{\mathsf{T}\,A \vee B}\mathsf{T}\vee I_2 \qquad \frac{\mathsf{F}\,B}{\mathsf{F}\,A \wedge B}\mathsf{F}\wedge I_2$$

provided that every parameter occurring in A already occurs above.

$$\frac{\begin{array}{c}\mathsf{T}\,A\\ \mathsf{F}\,B\end{array}}{\mathsf{F}\,A \rightarrow B}\mathsf{F}\rightarrow I \qquad \frac{\begin{array}{c}\mathsf{F}\,A\\ \mathsf{F}\,B\end{array}}{\mathsf{F}\,A \vee B}\mathsf{F}\vee I \qquad \frac{\begin{array}{c}\mathsf{T}\,A\\ \mathsf{T}\,B\end{array}}{\mathsf{T}\,A \wedge B}\mathsf{T}\wedge I$$

$$\frac{\mathsf{T}\,A}{\mathsf{F}\,\neg A}\mathsf{F}\neg I \qquad \frac{\mathsf{F}\,A}{\mathsf{T}\,\neg A}\mathsf{T}\neg I$$

Figure 3: Introduction rules for the propositional operators.

In essence, each bounded variable x in the premise of these rules can be instantiated at most once by a new parameter, denoting an unknown individual drawn from the W urn, although it can be instantiated by all the old parameters denoting known individuals in the box D.

Example 1. *Example of a* wrong *application of the* $\mathsf{T}\forall E$ *rule (the quantifier* $\forall y$ *has been used at step 3 to introduce the new parameter* b *and again at step 5 to introduce* *the new parameter c).*

1	$\mathsf{T}\,\forall x \forall y Rxy$	*Premise*
2	$\mathsf{T}\,\forall y Ray$	$\mathsf{T}\,\forall E, 1$
3	$\mathsf{T}\,Rab$	$\mathsf{T}\,\forall E, 2$
4	$\mathsf{T}\,\forall y Rby$	$\mathsf{T}\,\forall E, 1$
5	$\boldsymbol{\mathsf{T}\,Rbc}$	$\mathsf{T}\,\forall E, 4$
6	$\mathsf{T}\,Rab \wedge Rbc$	$\mathsf{T}\,\wedge I, 3, 5$
7	$\mathsf{T}\,\forall z(Rab \wedge Rbz)$	$\mathsf{T}\,\forall I, 6$
8	$\mathsf{T}\,\forall y \forall z(Ray \wedge Ryz)$	$\mathsf{T}\,\forall I, 7$
9	$\mathsf{T}\,\exists x \forall y \forall z(Rxy \wedge Ryz)$	$\mathsf{T}\,\exists I, 8$

$\mathsf{T}\,\exists$-elimination and $\mathsf{F}\,\forall$-elimination.

$$\frac{\mathsf{T}\,\exists x F}{\mathsf{T}\,F^x_a}\ \mathsf{T}\,\exists E \qquad\qquad \frac{\mathsf{F}\,\forall x F}{\mathsf{F}\,F^x_a}\ \mathsf{F}\,\forall E$$

provided that a is a *new* parameter and no other parameter has been already introduced above by an application of the same rule to a formula of the form $\mathsf{T}\,\exists x G$, respectively $\mathsf{F}\,\forall x G$.

In essence, each bounded variable x in the premise of these rules can be instantiated at most once by a new parameter, denoting the result of a search for an individual that fits the description in F.

When these rules are applied, we say that the new parameter a in the conclusion of the rule is *critical* and *depends* on all the other parameters occurring in F.

Example 2. *Example of a* wrong *application of the* $\mathsf{T}\exists E$ *rule (the quantifier* $\exists y$ *has been used at step 3 to introduce the new parameter b and again at step 5 to introduce* *the new parameter c).*

1	$\mathsf{T}\,\forall x \exists y Rxy$	*Premise*
2	$\mathsf{T}\,\exists y Ray$	$\mathsf{T}\forall E, 1$
3	$\mathsf{T}\,Rab$	$\mathsf{T}\,\exists E, 2$
4	$\mathsf{T}\,\exists y Rby$	$\mathsf{T}\forall E, 1$
5	$\boldsymbol{\mathsf{T}\,Rbc}$	$\mathsf{T}\,\exists E, 4$
6	$\mathsf{T}\,Rab \wedge Rbc$	$\mathsf{T}\wedge I, 3, 5$
7	$\mathsf{T}\,\exists z(Rab \wedge Rbz)$	$\mathsf{T}\,\exists I, 6$
8	$\mathsf{T}\,\exists y \exists z(Ray \wedge Ryz)$	$\mathsf{T}\,\exists I, 7$
9	$\mathsf{T}\,\forall x \exists y \exists z(Rxy \wedge Ryz)$	$\mathsf{T}\forall I, 8$

$\mathsf{T}\,\forall$-introduction and $\mathsf{F}\,\exists$-introduction.

$$\frac{\mathsf{T}\,F}{\mathsf{T}\,\forall x F^a_x}\,\mathsf{T}\forall I \qquad\qquad \frac{\mathsf{F}\,F}{\mathsf{F}\,\exists x F^a_x}\,\mathsf{F}\,\exists I$$

Provided that a is not critical and F does not contain critical parameters depending on a. Moreover, x is not bound in F.

Example 3. *Example of a* wrong *application of the* $\mathsf{T}\forall I$ *rule (b is a critical parameter depending on a).*

1	$\mathsf{T}\,\forall x \exists y Rxy$	*Premise*
2	$\mathsf{T}\,\exists y Ray$	$\mathsf{T}\forall E, 1$
3	$\mathsf{T}\,Rab$	$\mathsf{T}\,\exists E, 2$
4	$\boldsymbol{\mathsf{T}\,\forall x Rxb}$	$\mathsf{T}\forall I, 3$
5	$\mathsf{T}\,\exists y \forall x Rxy$	$\mathsf{T}\,\exists I, 4$

$\mathsf{T}\,\exists$-introduction and $\mathsf{F}\,\forall$-introduction.

$$\frac{\mathsf{T}\,F}{\mathsf{T}\,\exists x F[a/x]}\,\mathsf{T}\,\exists I \qquad\qquad \frac{\mathsf{F}\,F}{\mathsf{F}\,\forall x F[a/x]}\,\mathsf{F}\,\forall I$$

provided x is not bound in F.

General restriction on quantifier eliminations. In order to guarantee that the Principle of Quantificational Analyticity (PQA*) is always satisfied, we also need the following *general restriction* on the application of the quantifier elimination rules: their application is allowed only if *their premise is not the conclusion of an introduction.* It is easy to see how the violation of this general restriction may lead to violations of (PQA*). Consider, for example, the following proof:

1	$\mathsf{T}\,\forall x\exists yRxy$	premise
2	$\mathsf{T}\,\exists yRay$	from 1
3	$\mathsf{T}\,Rab$	from 2
4	$\mathsf{T}\,\exists zRaz$	from 3
5	$\mathsf{T}\,\forall x\exists zRxz$	from 4
6	$\mathsf{T}\,\exists zRbz$	from 5
7	$\mathsf{T}\,Rbc$	from 6
8	$\mathsf{T}\,Rab \wedge Rbc$	from 3 and 7
9	$\mathsf{T}\,\exists z(Rab \wedge Rbz)$	from 8
10	$\mathsf{T}\,\exists y\exists z(Ray \wedge Ryz)$	from 9
11	$\mathsf{T}\,\forall x\exists y\exists z(Rxy \wedge Ryz)$	from 10

Here, the number of paramenters occurring in the proof exceeds the Q-complexity of the premise. The proof is not analytic.

0-depth inferences (analytic sequences). An *analytic sequence based on* Γ, where Γ is in PPNF, is any sequence of signed formulae starting from the formulae in $\mathsf{T}\,\Gamma = \{\mathsf{T}\,B \mid B \in \Gamma\}$ and such that each subsequent signed formula results from signed formulae previously occurring in the sequence by means of an application of the intelim rules. An *analytic proof* of A *from* Γ is an analytic sequence based on $\mathsf{T}\,\Gamma$ that ends with $\mathsf{T}\,A$. We say that A is *deducible from* Γ *at depth* 0 when there is an analytic proof of A from Γ.

Note that $A, \neg A \vdash_0 B$ for any B, as shown by the following analytic sequence:

1	$\mathsf{T}\,A$	premise
2	$\mathsf{T}\,\neg A$	premise
3	$\mathsf{F}\,A$	from 2 by $\mathsf{T}\neg E$
4	$\mathsf{T}\,A \vee B$	from 1 by $\mathsf{T}\vee I$
5	$\mathsf{T}\,B$	from 4 and 3 by $\mathsf{T}\vee E$

However, this sequence is not "analytic" in one of the widespread senses of this word, in that it does not enjoy the subformula property (and indeed there is no 0-depth proof of B from $\{A, \neg A\}$ with the subformula property). If we want the subformula property to hold in general, we need to modify our definition of 0-depth deducibility as follows. An analytic sequence based on Γ is *closed* if it contains both $\mathsf{T}\, B$ and $\mathsf{F}\, B$ for some formula B. Otherwise we say that it is *open*. A 0*-depth refutation* of Γ is a closed analytic sequence based on Γ. Then we say that A is *deducible from* Γ *at depth* 0 when either there is 0-depth proof of A from Γ or a 0-depth refutation of Γ. On the other hand, if our notion of analytic proof is restricted to sequences with the subformula property, then the previous notion of analytic proof delivers a paraconsistent notion of 0-depth deducibility. Since our aim in this paper is to outline a depth-bounded approach to classical first-order logic, we shall adopt the amended definition of 0-depth deducibility. Note, however, that according to this definition, *not all classically inconsistent set of formulae are explosive*, but only those whose inconsistency can be detected at depth 0, i.e., by virtue of the surface informational meaning of the logical operators.[7]

The notion of 0-depth inference intends to capture the idea of an inference that is performed by virtue of the surface informational meaning of the quantifiers and makes no use of virtual information (no discharge of temporary hypothetical assumptions). The following example illustrate the restriction on the Boolean introduction rules that are needed to preserve the quantificational analyticity of proofs.

Example 4. *Example of a* wrong *application of the* $\mathsf{T} \vee I$ *rule (at step 4, a new parameter, namely c, occurs in the second disjunct of* $\mathsf{T} Rab \vee Rbc$*).*

1	$\mathsf{T}\, \forall x \exists y Rxy$	*Premise*
2	$\mathsf{T}\, \exists y Ray$	$\mathsf{T} \forall E, 1$
3	$\mathsf{T}\, Rab$	$\mathsf{T}\, \exists E, 2$
4	$\boldsymbol{\mathsf{T}\, Rab \vee Rbc}$	$\mathsf{T} \vee I, 3$
5	$\mathsf{T} \forall z(Rab \vee Rbz)$	$\mathsf{T} \forall I, 4$
6	$\mathsf{T}\, \exists y \forall z(Ray \vee Ryz)$	$\mathsf{T}\, \exists I, 5$
7	$\mathsf{T}\, \exists x \exists y \forall z(Rxy \vee Ryz)$	$\mathsf{T}\, \exists I, 6$

[7] For a further discussion of this point see [6, Section 8].

Example 5. $\mathsf{T}\forall x\exists y Rxy$, $\mathsf{T}\forall x\forall w(Rxw \to Rwx) \vdash_0 \mathsf{T}\forall x\exists y(Rxy \wedge Ryx)$

1	$\mathsf{T}\,\forall x\exists y Rxy$	*Premise*
2	$\mathsf{T}\,\forall x\forall w(Rxw \to Rwx)$	*Premise*
3	$\mathsf{T}\,\exists y Ray$	$\mathsf{T}\,\forall E, 1$
4	$\mathsf{T}\,Rab$	$\mathsf{T}\,\exists E, 3$
5	$\mathsf{T}\,\forall w(Raw \to Rwa)$	$\mathsf{T}\,\forall E, 2$
6	$\mathsf{T}\,Rab \to Rba$	$\mathsf{T}\,\forall E, 5$
7	$\mathsf{T}\,Rba$	$\mathsf{T} \to E, 4, 6$
8	$\mathsf{T}\,Rab \wedge Rba$	$\mathsf{T}\,\wedge I, 4, 7$
9	$\mathsf{T}\,\exists y(Ray \wedge Rya)$	$\mathsf{T}\,\exists I, 8$
10	$\mathsf{T}\,\forall x\exists y(Rxy \wedge Ryx)$	$\mathsf{T}\,\forall I, 9$

Let us write $\Gamma \vdash_0 A$ whenever A is 0-depth deducible from Γ, and let $\vdash_C$ denote the relation of deducibility in classical first order logic. The soundness of $\vdash_0$ with respect to $\vdash_C$ is trivial.

Proposition 2. $\Gamma \vdash_0 A \Longrightarrow \Gamma \vdash_C A$.

It can also be shown that:

Proposition 3. *If $\Gamma \vdash_0 A$, then there exists an analytic proof of A from Γ with the subformula property.*

For the propositional part, the proof can be found in [6]. Its extension to the first-order case is immediate given the general restriction on quantifier eliminations, according to which the premise of a quantifier elimination cannot be the conclusion of an introduction. In principle one could impose a similar restriction on the propositional elimination rules, so as to obtain only proofs with the subformula property.

Note that, as for its propositional counterpart, $\vdash_0$ is a Tarskian logic, i.e. it satisfies reflexivity, monotonicity, transitivity and substitution invariance. Moreover, it is not difficult to show that 0-depth inferences satisfy (PQA*), and that this fact implies the following:

Proposition 4. *The logic $\vdash_0$ is decidable.*

Given that $\vdash_0$ is tractable for its propositional fragment, we also conjecture that it is *tractable* also in the first-order case, but a proof of this conjecture will be the topic of future research.

To summarize, the 0-depth first-order logic captures a notion of analytic inference that makes no use of virtual information and satisfies the (PQA*) principle of quantificational analyticity.

7 Depth-bounded natural deduction for full first-order logic

Virtual information. The role of virtual information in the DBBL approach has been briefly illustrated in Section 4 and is discussed at length in [11, 10, 8, 7, 6]. In the context of this section it will be convenient to work with the relation of k-*depth derivability* between finite sets of *signed formulae* and *signed formulae* defined in the obvious way. Accordingly we can say that A is k-depth deducibile from Γ whenever $\mathsf{T}\,A$ is k-depth derivable from the set of signed formulae is $\mathsf{T}\,\Gamma$. However, the notion of k-depth derivability is defined for arbitrary finite sets of signed formulae (not necessarily in PPNF) and arbitrary formulae. We shall use X, Y, Z, etc. as variables ranging over finite sets of signed formulae and φ, ψ, χ, etc. as variables ranging over signed formulae.

Starting from 0-depth derivability, the transition from one degree of depth to the next is associated with the use of a *structural* rule that governs the use of virtual information in a proof. This is *the only* discharge rule of the system and takes the following form (where "S" stands either for "T" or for "F"):

If φ is k-depth derivable from $X \cup \{\mathsf{T}\,A\}$ and from $Y \cup \{\mathsf{F}\,A\}$ then φ is $k+1$-depth derivable from $X \cup Y$. (RB)

This rule simulates the transition from an information state in which we do not possess any information about the truth or falsity of A, to a richer one, in which the formula A is decided, that is, either we actually possess the information that it is true or we actually possess the information that it is false. It can be seen as a principle of *potential omniscience* and it is the informational version of the classical *principle of bivalence.* Accordingly we call this rule "Rule of Bivalence" (RB).

Given that the virtual assumptions $\mathsf{T}\,A$ and $\mathsf{F}\,A$ introduced by an application of this rule may contain parameters, its use suggests the need for a further restriction on the $\mathsf{T}\,\forall I$ and $\mathsf{F}\,\exists I$ rules, namely that *the parameter a does not occur in any undischarged virtual assumption on which the premise of the rule application depends.*

In the following examples we shall use boxes to represent the subproofs to which the RB rule is applied. The depth of a derivation is nothing but the maximum number of nested boxes occurring in it. We write $\Gamma \vdash_k A$ to mean that A is k-depth deducible from Γ ($\mathsf{T}\,A$ is k-depth derivable from $\mathsf{T}\,\Gamma$).

Example 6. $\mathsf{T}\forall x(Qx \to Rx)$, $\mathsf{T}\forall x(Rx \to Sx) \vdash_1 \mathsf{T}\forall x(Qx \to Sx)$

1	$\mathsf{T}\,\forall x(Qx \to Rx)$	*Premise*		
2	$\mathsf{T}\,\forall x(Rx \to Sx)$	*Premise*		
3	$\mathsf{T}\,Qa \to Ra$	$\mathsf{T}\forall E, 1$		
4	$\mathsf{T}\,Ra \to Sa$	$\mathsf{T}\forall E, 2$		
5	$\mathsf{T}\,Qa$		$\mathsf{F}\,Qa$	
6	$\mathsf{T}\,Ra$	$\mathsf{T}\to E, 3, 5$	$\mathsf{T}\,Qa \to Sa$	$\mathsf{T}\to I, 5$
7	$\mathsf{T}\,Sa$	$\mathsf{T}\to E, 4, 6$		
8	$\mathsf{T}\,Qa \to Sa$	$\mathsf{T}\to I, 7$		
9	$\mathsf{T}\,Qa \to Sa$			
10	$\mathsf{T}\,\forall x(Qx \to Sx)$	$\mathsf{T}\forall I, 9$		

Note that in the above proof the application of the rule $\mathsf{T}\,\forall I$ at step 10 is allowed because the virtual assumptions containing the parameter a have already been discharged.

Example 7. *Consider again the example discussed in Section 3. We transform the premises of the argument in PPNF and obtain the following set $\Gamma = \{\forall x\forall y\exists z(Rxy \to (Gxz \wedge Gzy)), \forall x\forall y\exists w(Gxy \to (Bxw \wedge Bwy)), \forall x\forall y((Bxy \wedge Cx) \to Cy)\}$. Figure 4 illustrates the configuration of the premises in Γ. The derivation of the conclusion $\mathsf{T}\,\forall x\forall y((Rxy \wedge Cx) \to Cy)$ from the premises $\mathsf{T}\,\Gamma$, which is shown in Figure 5, has depth 2. Notice that, unlike the former, the latter application of the RB introduces a new quantifier $\exists v$, the elimination of which permits the introduction into the argument of a new individual e. As a result, this derivation violates* (PQA) *and vindicates Hintikka's insights.*

Liberalized introduction rules. In [6], D'Agostino, Gabbay and Modgil have shown that, towards the normalization result, it is convenient to prove that every derivation can be transformed into its RB-canonical form, i.e. into a derivation in which there is no application of a rule below the conclusion of an application of RB. In that paper, the authors have shown that this outcome can be achieved by applying the transformation depicted in Figure 6, where χ is the conclusion of a rule

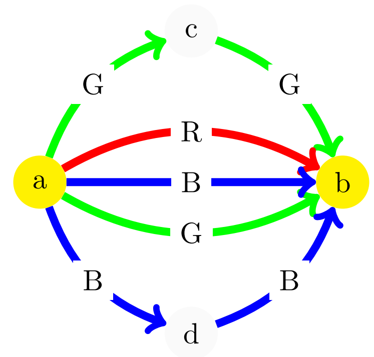

Figure 4: This figure illustrates the configuration of the premises in PPNF of the example shown in Section 3.

having as its premise(s) ψ (and φ). The iterated application of this transformation results in pushing downwards all the applications of RB so that, eventually, the conclusion of an application of RB is never used as a premise of a rule and must be identical to the conclusion of the whole derivation.

This theorem concerning RB-canonical derivations, that D'Agostino, Gabbay and Modgil proved for *propositional* logic, might be useful also in DBFOLs. However, the result of the transformation shown in Figure 7 is not sound, because, as we have seen above, the $\mathsf{T}\,\forall I$ rule might be applied to a formula such as $\mathsf{T}\,B(c)$ whenever c does not occur in any undischarged assumption on which the premise of the rule application depends. Therefore, if we want to prove that every derivation of DBFOLs can be transformed into an equivalent one in RB-canonical form, we are required to liberalize the use of the rule $\mathsf{T}\,\forall I$ (and of the rule $\mathsf{F}\,\exists I$), in such a way that the transformation shown in Figure 7 turns out to be sound.

We say that an individual denoted by a is *arbitrary* for a certain property $F(x)$ expressed by an open formula with a free variable x, if *either* F^x_a is false, *or* F^x_a is true for *every* individual a in the domain. Note that we can always assume, with no loss of generality, that a *new* parameter introduced in a virtual assumption via an application of the RB denotes an individual that is arbitrary for a certain property $F(x)$. The crucial point is that if a is arbitrary for $F(x)$, then it might not be arbitrary for a syntactically distinct property $G(x)$. The idea behind the liberalized versions of $\mathsf{T}\,\forall I$ and $\mathsf{F}\,\exists I$ is that if an individual denoted by the parameter a is arbitrary for a certain property, the same individual cannot be arbitrary for a different property:

$\mathsf{T}\ \forall x \forall y \exists z(Rxy \to (Gxz \wedge Gzy))$ Premise
$\mathsf{T}\ \forall x \forall y \exists w(Gxy \to (Bxw \wedge Bwy))$ Premise
$\mathsf{T}\ \forall x \forall y((Bxy \wedge Cx) \to Cy)$ Premise
$\mathsf{T}\ \forall y \exists z(Ray \to (Gaz \wedge Gzy))$
$\mathsf{T}\ \exists z(Rab \to (Gaz \wedge Gzb))$
$\mathsf{T}\ Rab \to (Gac \wedge Gcb)$
$\mathsf{T}\ \forall y \exists w(Gay \to (Baw \wedge Bwy))$
$\mathsf{T}\ \exists w(Gac \to (Baw \wedge Bwc))$
$\mathsf{T}\ Gac \to (Bad \wedge Bdc)$
$\mathsf{T}\ \forall y \exists w(Gcy \to (Bcw \wedge Bwy))$
$\mathsf{T}\ \exists w(Gcb \to (Bcw \wedge Bwb))$
$\mathsf{T}\ \forall y((Bay \wedge Ca) \to Cy)$
$\mathsf{T}\ (Bad \wedge Ca) \to Cd$
$\mathsf{T}\ \forall y((Bdy \wedge Cd) \to Cy)$
$\mathsf{T}\ (Bdc \wedge Cd) \to Cc$

$\mathsf{T}\ Rab \wedge Ca$
$\mathsf{T}\ Rab$
$\mathsf{T}\ Ca$
$\mathsf{T}\ Gac \wedge Gcb$
$\mathsf{T}\ Gac$
$\mathsf{T}\ Gcb$
$\mathsf{T}\ Bad \wedge Bdc$
$\mathsf{T}\ Bad$
$\mathsf{T}\ Bdc$
$\mathsf{T}\ Bad \wedge Ca$
$\mathsf{T}\ Cd$
$\mathsf{T}\ Bdc \wedge Cd$
$\mathsf{T}\ Cc$

$\mathsf{T}\ \exists v(Gcb \to (Bcv \wedge Bvb))$
$\mathsf{T}\ Gcb \to (Bce \wedge Beb)$
$\mathsf{T}\ Bce \wedge Beb$
$\mathsf{T}\ Bce$
$\mathsf{T}\ Beb$
$\mathsf{T}\ \forall y((Bcy \wedge Cc) \to Cy)$
$\mathsf{T}\ (Bce \wedge Cc) \to Ce$
$\mathsf{T}\ Bce \wedge Cc$
$\mathsf{T}\ Ce$
$\mathsf{T}\ \forall y((Bey \wedge Ce) \to Cy)$
$\mathsf{T}\ (Beb \wedge Ce) \to Cb$
$\mathsf{T}\ Beb \wedge Ce$
$\mathsf{T}\ Cb$
$\mathsf{T}\ (Rab \wedge Ca) \to Cb$

$\mathsf{F}\ \exists v(Gcb \to (Bcv \wedge Bvb))$
$\mathsf{F}\ (Gcb \to (Bce \wedge Beb))$
$\mathsf{F}\ \exists w(Gcb \to (Bcw \wedge Bwb))$
$\times$

$\mathsf{T}\ (Rab \wedge Ca) \to Cb$

$\mathsf{F}\ Rab \wedge Ca$
$\mathsf{T}\ (Rab \wedge Ca) \to Cb$

$\mathsf{T}\ (Rab \wedge Ca) \to Cb$
$\mathsf{T}\ \forall y((Ray \wedge Ca) \to Cy)$
$\mathsf{T}\ \forall x \forall y((Rxy \wedge Cx) \to Cy)$

Figure 5: Example of a derivation of depth 2. Due to space restrictions the justifications of the steps are omitted.

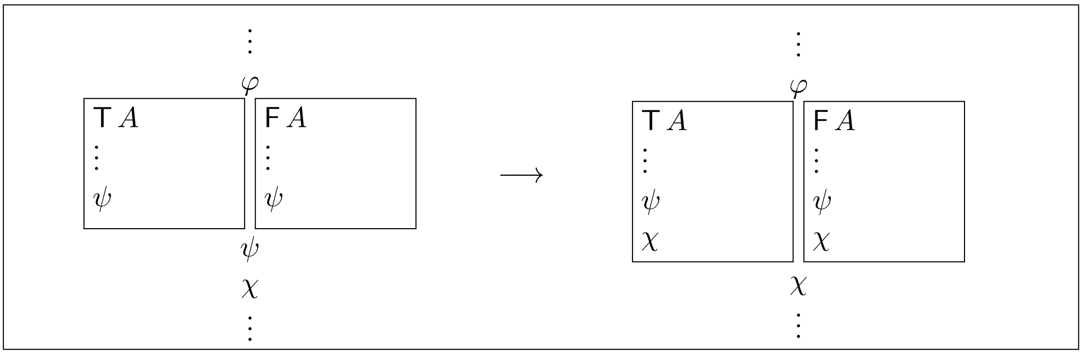

Figure 6: Iterated applications of this transformation turn any derivation into an RB-canonical one: χ is the conclusion of an *intelim*-rule having as its premise(s) ψ (and φ) [6].

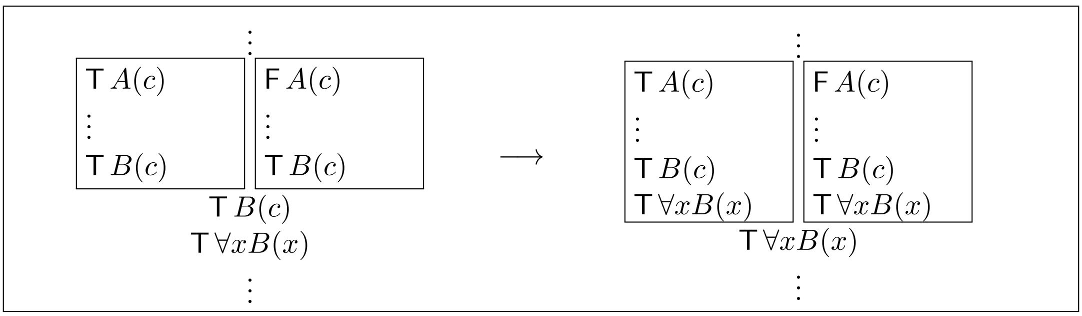

Figure 7: The result of this transformation shows an application of the liberalized $\mathsf{T}\,\forall I$ rule.

$$\frac{\mathsf{T}\,F}{\mathsf{T}\,\forall x F^a_x}\ \mathsf{T}\,\forall I \qquad\qquad \frac{\mathsf{F}\,F}{\mathsf{F}\,\exists x F^a_x}\ \mathsf{F}\,\exists I$$

Provided that a is not critical, F does not contain critical parameters depending on a and x is not bound in F. Moreover, the rule $\mathsf{T}\,\forall I$ (respectively $\mathsf{F}\,\exists I$) is not applied to $\mathsf{T}\,G(a)$ (respectively, $\mathsf{F}\,G(a)$) obtaining $\mathsf{T}\,\forall x G^a_x$ (respectively, $\mathsf{F}\,\exists x G^a_x$) for any formula G syntactically distinct from F.

Example 8. *Example of a* wrong *application of the liberalized I-rule. The $\mathsf{T}\,\forall I$ rule cannot be applied both at step 2 left and at step 3 right, because the individual* *denoted by c cannot be arbitrary for both $\mathsf{T}\,Ac$ and $\mathsf{T}\,\neg Ac$.*

1	$\mathsf{T}\,Ac$		$\mathsf{F}\,Ac$	
2	$\mathsf{T}\,\forall x Ax$	$\mathsf{T}\,\forall I, 1$	$\mathsf{T}\,\neg Ac$	$\mathsf{T}\,\neg I, 1$
3	$\mathsf{T}\,\forall x Ax \vee \forall x \neg Ax$	$\mathsf{T}\,\vee I, 2$	$\boldsymbol{\mathsf{T}\,\forall x \neg Ax}$	$\mathsf{T}\,\forall I, 2$
4			$\mathsf{T}\,\forall x Ax \vee \forall x \neg Ax$	$\mathsf{T}\,\vee I, 3$
5	$\mathsf{T}\,\forall x Ax \vee \forall x \neg Ax$			

Example 9. *Example of a correct application of the liberalized I-rule:*
$\vdash_1 \mathsf{T}\,\forall x Ax \vee \exists x \neg Ax$

1	$\mathsf{T}\,Ac$		$\mathsf{F}\,Ac$	
2	$\mathsf{T}\,\forall x Ax$	$\mathsf{T}\,\forall I, 1$	$\mathsf{T}\,\neg Ac$	$\mathsf{T}\,\neg I, 1$
3	$\mathsf{T}\,\forall x Ax \vee \exists x \neg Ax$	$\mathsf{T}\,\vee I, 2$	$\mathsf{T}\,\exists x \neg Ax$	$\mathsf{T}\,\exists I, 2$
4			$\mathsf{T}\,\forall x Ax \vee \exists x \neg Ax$	$\mathsf{T}\,\vee I, 3$
5	$\mathsf{T}\,\forall x Ax \vee \exists x \neg Ax$			

Example 10. *Example of a correct application of the liberalized I-rule:*
$\mathsf{T}\,\forall x \exists y(\neg Ay \vee Ax) \vdash_1 \mathsf{T}\,\exists y \neg Ay \vee \forall x Ax$

1	$\mathsf{T}\,\forall x \exists y(\neg Ay \vee Ax)$	*Premise*		
2	$\mathsf{T}\,\exists y(\neg Ay \vee Aa)$	$\mathsf{T}\,\forall E, 1$		
3	$\mathsf{T}\,\neg Ab \vee Aa$	$\mathsf{T}\,\exists E, 2$		
4	$\mathsf{T}\,Aa$		$\mathsf{F}\,Aa$	
5	$\mathsf{T}\,\forall x Ax$	$\mathsf{T}\,\forall I, 4$	$\mathsf{T}\,\neg Ab$	$\mathsf{T}\,\vee E, 3, 4$
6	$\mathsf{T}\,\exists y \neg Ay \vee \forall x Ax$	$\mathsf{T}\,\vee I, 5$	$\mathsf{T}\,\exists y \neg Ay$	$\mathsf{T}\,\exists I, 5$
7			$\mathsf{T}\,\exists y \neg Ay \vee \forall x Ax$	$\mathsf{T}\,\vee I, 6$
8	$\mathsf{T}\,\exists y \neg Ay \vee \forall x Ax$			

The propositions stated above, at the end of Section 6 for the notion of 0-depth deducibility, can be extended to the general case. Here we state them without proof.

Proposition 5. *A formula A is a classical consequence of Γ if and only if there is a k-depth proof of P from Γ for some $k \in \mathbb{N}$.*

We just observe that completeness can be proven via simulation of a classical natural deduction system (with or without the liberalized introduction rules). Applications of the standard quantifier eliminations that increase the number of distinct parameters beyond the Q-complexity of the premises can be simulated via suitable applications of the RB-rule introducing virtual information and so increasing the depth of the proof.

Proposition 6. *If A is a classical consequence of Γ, there is a k-depth proof of A from Γ with the subformula property for some $k \in \mathbb{N}$.*

A detailed proof-theoretical investigation of the normalization problem will be the topic of a future paper.

Proposition 7. *The notion of k-depth inference is decidable for every fixed k.*

Decidability follows from depth-boundedness: only a finite number of new parameters can be introduced by increasing the depth of the proof. It is open whether k-depth inference (for normal proofs with the subformula property) is tractable, i.e., if there exists a polyniomial time decision procedure. This problem will also be a crucial topic of further research.

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