

Revisiting Preferences and Argumentation: Technical Report

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Abstract

The *ASPIC*⁺ framework is intermediate in abstraction between Dung’s argumentation framework and concrete instantiating logics. This paper generalises *ASPIC*⁺ to accommodate classical logic instantiations, and adopts a new proposal for evaluating extensions: attacks are used to define the notion of conflict-free sets, while the defeats obtained by applying preferences to attacks, are exclusively used to determine the acceptability of arguments. Key properties and rationality postulates are then shown to hold for the new framework.

1 Introduction

A Dung *argumentation framework* (*AF*) [Dung, 1995] consists of a binary *attack* relation on a set of arguments. The justified arguments are then evaluated based on the framework’s extensions: conflict-free sets of arguments (sets that do not contain arguments that attack) that can be defended against attacks (and so are said to be *acceptable*). The abstract nature of Dung’s theory provides for a general and intuitive semantics for the consequence notions of argumentation logics and for nonmonotonic logics in general: an *AF* can be instantiated by the arguments and conflict-based attacks defined by a theory in a logic, and the theory’s inferences are then defined in terms of the claims of the justified arguments.

Several works augment *AF*s with preferences and/or values [Amgoud and Cayrol, 2002; Bench-Capon, 2003; Modgil, 2009], so that the conflict-free and acceptable sets of arguments are evaluated only w.r.t the *successful attacks* (*defeats*), where an argument *X*’s attack on *Y* is successful only if *Y* is not preferred to *X*. However, we argue in this paper that it is conceptually more intuitive to continue to define conflict-free sets in terms of those that do not contain attacking arguments. Defeats then encode the preference-dependent success of attacks as they are deployed in the dialectical evaluation of arguments, and so should only be used to determine the acceptability of arguments.

We explore this proposal in the context of the *ASPIC* framework [Amgoud *et al.*, 2006]. The very abstract nature of Dung’s framework precludes giving guidance to ensure that the instantiating theory’s defined inferences satisfy intuitively rational properties, and so *ASPIC* provided abstract

accounts of the structure of arguments, the nature of attack, and the use of preferences. [Caminada and Amgoud, 2007] then formulated consistency and closure postulates that cannot be formulated at Dung’s fully abstract level, and showed these postulates to hold for a special case of *ASPIC*; one in which preferences were *not* accounted for. More recently, *ASPIC*⁺ [Prakken, 2010] generalised *ASPIC* and showed that the postulates were satisfied when applying preferences and evaluating the justified arguments on the basis of the derived defeat relation.

This paper makes the following contributions. Firstly, we redefine evaluation of the justified arguments of the *ASPIC*⁺ framework under our proposed distinct use of attacks and defeats, and show satisfaction of the key properties of Dung frameworks, and [Caminada and Amgoud, 2007]’s rationality postulates. This is significant given that *ASPIC*⁺ captures a broad range of instantiating logics and argumentation systems, extending those captured by *ASPIC* (e.g., to include assumption-based argumentation [Bondarenko *et al.*, 1997] and systems using argument schemes). However, *ASPIC*⁺’s generality is compromised in that it does not capture classical logic approaches to argumentation [Amgoud and Cayrol, 2002; Besnard and Hunter, 2008; Amgoud and Besnard, 2009], since it does not require that the premises of an argument are consistent. This paper’s second contribution is to therefore adapt *ASPIC*⁺ so as to capture these approaches, and so demonstrate satisfaction of the rationality postulates for classical logic approaches that accommodate preferences; a result that to the best of our knowledge has hitherto not been shown for the full range of Dung’s original semantics.

The paper is organised as follows. Section 2 reviews background concepts. Section 3 adapts *ASPIC*⁺ to include classical logic approaches, and defines extensions under the new proposal outlined above. Section 4 shows that key properties of Dung frameworks and the rationality postulates are satisfied¹. Section 5 discusses related and future work.

2 Background

A *Dung argumentation framework* (*AF*) [Dung, 1995] is a tuple $(\mathcal{A}, \mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{A}$ is an attack relation on the arguments \mathcal{A} . $S \subseteq \mathcal{A}$ is *conflict free* iff $\forall X, Y \in S, (X, Y)$

¹Only proofs of the main results are shown in the paper. Proofs not given here can be found in [Modgil and Prakken, 2011].

$\notin \mathcal{C}$. An argument $X \in \mathcal{A}$ is acceptable w.r.t. some $S \subseteq \mathcal{A}$ iff $\forall Y$ s.t. $(Y, X) \in \mathcal{C}$ implies $\exists Z \in S$ s.t. $(Z, Y) \in \mathcal{C}$. Then:

Definition 1 Let $(\mathcal{A}, \mathcal{C})$ be a AF. Then a *conflict free* $S \subseteq \mathcal{A}$ is : an *admissible* extension iff $X \in S$ implies X is acceptable w.r.t. S ; a *complete* extension iff $X \in S$ iff X is acceptable w.r.t. S ; a *preferred* extension iff it is a set inclusion maximal complete extension; the *grounded* extension iff it is the set inclusion minimal complete extension; a *stable* extension iff it is preferred and $\forall Y \notin S, \exists X \in S$ s.t. $(X, Y) \in \mathcal{C}$.

For $s \in \{\text{complete, preferred, grounded, stable}\}$, X is *sceptically* or *credulously* justified under the s semantics if X belongs to all, respectively at least one, s extension.

Preference-based AFs (PAFs) [Amgoud and Cayrol, 2002] are tuples $(\mathcal{A}, \mathcal{C}, \mathcal{P})$, where the preference pre-ordering $\mathcal{P} \subseteq \mathcal{A} \times \mathcal{A}$ determines which attacks succeed as defeats. With the corresponding strict ordering $-Y >_{\mathcal{P}} X$ iff $(Y, X) \in \mathcal{P}$ and $(X, Y) \notin \mathcal{P}$ – then $(X, Y) \in \mathcal{D}$ (i.e., X defeats Y) iff $(X, Y) \in \mathcal{C}$ and $Y \not>_{\mathcal{P}} X$. The extensions of $(\mathcal{A}, \mathcal{C}, \mathcal{P})$ are then defined as the extensions of the Dung framework $(\mathcal{A}, \mathcal{D})$.

[Prakken, 2010]’s ASPIC⁺ framework instantiates Dung’s abstract approach by assuming an unspecified logical language \mathcal{L} , and by defining arguments as inference trees formed by applying strict or defeasible inference rules of the form $\varphi_1, \dots, \varphi_n \rightarrow \varphi$ and $\varphi_1, \dots, \varphi_n \Rightarrow \varphi$, interpreted as ‘if the antecedents $\varphi_1, \dots, \varphi_n$ hold, then *without exception*, respectively *presumably*, the consequent φ holds’. There are two ways to use these rules: they could encode domain-specific information (as in e.g. default logic) but they could also express general laws of reasoning. For example, the defeasible rules could express argument schemes and the strict rules could consist of all classically valid inferences or could more generally conform to any Tarskian consequence notion (cf. [Amgoud and Besnard, 2009]). In order to define attacks, some minimal assumptions on \mathcal{L} are made; namely that certain wff (well formed formulae) are a contrary or contradictory of certain other wff. Apart from this the framework is still abstract: it applies to any set of strict and defeasible inference rules, and to any logical language with a defined contrary relation.

The basic notion of ASPIC⁺ is that of an argumentation system. Arguments are then constructed w.r.t a knowledge base that is assumed to contain three kinds of formulas.

Definition 2 An *argumentation system* is a tuple $AS = (\mathcal{L}, \bar{\cdot}, \mathcal{R}, \leq)$ where:

- \mathcal{L} is a logical language.
- $\bar{\cdot}$ is a contrariness function from \mathcal{L} to $2^{\mathcal{L}}$, such that:
 - φ is a *contrary* of ψ if $\varphi \in \bar{\psi}, \psi \notin \bar{\varphi}$;
 - φ is a *contradictory* of ψ (denoted by ‘ $\varphi = -\psi$ ’), if $\varphi \in \bar{\psi}, \psi \in \bar{\varphi}$.
- $\mathcal{R} = \mathcal{R}_s \cup \mathcal{R}_d$ is a set of strict (\mathcal{R}_s) and defeasible (\mathcal{R}_d) inference rules such that $\mathcal{R}_s \cap \mathcal{R}_d = \emptyset$.
- \leq is a preordering on \mathcal{R}_d .

Henceforth, a set $S \subseteq \mathcal{L}$ is said to be consistent iff $\nexists \psi, \varphi \in S$ such that $\psi \in \bar{\varphi}$, otherwise it is *inconsistent*.

A *knowledge base* in an argumentation system $(\mathcal{L}, \bar{\cdot}, \mathcal{R}, \leq)$ is a pair (\mathcal{K}, \leq') where $\mathcal{K} \subseteq \mathcal{L}$ and \leq' is a preordering on

$\mathcal{K} \setminus \mathcal{K}_n$. Here, $\mathcal{K} = \mathcal{K}_n \cup \mathcal{K}_p \cup \mathcal{K}_a$ where these subsets of \mathcal{K} are disjoint: \mathcal{K}_n is the (necessary) *axioms* (which cannot be attacked); \mathcal{K}_p is the *ordinary premises* (on which attacks succeed contingent upon preferences), and; \mathcal{K}_a is the *assumptions* (on which attacks are always successful, c.f. assumptions in [Bondarenko et al., 1997]).

The orderings on defeasible rules and non-axiom premises (we assume their strict counterparts defined in the usual way, i.e., $l < l'$ iff $l \leq l'$ and $l' \not\leq l$) are assumed to be used in defining an ordering \preceq on the constructed arguments (see Section 4). Henceforth, we assume the strict counterpart \prec of \preceq defined in the usual way. Arguments are now defined, where for any argument A , Prem returns all the formulas of \mathcal{K} (*premises*) used to build A , Conc returns A ’s conclusion, Sub returns all of A ’s sub-arguments, and DefRules returns all defeasible rules in A .

Definition 3 An *argument* A on the basis of a knowledge base (\mathcal{K}, \leq') in an argumentation system $(\mathcal{L}, \bar{\cdot}, \mathcal{R}, \leq)$ is:

1. φ if $\varphi \in \mathcal{K}$ with: Prem(A) = $\{\varphi\}$; Conc(A) = φ ; Sub(A) = $\{\varphi\}$; Rules(A) = \emptyset ; TopRule(A) = undefined.
2. $A_1, \dots, A_n \rightarrow/\Rightarrow \psi$ if A_1, \dots, A_n are arguments such that there exists a strict/defeasible rule Conc(A_1), \dots , Conc(A_n) $\rightarrow/\Rightarrow \psi$ in $\mathcal{R}_s/\mathcal{R}_d$.
Prem(A) = Prem(A_1) $\cup \dots \cup$ Prem(A_n),
Conc(A) = ψ ,
Sub(A) = Sub(A_1) $\cup \dots \cup$ Sub(A_n) $\cup \{A\}$.
Rules(A) = Rules(A_1) $\cup \dots \cup$ Rules(A_n) \cup
 $\{\text{Conc}(A_1), \dots, \text{Conc}(A_n) \rightarrow/\Rightarrow \psi\}$
DefRules(A) = $\{r | r \in \text{Rules}(A), r \in \mathcal{R}_d\}$
TopRule(A) = Conc(A_1), \dots , Conc(A_n) $\rightarrow/\Rightarrow \psi$

Furthermore, A is: *strict* if DefRules(A) = \emptyset ; *defeasible* if DefRules(A) $\neq \emptyset$; *firm* if Prem(A) $\subseteq \mathcal{K}_n$; *plausible* if Prem(A) $\not\subseteq \mathcal{K}_n$.

For any $P \subseteq \mathcal{L}$, the closure of P under strict rules, denoted $Cl_{\mathcal{R}_s}(P)$, is the smallest set containing P and the consequent of any strict rule in \mathcal{R}_s whose antecedents are in $Cl_{\mathcal{R}_s}(P)$. Also, we write $S \vdash \varphi$ if there is a strict argument A such that Conc(A) = φ , with all premises taken from S . Given the intuitive meaning of strict rules, axioms and assumptions, we think that any argumentation system should:

- be *closed under contraposition* or *transposition* (as in [Caminada and Amgoud, 2007]). The former implies that for all $S \subseteq \mathcal{L}$, $s \in S$ and ϕ , if $S \vdash \phi$, then $S \setminus \{s\} \cup \{-\phi\} \vdash -s$. The latter implies that if $\phi_1, \dots, \phi_n \rightarrow \psi \in \mathcal{R}_s$, then for $i = 1 \dots n$, $\phi_1, \phi_{i-1}, -\psi, \phi_{i+1}, \dots, \phi_n \rightarrow -\phi_i \in \mathcal{R}_s$;
- be *axiom consistent*, i.e., $Cl_{\mathcal{R}_s}(\mathcal{K}_n)$ is consistent.
- be *well formed*, i.e., if φ is a contrary of ψ then $\psi \notin \mathcal{K}_n$ and ψ is not the consequent of a strict rule (since as we will see, attacks by contraries characterise attacks that always succeed; e.g. when ψ is of the form *not* φ in logic programming).

Henceforth, argumentation systems are assumed to satisfy these properties. It should be obvious to see that if the strict rules conform to a Tarskian consequence operator (cf. [Amgoud and Besnard, 2009]), for example, if they consist of all valid propositional or first-order inferences over \mathcal{L} , then the first property is always satisfied.

When arguments are inference trees, three syntactic forms of attack from an argument B to an argument A are possible: attacking a premise of A , a conclusion of A , or an inference step in A , respectively called undermining, rebutting and undercutting attacks. To model undercutting attacks on inferences, it is assumed that applications of inference rules can be expressed in the object language; the precise nature of this naming convention will be left implicit. Apart from undercut attacks and attacks on contraries, the success of attacks as defeats is contingent upon preferences.

Definition 4 A attacks B iff A undercuts, rebuts or undermines B , where:

- A undercuts argument B (on B') iff $\text{Conc}(A) \in \overline{B'}$ for some $B' \in \text{Sub}(B)$ of the form $B'_1, \dots, B'_n \Rightarrow \psi$.
- A rebuts argument B (on B') iff $\text{Conc}(A) \in \overline{\varphi}$ for some $B' \in \text{Sub}(B)$ of the form $B'_1, \dots, B'_n \Rightarrow \varphi$. In such a case A contrary-rebuts B iff $\text{Conc}(A)$ is a contrary of φ .
- Argument A undermines B (on B') iff $\text{Conc}(A) \in \overline{\varphi}$ for some $B' = \varphi$, $\varphi \in \text{Prem}(B) \setminus \mathcal{K}_n$. In such a case A contrary-undermines B iff $\text{Conc}(A)$ is a contrary of φ or if $\varphi \in \mathcal{K}_a$.

A defeats B iff A undercuts B or A rebuts/undermines B on B' and either A contrary rebuts/undermines B , or $A \not\prec B'$.

3 Argumentation, Logic and Preferences: Revisiting and Generalising *ASPIC*⁺

As with other works augmenting AFs with *preferences* and/or *values*, *ASPIC*⁺ evaluates the notions of conflict free and acceptability on the basis of the arguments and *defeats*. [Prakken, 2010] then shows that [Caminada and Amgoud, 2007]’s closure and consistency postulates are satisfied under a number of assumptions. In particular, *direct consistency* is shown by proving that no complete extension yields arguments with conclusions that are the contraries of other arguments. Essentially, this amounts to showing that no complete extension contains arguments that attack. This then begs the question as to why one should not define conflict-free sets as those that do not contain attacking arguments. Intuitively, the success of an attack as a defeat, contingent upon the preference relation, has no bearing on whether an attacking argument is incompatible with the attacked argument, but rather on the dialectical relationship of the former to the latter. Defeats should therefore be reserved for determining whether an attacking argument can be successfully deployed as a counter-argument. That is, they should only be used when determining the acceptability of arguments.

Adopting this new distinct use of attacks and defeats, we now link Section 2’s *ASPIC*⁺ concepts to abstract argumentation frameworks, and define their extensions:

Definition 5 Let \mathcal{A} be a set of arguments as defined in Definition 3. A *structured abstract argumentation framework (SAF)* is a tuple $\langle \mathcal{A}, \mathcal{C}, \preceq \rangle$ such that $(X, Y) \in \mathcal{C}$ iff X attacks Y as defined in Definition 4, and \preceq is a preordering on \mathcal{A} .

- Let $\mathcal{D} \subseteq \mathcal{A} \times \mathcal{A}$, where $(X, Y) \in \mathcal{D}$ iff X defeats Y as defined in Definition 4
- $S \subseteq \mathcal{A}$ is conflict free iff $\forall X, Y \in S, (X, Y) \notin \mathcal{C}$.

- The extensions of a SAF $\langle \mathcal{A}, \mathcal{C}, \preceq \rangle$ are the extensions of the Dung framework $(\mathcal{A}, \mathcal{D})$ as defined in Definition 1.

We now also adapt *ASPIC*⁺ to accommodate classical logic approaches to argumentation [Amgoud and Cayrol, 2002; Besnard and Hunter, 2008]. These assume a propositional or first-order predicate logic and arguments as pairs (S, φ) where S is a consistent and minimal set of wffs that classically entails φ . We therefore identify a class of SAF that places restrictions on the instantiating arguments:

Definition 6 A set $S \subseteq \mathcal{L}$ is *c-consistent* if for no φ it holds that $S \vdash \varphi, \neg\varphi$ (i.e., no strict arguments with contradictory conclusions can be built from premises S). Otherwise S is *c-inconsistent*.

A SAF $\langle \mathcal{A}, \mathcal{C}, \preceq \rangle$ is said to be *c-consistent* if the arguments \mathcal{A} are as defined in Definition 3, with the added condition that for any $A \in \mathcal{A}$, $\text{Prem}(A)$ is *c-consistent*.

Note that we use the term ‘*c-consistent*’ to distinguish the notion of consistency in Definition 2. We can now instantiate *c-consistent SAFs* (*c-SAFs* for short) with arguments defined by classical approaches (we don’t need a minimality condition on arguments since Definition 3 guarantees that an argument has no unused premises). The language \mathcal{L} is a first-order language, the contrariness function corresponds to classical negation, and R_d is empty while R_s consists of all valid first-order inferences over \mathcal{L} . Furthermore, all elements of a knowledge base are in \mathcal{K}_p . Both [Amgoud and Cayrol, 2002; Besnard and Hunter, 2008] consider several notions of attack, one of which corresponds to the present notion of undermining attack.

Henceforth, we assume *c-SAFs* are *c-classical*:

Definition 7 Let $S \subseteq \mathcal{L}$ be a minimal *c-inconsistent* set iff $\forall S' \subset S, S'$ is *c-consistent*. A *c-SAF* is *c-classical* iff for any minimal *c-inconsistent* S and any $\varphi \in S$ it holds that $S \setminus \{\varphi\} \vdash \neg\varphi$ (i.e., amongst all arguments defined there exists a strict argument with conclusion $\neg\varphi$ with all premises taken from $S \setminus \{\varphi\}$).

If the strict rules in a *c-SAF* conform to a Tarskian consequence operator (cf. [Amgoud and Besnard, 2009]) then it should be obvious to see that the *cSAF* is *c-classical*.

4 Properties of SAFs

4.1 Properties of Dung Frameworks

We now prove some fundamental results for SAFs and *c-SAFs*. Retaining attacks when defining conflict-free sets potentially undermines some key results shown for Dung frameworks. It may be that A is acceptable w.r.t. an admissible set S , but $S \cup \{A\}$ is not conflict free, so that the *fundamental lemma* [Dung, 1995] does not hold. To illustrate, suppose a SAF containing A, B , where B attacks A and $B \prec A$. Then $\{B\}$ is admissible, A is acceptable w.r.t. $\{B\}$, but $\{A, B\}$ is not conflict free. However, under intuitive assumptions on the argument ordering, we show that the result holds. Prior to this we recall some notation from [Prakken, 2010] and then define here the notion of a strict extension of a set of arguments.

Notation 1 Let $M(B)$ denotes the maximal fallible sub-arguments of B , where for any $B' \in \text{Sub}(B)$, $B' \in M(B)$

iff: 1) B' final inference is defeasible or B' is a non-axiom premise, and; 2) there is no $B'' \in \text{Sub}(B)$ s.t. $B'' \neq B$ and $B' \in \text{Sub}(B'')$ and B'' satisfies 1).

E.g., $M([\Rightarrow r; q; r, q \rightarrow \neg p]) = [\Rightarrow r]$ and $[q]$ (assuming $q \in \mathcal{K}_P$ and $\Rightarrow r$ is a defeasible rule with empty antecedent).

Definition 8 Let \mathcal{A} be as defined in Definition 3 or 6. For any $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$, $A \in \mathcal{A}$ is a *strict extension* of $\{A_1, \dots, A_n\}$ iff:

- the ordinary and assumption premises in A are exactly those in $\{A_1, \dots, A_n\}$;
- the defeasible rules in A are exactly those in $\{A_1, \dots, A_n\}$;
- the strict rules and axiom premises of A are a superset of the strict rules and axiom premises in $\{A_1, \dots, A_n\}$.

Notice that if B defeats some strict extension A of $\{A_1, \dots, A_n\}$ then the defeat must be on some A_i . Hence:

Lemma 2 Let $(\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF. Let $A \in \mathcal{A}$ be a strict extension of $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$, and for $i = 1 \dots n$, A_i is acceptable w.r.t. $E \subseteq \mathcal{A}$. Then A is acceptable w.r.t. E .

PROOF. Let B be any argument s.t. $B \rightarrow^E A$. By Definition 4, B attacks A by undercutting or rebutting on defeasible rules in A or undermining on a non-axiom premise in A . Hence, by definition of strict extensions, it must be that $(B, A) \in \mathcal{C}$ iff $(B, A_i) \in \mathcal{C}$ for some (possibly more than one) $A_i \in \{A_1, \dots, A_n\}$. Either:

1. B undercuts or contrary undermines or contrary rebuts some A_i , and so by Definition 4, B defeats A_i .
2. B does not undercut or contrary undermine or contrary rebut some A_i . Suppose for all A_i , for all sub-arguments A'_i of A_i s.t. B rebuts or undermines A_i on A'_i , $B \prec A'_i$. This contradicts B defeats A . Hence, for some A_i , B defeats A_i .

Hence, if B defeats A then B defeats A_i for some A_i . By assumption, A_i is acceptable w.r.t. E . Hence, A is acceptable w.r.t. E . QED

We now state under what assumptions a preordering on arguments is said to be *reasonable*.

Definition 9 \preceq is said to be *reasonable* iff:

1. i) $\forall A, B$, if A is strict and firm and B is plausible or defeasible, then $B \prec A$;
- ii) $\forall A, B$, if B is strict and firm then $B \not\prec A$;
- ii) $\forall A, A', B, C$ such that $A \prec B$, $C \prec A$, and A' is a strict extension of A , then $A' \prec B$, $C \prec A'$ (i.e., strict inferences and axiom premises do not change the strength of arguments)

2. Let $\{C_1, \dots, C_n\}$ be a finite subset of \mathcal{A} , and:
 - for $i = 1 \dots n$, let $C^{+/i}$ be some strict extension of $\{C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n\}$.

Then it is not the case that: $\forall i, C^{+/i} \prec C_i$.

The second condition is essentially a weaker form of the property satisfied by any partial ordering defined over a finite set, viz. that there exists a minimal element.

In [Prakken, 2010], an ordering \prec_s is defined over sets: $\Gamma \prec_s \Gamma'$ iff there is a member of Γ that is strictly less than ($<$)

all members of Γ' , where $<$ is the strict counterpart of the pre-ordering \leq on the defeasible rules or non-axiom premises. Based on \prec_s , the commonly used *last* and *weakest* link principles are then used to define example argument orderings \preceq . Essentially, $B \prec A$ by the last link principle if the set of top defeasible rules in B is \prec_s the set of top defeasible rules in A , and if both these sets are empty, then the set of non-axiom premises in B is \prec_s the set of non-axiom premises in A . $B \prec A$ by the weakest link principle if the set of defeasible rules in B is \prec_s the set of defeasible rules in A , and the set of non-axiom premises in B is \prec_s the set of non-axiom premises in A .

In what follows, we make use of the following notation and definitions of the weakest and last link principles:

Notation 3 For any argument A :

- $\text{LastDefRules}(A) = \emptyset$ if $\text{DefRules}(A) = \emptyset$.
- If $A = A_1, \dots, A_n \Rightarrow \phi$ then $\text{LastDefRules}(A) = \text{Conc}(A_1), \dots, \text{Conc}(A_n) \Rightarrow \phi$, otherwise $\text{LastDefRules}(A) = \text{LastDefRules}(A_1) \cup \dots \cup \text{LastDefRules}(A_n)$.
- $\text{Prem}_{\text{ap}}(A)$ denotes the set $\text{Prem}(A) \cap (\mathcal{K}_a \cup \mathcal{K}_p)$, and $\text{Prem}_n(A)$ denotes the set $\text{Prem}(A) \cap \mathcal{K}_n$.

Henceforth we will write LDR as an abbreviation for LastDefRules , and DR as an abbreviation for DefRules .

The last and weakest link principles then make use of an ordering \prec_s over sets of wff of \mathcal{L} , under the assumption of a pre-order \leq on \mathcal{L} .

Definition 10 Let $\Gamma = l_1, \dots, l_n$, $\Gamma' \subseteq \Gamma$, $\Gamma'' \subseteq \Gamma$, and \leq a pre-ordering on Γ , where the strict counterpart of \leq is defined as $l < l'$ iff $l \leq l'$ and $l' \not\leq l$. Then, $\Gamma' \prec_s \Gamma''$ iff $\exists l' \in \Gamma'$ s.t. $\forall l'' \in \Gamma'', l' < l''$.

Note that:

Lemma 4 \prec_s is transitive and asymmetric.

PROOF. *Transitivity*: suppose $\Gamma \prec_s \Gamma'$ and $\Gamma' \prec_s \Gamma''$. Then by definition of \prec_s , Γ, Γ' must be non-empty. If $\Gamma'' = \emptyset$, then trivially, by definition of \prec_s , $\Gamma \prec_s \Gamma''$. Else let $\Gamma'' = \{l''_1 \dots l''_n\}$. By definition of \prec_s : $\exists l' \in \Gamma'$, for $j = 1 \dots n$, $l' < l''_j$; $\exists l \in \Gamma$, $l < l'$. Hence, for $j = 1 \dots n$, $l < l''_j$, i.e., $\Gamma \prec_s \Gamma''$.

Asymmetry: Suppose $\Gamma \prec_s \Gamma'$ and $\Gamma' \prec_s \Gamma$. Then, by definition of \prec_s , $\exists l \in \Gamma$, $\exists l' \in \Gamma'$ s.t. $l < l'$ and $l' < l$, contradicting the asymmetry of $<$. QED

Definition 11 [last link principle] $B \preceq A$ under the last link principle iff

1. $\text{LastDefRules}(B) \prec_s \text{LastDefRules}(A)$; or
2. $\text{LastDefRules}(B) = \emptyset$, $\text{LastDefRules}(A) = \emptyset$, and $\text{Prem}_{\text{ap}}(B) \prec_s \text{Prem}_{\text{ap}}(A)$

Definition 12 [weakest link principle] $B \preceq A$ under the weakest link principle iff:

1. $\text{DefRules}(B) \prec_s \text{DefRules}(A)$, and
2. $\text{Prem}_{\text{ap}}(B) \prec_s \text{Prem}_{\text{ap}}(A)$.

Lemma 5 If \preceq is defined under the last link principle, then \preceq is transitive.

PROOF. Suppose $C \prec B$ and $B \prec A$. Consider the following two cases:

1) Suppose $\text{LDR}(A) \neq \emptyset$. $B \prec A$ must therefore be based on condition 1 of Definition 11; i.e., $\text{LDR}(B) \prec_s \text{LDR}(A)$, and so it must be that $\text{LDR}(B) \neq \emptyset$. Hence, by the same reasoning, $\text{LDR}(C) \prec_s \text{LDR}(B)$, and $\text{LDR}(C) \neq \emptyset$.

By Lemma 4, $\text{LDR}(C) \prec_s \text{LDR}(A)$ and so $C \prec A$.

2) Suppose $\text{LDR}(A) = \emptyset$.

– If $\text{LDR}(C) \neq \emptyset$ then by definition of \prec_s , $C \prec A$ by condition 1 of Definition 11.

– If $\text{LDR}(C) = \emptyset$ then it must be that $\text{LDR}(B) = \emptyset$ (if $\text{LDR}(B) \neq \emptyset$ one could not conclude $C \prec B$ by condition 1 or 2 of Definition 11). Hence $B \prec A$ and $C \prec B$, are based on condition 2 of Definition 11. That is to say, $\text{Prem}_{\text{ap}}(C) \prec_s \text{Prem}_{\text{ap}}(B)$, $\text{Prem}_{\text{ap}}(B) \prec_s \text{Prem}_{\text{ap}}(A)$. By Lemma 4, $\text{Prem}_{\text{ap}}(C) \prec_s \text{Prem}_{\text{ap}}(A)$ and so $C \prec A$.

QED

We can now show that:

Proposition 6 Let \preceq be defined according to the last or weakest link principle. Then \preceq is *reasonable*.

PROOF. *Proof of the first condition of reasonableness:*

1-i) Under the last or weakest link principles, since $\text{LDR}(A) = \emptyset$, respectively $\text{DR}(A) = \emptyset$, and $\text{Prem}_{\text{ap}}(A) = \emptyset$, it cannot be that $A \prec B$, by the first and second conditions of Definitions 11 and 12 respectively.

1-ii) Under the last or weakest link principles, since $\text{LDR}(B) = \emptyset$, respectively $\text{DR}(B) = \emptyset$, and $\text{Prem}_{\text{ap}}(B) = \emptyset$, it cannot be that $B \prec A$, by the first and second conditions of Definitions 11 and 12 respectively.

1-iii) Follows straightforwardly from Definitions 11 and 12 respectively, given that A' differs from A only in its strict rules and/or axiom premises.

Proof of the second condition of reasonableness for the last link principle:

Suppose for contradiction that $\forall i$, there is a strict extension $C^{+/i}$ of $\{C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n\}$ such that $C^{+/i} \prec C_i$. Then $\forall i$:

$$\begin{aligned} & \text{LDR}(C_1) \cup \dots \cup \text{LDR}(C_{i-1}) \cup \text{LDR}(C_{i+1}) \dots \cup \\ & \text{LDR}(C_n) \prec_s \text{LDR}(C_i), \text{ or;} \\ & \text{Prem}_{\text{ap}}(C_1) \cup \dots \cup \text{Prem}_{\text{ap}}(C_{i-1}) \cup \\ & \text{Prem}_{\text{ap}}(C_{i+1}) \dots \cup \text{Prem}_{\text{ap}}(C_n) \prec_s \text{Prem}_{\text{ap}}(C_i). \end{aligned}$$

Hence, by definition of \prec_s , it follows that for $\forall i$, $\exists j$ s.t. $\text{LDR}(C_j) \prec_s \text{LDR}(C_i)$ or $\text{Prem}_{\text{ap}}(C_j) \prec_s \text{Prem}_{\text{ap}}(C_i)$. Hence:

$$\forall i, \exists j, C_j \prec C_i \quad \mathbf{i)}$$

But then given $\{C_1, \dots, C_n\}$ is finite, **i)** implies the existence of a cycle in $\{C_1, \dots, C_n\}$. Hence, by transitivity (Lemma 5), for some i, j , $C_i \prec C_j$ and $C_j \prec C_i$. But then if it is not the case that $\text{LDR}(C_i) = \emptyset$ and $\text{LDR}(C_j) = \emptyset$, then $\text{LDR}(C_i) \prec_s \text{LDR}(C_j)$ and $\text{LDR}(C_j) \prec_s C_i(B)$, contradicting the asymmetry of \prec_s . If $\text{LDR}(C_i) = \emptyset$ and $\text{LDR}(C_j) = \emptyset$, then $\text{Prem}_{\text{ap}}(C_i) \prec_s \text{Prem}_{\text{ap}}(C_j)$ and $\text{Prem}_{\text{ap}}(C_j) \prec_s \text{Prem}_{\text{ap}}(C_i)$,

contradicting the asymmetry of \prec_s .

Proof of the second condition of reasonableness for the weakest link principle:

Suppose for contradiction that $\forall i$, there is a strict extension $C^{+/i}$ of $\{C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n\}$ such that $C^{+/i} \prec C_i$. By 1-ii), $\forall i$, $C^{+/i}$ is not strict and firm. Hence, we consider two cases:

1) For some $i = 1 \dots n$, $\text{DR}(C_i) \neq \emptyset$, and so $\text{DR}(C_1) \cup \dots \cup \text{DR}(C_{i-1}) \cup \text{DR}(C_{i+1}) \dots \cup \text{DR}(C_n) \prec_s \text{DR}(C_i)$, and so $\text{DR}(C_j) \prec_s \text{DR}(C_i)$ for some $j = 1 \dots i-1, i+1 \dots n$. By definition of \prec_s , it must be that $\text{DR}(C_j) \neq \emptyset$. Then, since $C^{+/j} \prec C_j$, $\text{DR}(C_k) \prec_s \text{DR}(C_j)$ for some $k = 1 \dots j-1, j+1 \dots n$, where $\text{DR}(C_k) \neq \emptyset$. We can continue to reason in the same way, thus obtaining that for some non-empty subset Π of $\{\text{DR}(C_1), \dots, \text{DR}(C_n)\}$:

$$\forall i, \exists j \text{ s.t. } \text{DR}(C_j) \prec_s \text{DR}(C_i) \quad \mathbf{ii)}$$

But then given $\{C_1, \dots, C_n\}$ is finite, **ii)** implies the existence of a cycle in $\{\text{DR}(C_1), \dots, \text{DR}(C_n)\}$. Hence, by Lemma 4 (transitivity of \prec_s), $\exists i, j$ s.t. $\text{DR}(C_i) \prec_s \text{DR}(C_j)$ and $\text{DR}(C_j) \prec_s \text{DR}(C_i)$, contradicting the asymmetry of \prec_s .

2) For $i = 1 \dots n$, $\text{DR}(C_i) = \emptyset$. Then since no $C^{+/i}$ can be strict and firm, at least one argument in $\{C_1, \dots, C_n\}$ must contain a non-empty set of non-axiom premises. We can then reason in exactly the same way as in case 1, obtaining some non-empty subset Π of $\{\text{Prem}_{\text{ap}}(C_1), \dots, \text{Prem}_{\text{ap}}(C_n)\}$, s.t.

$$\forall i, \exists j \text{ s.t. } \text{Prem}_{\text{ap}}(C_j) \prec_s \text{Prem}_{\text{ap}}(C_i) \quad \mathbf{iii)}$$

As in case 1, **iii)** leads to a contradiction. QED

Observe now that given arguments $A = [\Rightarrow p]$, $B_1 = [\Rightarrow r]$, $B_2 = [\Rightarrow q]$, $B = [B_1; B_2; r, q \rightarrow \neg p]$, then B asymmetrically attacks A , so that if $B \prec A$ then neither argument defeats the other. However, since we assume closure of the strict rules under transposition or contraposition, one can construct strict extensions $A_1^+ = [\Rightarrow p; \Rightarrow q; p, q \rightarrow \neg r]$ of $\{A, B_2\}$ and $A_2^+ = [\Rightarrow p; \Rightarrow r; p, r \rightarrow \neg q]$ of $\{A, B_1\}$, where A_1^+ attacks B on B_1 , and A_2^+ attacks B on B_2 . Given that \preceq is *reasonable*, and that B is a strict extension of $\{B_1, B_2\}$, then it cannot be that $A_1^+ \prec B_1$, $A_2^+ \prec B_2$ and $B \prec A$. Since by assumption $B \prec A$, then it must be that $A_1^+ \not\prec B_1$ or $A_2^+ \not\prec B_2$, and so either A_1^+ or A_2^+ 's attack on B succeeds as a defeat. In fact, the following general result can be shown:

Proposition 7 Let A and B be arguments where B is plausible or defeasible and A and B have contradictory conclusions, and if A and B are defined as in Definition 6, then $\text{Prem}(A) \cup \text{Prem}(B)$ is c-consistent. Then:

1. For all $B' \in M(B)$, there exists a strict extension $A_{B'}^+$ of $(M(B) \setminus \{B'\}) \cup M(A)$ such that $A_{B'}^+$ rebuts or undermines B on B' .
2. If $B \prec A$, and \preceq is *reasonable*, then for some $B' \in M(B)$, $A_{B'}^+$ defeats B .

PROOF.

1. Consider first systems closed under contraposition. Observe first that $\text{Conc}(M(B)) \cup \text{Prem}_n(B) \vdash \text{Conc}(B)$ (i.e., one can construct a strict argument concluding

$\text{Conc}(B)$ with all premises taken from $\text{Conc}(M(B))$ and the axiom premises in B). By contraposition, and since $\text{Conc}(A)$ and $\text{Conc}(B)$ contradict each other, we have that for any $B_i \in M(B)$: $\text{Conc}(M(B) \setminus \{B_i\}) \cup \text{Prem}_n(B) \cup \text{Conc}(A) \vdash \neg \text{Conc}(B_i)$. Hence, one can construct a strict extension $A_{B_i}^+$ that extends $\{A\} \cup M(B) \setminus \{B_i\} \cup \text{Prem}_n(B)$ with strict rules, and that concludes $\neg \text{Conc}(B_i)$. By construction, $M(B) \setminus \{B_i\}$ and $M(A)$ are the maximal fallible sub-arguments of $A_{B_i}^+$, and $\text{Prem}(A_{B_i}^+) \subseteq \text{Prem}(A) \cup \text{Prem}(B)$. Since by construction of $M(B)$ either B_i is a non-axiom premise or ends with a defeasible inference, $A_{B_i}^+$ either undermines or rebuts B_i . But then $A_{B_i}^+$ also undermines or rebuts B . For systems closed under transposition the existence of arguments $A_{B_i}^+$ and B_i is proven by straightforward generalisation of Lemma 6 of [Caminada and Amgoud, 2007]. Then the proof can be completed as above.

In the case that A and B are defined as in Definition 6, one only needs to additionally show that $\text{Prem}(A_{B_i}^+)$ is c-consistent, which follows given that by construction of $A_{B_i}^+$, $\text{Prem}(A_{B_i}^+) \subseteq \text{Prem}(A) \cup \text{Prem}(B)$, and by assumption, $\text{Prem}(A) \cup \text{Prem}(B)$ is c-consistent.

2. Note that in the above, by construction, each B' extension $A_{B'}^+$ of A is a strict extension of $\{A\} \cup M(B) \setminus \{B'\} \cup \text{Prem}_n(B)$. Hence, we have $\{B_1, \dots, B_n, A\}$, where each $A_{B_i}^+$ is a strict extension of $\{B_1, \dots, B_{i-1}, B_{i+1}, B_n, A\}$. Also, B is a strict extension of $\{B_1, \dots, B_n\}$. Hence, since \preceq is reasonable, it cannot be that: $B \prec A$ and $A_{B_1}^+ \prec B_1$ and \dots and $A_{B_n}^+ \prec B_n$. Since by assumption $B \prec A$, then for some i , $A_{B_i}^+$ rebuts or undermines B on B_i , $A_{B_i}^+ \not\prec B_i$, and so $A_{B_i}^+$ defeats B .

QED

We now state some useful results that will be used for proving the main results in this section. Note that henceforth, $X \rightarrow Y$ denotes X attacks Y and $X \leftrightarrow Y$ denotes X defeats Y .

Lemma 8 Let $(\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF:

1. If A is acceptable w.r.t. $S \subseteq \mathcal{A}$ then A is acceptable w.r.t. any superset of S .
2. If $A \rightleftharpoons B$ then $A \leftrightarrow B$ or $B \leftrightarrow A$.
3. If $A \leftrightarrow B$, then $A \leftrightarrow B'$ for some $B' \in \text{Sub}(B)$, and if $A \leftrightarrow B'$, $B' \in \text{Sub}(B)$, then $A \leftrightarrow B$.
4. If A is acceptable w.r.t. $S \subseteq \mathcal{A}$, $A' \in \text{Sub}(A)$, then A' is acceptable w.r.t. S .

PROOF. Proofs of 8-1, 8-2 and 8-3 are straightforward given the definitions of acceptability and defeat. For 8-4, suppose $B \leftrightarrow A'$. By 8-3, $B \leftrightarrow A$, and so $\exists C \in S$ s.t. $C \leftrightarrow A$. Hence A' is acceptable w.r.t. S . QED

Lemma 9 Suppose $B \rightarrow A$, and not $A \rightarrow B$. If not $B \leftrightarrow A$ then either:

1. $\exists A' \in \text{Sub}(A)$ s.t. $A' \leftrightarrow B$, or;

2. There is a strict extension $A_{B'}^+$ of $(M(B) \setminus \{B'\}) \cup M(A)$ s.t. $A_{B'}^+ \leftrightarrow B$, given that if A and B are defined as in Def. 6, then $\text{Prem}(A) \cup \text{Prem}(B)$ is c-consistent.

PROOF. Let A' be any sub-argument of A on which B attacks. Since not $B \leftrightarrow A$, then: B rebuts on the conclusion φ of A' where A' 's top rule is defeasible, or B undermines the ordinary premise $A' = \varphi$, and $B \prec A'$. Also, $\text{Conc}(B)$ must be a contradictory of φ since otherwise $\text{Conc}(B)$ would be a contrary of φ implying that $B \leftrightarrow A$. Also, B must be plausible or defeasible since for $B \prec A'$ to be the case, B cannot be strict and firm (since \preceq is reasonable).

- 1) If B is an ordinary or assumption premise or has a defeasible top rule, $A' \rightarrow B$, and since $B \prec A'$, $A' \leftrightarrow B$.
- 2) If B has a strict top rule, then by Proposition 7 there exists a strict extension $A_{B'}^+$ s.t. $A_{B'}^+ \leftrightarrow B$. QED

Lemma 10 Let $(\mathcal{A}, \mathcal{C}, \preceq)$ be a c-SAF. If A_1, \dots, A_n are acceptable w.r.t. some conflict-free $E \subseteq \mathcal{A}$, then $\bigcup_{i=1}^n \text{Prem}(A_i)$ is c-consistent.

PROOF. Suppose for contradiction otherwise, and let S be any minimally c-inconsistent subset of $\bigcup_{i=1}^n \text{Prem}(A_i)$. By assumption of c-classicality, for all $\varphi \in S$, $S \setminus \{\varphi\} \vdash \neg \varphi$ and $S \setminus \{\varphi\}$ is c-consistent. We thus have the set of non-axiom premises $\{\varphi_1, \dots, \varphi_m\} \subseteq \bigcup_{i=1}^n \text{Prem}(A_i)$ (that must be non-empty given that \mathcal{K}_n is c-consistent by assumption of axiom consistency), such that for $i = 1 \dots m$, there is a strict extension $B^{+/i}$ of $\{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \varphi_m\}$ s.t. $B^{+/i} \rightarrow \varphi_i$. Since \preceq is reasonable, for some i , $B^{+/i} \not\prec \varphi_i$ and so $B^{+/i} \leftrightarrow \varphi_i$. Since for $i = 1 \dots n$, A_i is acceptable w.r.t. E , then:

since $\varphi_i \in \bigcup_{i=1}^n \text{Prem}(A_i)$, then by Lemma 8-4, φ_i is acceptable w.r.t. E ;
since $B^{+/i}$ is a strict extension of some subset of $\bigcup_{i=1}^n \text{Prem}(A_i)$, then by Lemmas 8-4 and 2, $B^{+/i}$ is acceptable w.r.t. E .

But then since $B^{+/i} \leftrightarrow \varphi_i$, $\exists X, Y \in E$ s.t. $Y \leftrightarrow B^{+/i}$, $X \leftrightarrow Y$, contradicting E is conflict free. QED

Lemma 11 Let A be acceptable w.r.t an admissible extension S of $(\mathcal{A}, \mathcal{C}, \preceq)$. Then $\forall B \in S \cup \{A\}$, neither $B \leftrightarrow A$ or $A \leftrightarrow B$.

PROOF. Suppose for contradiction that: 1) $A \leftrightarrow B$, $B \in S \cup \{A\}$. By assumption of acceptability of B , $\exists C \in S$ s.t. $C \leftrightarrow A$, and by acceptability of A , $\exists D \in S$ s.t. $D \leftrightarrow C$, hence $D \rightarrow C$, contradicting S is conflict free; 2) $B \leftrightarrow A$, $B \in S$. By acceptability of A , $\exists D \in S$ s.t. $D \leftrightarrow B$, hence $D \rightarrow B$, contradicting S is conflict free. QED

We now state the two main results of this section:

Proposition 12 Let A be acceptable w.r.t an admissible extension S of $(\mathcal{A}, \mathcal{C}, \preceq)$. Then $S' = S \cup \{A\}$ is conflict free.

PROOF. Firstly, since for any $B \in S$, B is acceptable w.r.t. S , then by Lemma 10, $\text{Prem}(A) \cup \text{Prem}(B)$ is c-consistent.

Now, suppose for contradiction that S' is not conflict free. If $\exists B \in S'$ s.t. $A \rightleftharpoons B$ (this accounts for the case that $B = A$), then by Lemma 8-2, either $A \leftrightarrow B$ or $B \leftrightarrow A$, contradicting Lemma 11. Else:

- 1) $\exists B \in S$, $B \rightarrow A$, $B \prec A$, and not $A \rightarrow B$. By Lemma 9

either:

1.1 some sub-argument A' of A defeats B , hence (by acceptability of B) $\exists C \in S$ s.t. $C \hookrightarrow A'$, and so (by Lemma 8-3) $C \hookrightarrow A$, contradicting Lemma 11, or;

1.2 $\exists A_{B'}^+, A_{B'}^+ \hookrightarrow B$, hence $\exists C \in S$ s.t. $C \hookrightarrow A_{B'}^+$. By construction of $A_{B'}^+$, it must be that $C \hookrightarrow Z$, $Z \in \text{Sub}(A) \cup \text{Sub}(B)$. Hence, (by Lemma 8-3) either $C \hookrightarrow B$, contradicting S is conflict free, or $C \hookrightarrow A$, contradicting Lemma 11.

2 $\exists B \in S$, $A \rightarrow B$, $A \prec B$, and not $B \rightarrow A$. By Lemma 9 either:

2.1 some sub-argument B' of B defeats A , hence (by acceptability of A) $\exists C \in S$ s.t. $C \hookrightarrow B'$ and so (by Lemma 8-3) $C \hookrightarrow B$, contradicting S is conflict free, or;

2.2 $\exists B_{A'}^+, B_{A'}^+ \hookrightarrow A$, hence $\exists C \in S$ s.t. $C \hookrightarrow B_{A'}^+$. By construction of $B_{A'}^+$, it must be that $C \hookrightarrow Z$, $Z \in \text{Sub}(A) \cup \text{Sub}(B)$, leading to a contradiction as in **1.2**). QED

Proposition 13 Let A, A' be acceptable w.r.t an admissible extension S of $(\mathcal{A}, \mathcal{C}, \preceq)$. Then:

1. $S' = S \cup \{A\}$ is admissible
2. A' is acceptable w.r.t. S' .

PROOF. 1) By Lemma 8-1, all arguments in S' are acceptable w.r.t. S' . By Proposition 12, S' is conflict free, and hence admissible. 2) By Lemma 8-1, A' is acceptable w.r.t. S' . QED

We have shown that under intuitive assumptions on argument orderings (satisfied by the commonly used weakest and last link principles), and closure of strict rules under transposition or contraposition, SAFs and c-SAFs satisfy Dung's fundamental lemma (implying, for example, that every admissible extension is a subset of a preferred extension). Also, note that Lemma 8-1 implies monotonicity of the characteristic function, so that a SAF's (c-SAF's) grounded extension can be identified by the function's least fixed point.

4.2 Rationality Postulates

We now show that SAFs and c-SAFs satisfy [Caminada and Amgoud, 2007]'s rationality postulates for the semantics defined in Definition 1 (it suffices to show they are satisfied by complete extensions).

Theorem 14 [Sub-argument Closure] Let $\Delta = (\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF and E a complete extension of Δ . Then for all $A \in E$: if $A' \in \text{Sub}(A)$ then $A' \in E$.

PROOF. A' is acceptable w.r.t. E by Lemma 8-4, $E \cup \{A'\}$ is conflict free by Prop.12, and so $A' \in E$. QED

Theorem 15 [Closure under Strict Rules] Let $\Delta = (\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF and E a complete extension of Δ . Then $\{\text{Conc}(A) | A \in E\} = \text{Cl}_{R_s}(\{\text{Conc}(A) | A \in E\})$.

PROOF. It suffices to show that any strict extension X of $\{A | A \in E\}$ is in E , which follows given that by Lemma 2, X is acceptable w.r.t. E , and by Proposition 12, $E \cup \{X\}$ is conflict free. If Δ is a c-SAF, Lemma 10 guarantees that X is a valid argument in the sense that its premises are c-consistent. QED

Theorem 16 [Direct Consistency] Let $\Delta = (\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF and E a complete extension of Δ . Then $\{\text{Conc}(A) | A \in E\}$ is consistent.

PROOF. We show that if $A, B \in E$, $\text{Conc}(A) \in \overline{\text{Conc}(B)}$, then this leads to a contradiction:

1. A is firm and strict, and:

1.1 if B is firm and strict then this contradicts assumption of *axiom consistency*;

1.2 if B is plausible or defeasible, and **1.2.1** B is an ordinary/assumption premise or has a defeasible top rule, then $A \rightarrow B$, contradicting E is conflict free, or **1.2.2** B has a strict top rule (see **3** below).

2. A is plausible or defeasible, and:

2.1 if B is firm and strict then under the *well-formed* assumption $\text{Conc}(A)$ cannot be a contrary of $\text{Conc}(B)$, and so they are contradictory (i.e., a contrary of each other), and **2.1.1** A is an ordinary/assumption premise or has a defeasible top rule, in which case $B \rightarrow A$, contradicting E is conflict free, or **2.1.2** A has a strict top rule (see **3** below);

2.2 if B is plausible or defeasible and **2.2.1** B is an ordinary/assumption premise or has a defeasible top rule then $A \rightarrow B$, contradicting E is conflict free, or **2.2.2** B has a strict top rule (see **3** below).

3. Each of **1.2.2**, **2.1.2** and **2.2.2** describes the case where $X, Y \in E$, $\text{Conc}(X) \in \overline{\text{Conc}(Y)}$, Y is defeasible or plausible and has a strict top rule, and by the *well-formed* assumption $\text{Conc}(X)$ and $\text{Conc}(Y)$ must be contradictory. In the case that Δ is a c-SAF, since $X, Y \in E$, X, Y are acceptable w.r.t. E , and so by Lemma 10, $\text{Prem}(A) \cup \text{Prem}(B)$ is c-consistent. By Prop 7 there is a strict extension $X_{Y'}^+$ of $M(Y) \setminus \{Y'\} \cup M(X)$ s.t. $X_{Y'}^+ \rightarrow Y$. By Lemma 2 $X_{Y'}^+$ is acceptable w.r.t. E , and by Prop. 12, $E \cup \{X_{Y'}^+\}$ is conflict free, contradicting $X_{Y'}^+ \rightarrow Y$. QED

Theorem 17 [Indirect Consistency] Let $\Delta = (\mathcal{A}, \mathcal{C}, \preceq)$ be a SAF or c-SAF and E a complete extension of Δ . Then $\text{Cl}_{R_s}(\{\text{Conc}(A) | A \in E\})$ is consistent.

PROOF. Follows from Theorems 15 and 16. QED

5 Conclusions

This paper has proposed that in contrast with existing works that obtain defeats by application of preferences, one should retain the attack-based definition of conflict-free sets, given that attacks between arguments indicate the mutual inconsistency of the logical instantiations of the attacking arguments; defeats encode the preference dependent use of attacks in a dialectical context, and so should only be deployed when evaluating the acceptability of arguments. We have reformulated the *ASPIC*⁺ framework under the new proposal, and generalised *ASPIC*⁺, thus obtaining structured argumentation frameworks that not only subsume argumentation systems such as [Bondarenko *et al.*, 1997], but also now captures classical logic approaches to argumentation. We have then shown that under some intuitive assumptions, key properties of Dung frameworks, and [Caminada and Amgoud, 2007]'s rationality postulates are satisfied. We are thus the first to demonstrate the consistency of extensions (under *any* semantics subsumed by the complete semantics) obtained by classical logic approaches augmented with preferences². This is

²Note that [Amgoud and Vesic, 2010] apply preferences to sets of arguments, rather than to individual attacks, and then show con-

of particular importance given that classical-logic approaches without preferences yield (stable/preferred) extensions that simply correspond to the maximal consistent subsets of the instantiating classical logic theories [Cayrol, 1995] (in classical logic preferences thus play a particularly important role in arbitrating between conflicts).

This paper’s generalisation and reformulation of *ASPIC*⁺ treats unsuccessful asymmetric attacks in a different way to [Prakken, 2010]. Such attacks occur when a defeasible or plausible B with strict top rule, rebuts a defeasible A with defeasible top rule, and $B \prec A$. Note that given the intended meaning of strict rules as deductive inferences, it is entirely intuitive that A does not rebut B on the conclusion of B ’s strict top rule: a basic principle of deductive reasoning is that if one does not like the conclusion of a deductively derived inference, one must *give up* (which in dialectical setting equates with *deploy an attack against*) one of the defeasible premises. In [Prakken, 2010], the set $\{A, B\}$ is conflict-free, although $\{A, B\}$ cannot be the subset of a complete extension since such an extension would necessarily contain an argument A_B^+ , that defeats B (as described in Proposition 7). In this paper’s version of *ASPIC*⁺, $\{A, B\}$ is *not* conflict-free, which is conceptually more satisfactory given the logical incompatibility of the information in the arguments.

[Kaci, 2010] argues that to ensure that the use of defeats does not violate consistency, all attack relations should be symmetric, but we agree with [Amgoud and Besnard, 2009] that this would lead to problems. [Amgoud and Vesic, 2009; 2010] claim that unsuccessful asymmetric attacks should result in rejection of the attacker, so that in our example only $\{A\}$ would be admissible. However, our consistency results obviate the need for this. [Amgoud and Vesic, 2010] also motivate their approach by a counterexample to extension consistency but this does not apply to *ASPIC*⁺ because of its different definition of undermining defeat. Moreover, unlike in *PAFs* [Amgoud and Cayrol, 2002], where all attacks require preferences to be successful, we can make undercutting attack successful irrespective of preferences. This is desirable since unlike with asymmetric rebutting attack, with undercutting attack the attacked argument can, even if R_s is closed, never be extended to yield an attacker of the undercutter, so there is no intuitive sense in which an undercut argument attacks its undercutter. Thus we can, unlike [Amgoud and Vesic, 2009; 2010], preserve the basic principle of argumentation that an un-attacked argument is always justified.

Much work on value based argumentation frameworks (*VAFs*) [Bench-Capon, 2003] considers extensions that contain arguments that asymmetrically attack but do not defeat after accounting for the ranking of the arguments’ values. However, little work has been done on logical instantiations of *VAFs*. Since *ASPIC*⁺ *explicitly* relates the notion of attack to conflict in some (unspecified) instantiating logic, then such extensions are clearly undesirable.

Finally, future work will extend this paper’s work in order to structure Extended Argumentation Frameworks (*EAFs*) [Modgil, 2009]. Initial investigations suggest that if one retains the attack-based definition of conflict free for *EAFs*,

then the characteristic functions of such structured *EAFs* will be monotonic (in [Modgil, 2009] monotonicity is only shown for the characteristic functions of *hierarchical EAFs*).

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