Normality, Non-Contamination and Logical Depth in Classical Natural Deduction

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Abstract

In this paper we provide a detailed proof-theoretical analysis of a natural deduction system for classical propositional logic that (i) represents classical proofs in a more natural way than standard Gentzen-style natural deduction, (ii) admits of a simple normalization procedure such that normal proofs enjoy the Weak Subformula Property, (iii) provides the means to prove a Non-Contamination Property of normal proofs that is not satisfied by normal proofs in the Gentzen tradition and is useful for applications, especially to formal argumentation, (iv) naturally leads to defining a notion of *depth* of a proof, to the effect that, for every fixed natural k, normal k-depth deducibility is a tractable problem and converges to classical deducibility as k tends to infinity.

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1 Introduction

Gentzen introduced his natural deduction systems NJ and NK in his [1935] paper in order to "set up a formal system that comes as close as possible to actual reasoning" [p. 68]. In the same paper, Gentzen also introduced his sequent calculi LJ and LK to overcome some technical difficulties in the proof of his "main result" (*Haupsatz*), namely the subformula theorem for first order logic. The theorem ensures that the search for proofs can be pursued by *analytic* methods, i.e. by considering only inference steps involving formulae that are "contained" in the assumptions or in the conclusion.[1] Thus, no particular ingenuity is required, at least in principle, to construct such analytic arguments and their search is amenable to algorithmic treatment.

[1]"No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result" [Szabo, 1969, p. 69]; "the final result is, as it were, gradually built up from its constituent elements" [Szabo, 1969, p. 88].

Gentzen proved the subformula property in the context of his sequent calculi by means of the celebrated "cut-elimination" theorem. The proof of the same result for the natural deduction systems, usually ascribed to Dag Prawitz [2] [1965], makes use of a "normalization" procedure by means of which any proof can be reduced to one with a specified (albeit not unique) normal form. Dag Prawitz [1965] proved a normalization theorem (with a weak subformula property) for the fragment without $\vee$ and $\exists$. Normalization procedures for the full system were provided in [Statman, 1974, Stålmarck, 1991]. More recenlty Jan Von Plato [2012] has proved a full normalization theorem using a considerably simpler procedure.

Natural deduction is now widely used in the teaching of logic,[3] and has been thoroughly investigated in the philosophical and proof-theoretical literature.[4] In automated deduction circles, however, it has been considerably neglected in favour of alternative methods such as resolution, the sequent calculus, the method of analytic tableaux or variants thereof. The official motivation — namely that natural deduction is less amenable to the development of automated proof search methods — can be, and has been, challenged, especially in view of the normalization theorem and the related subformula property of proofs.[5] Moreover, owing to recent advances in Artificial Intelligence and Human-Oriented Computing, most notably the research program on Formal Argumentation Theory pioneered by [Dung, 1995], there is a growing need for automated proof procedures based on natural inference patterns that, in the spirit of Gentzen's work, come "as close as possible to actual reasoning" and can therefore be fruitfully employed in human-computer interaction [Modgil et al., 2013].

Our starting point in this paper is the well-known fact that Gentzen-style natural deduction is natural from the point of view of intuitionistic logic, but not from the point of view of classical logic, for its introduction and elimination rules are faithful to the intuitionistic meaning of the logical operators, but not to their classical meaning. As a result they are unable to adequately represent the natural inference patterns that exploit the inner symmetries of classical logic. This is a serious hindrance towards providing adequate alternatives to the more popular methods used in the field of automated deduction for classical logic.

The main purpose of this paper is an in-depth proof-theoretical analysis of a non-standard natural deduction system for classical propositional logic that we call *C-intelim* — where "intelim" stands for "introduction and elimination"[6] — which is based

[2]In the same year Andrés Raggio also published a short paper containing a proof of Gentzen's *Hauptsatz* [Raggio, 1965]. Ian Von Plato has recently discovered that, in fact, Gentzen had a proof also for his natural deduction system NJ that he decided not to publish [von Plato, 2008] probably because he despaired to show a similar result for classical logic. For an excellent overview of Prawitz's work in proof theory see [Wansing, 2015].

[3]In fact, for this purpose the Jaśkoswski-Fitch presentation of proofs is much more popular than the original Gentzen-Prawitz presentation which, on the other hand, is dominant in proof-theoretical studies. For a comparison see [Pelletier and Hazen, 2012, Hazen and Pelletier, 2014].

[4] For insightful and comprehensive treatments see [Tennant, 1990, Negri and von Plato, 2001]). For the history of natural deduction see [Pelletier and Hazen, 2012].

[5]See, for example, [Tennant, 1992, Sieg and Pfenning, 1998]; for more recent examples see [Indrzejczak, 2010, Ferrari and Fiorentini, 2015, Maretić, 2018]. For a more comprehensive list of references see [Indrzejczak, 2010, pp. 96–97].

[6]As a reviewer pointed out to us, the first use of the term "intelim" is probably due to Frederik B. Fitch [1952].

on the *classical* truth-table meaning of the logical operators and, while sharing some interesting features with Gentzen-style natural deduction, brings it somewhat closer to the method of analytic tableaux. This allows us to prove a normalization theorem that is more informative than the ones that are usually shown for Gentzen-style systems. The rules of this system were first proposed in [Mondadori, 1989] and discussed in [D'Agostino, 2005].[7] This is, however, the first time these rules are the object of a detailed proof-theoretical investigation aimed at a direct comparison with the large body of literature on Gentzen-style natural deduction.

The restriction of C-intelim to normal proofs enforces a stricter control discipline on proof-construction, to the effect that normal proofs, besides enjoying the *subformula property*, enjoy also a kind of weak relevance property that we call *non-contamination*. We say that a proof of A depending on Γ is "contaminated" if one of the following conditions holds: (i) A is equal to the special "falsum" symbol $\curlywedge$ (i.e., the proof is refutation of Γ) and, for some $\Delta \subset \Gamma$, the formulae in Δ are syntactically disjoint[8] from those in $\Gamma \setminus \Delta$; (ii) $A \neq \curlywedge$ and, for some $\Delta \subseteq \Gamma$, the formulae in Δ are syntactically disjoint from those in $(\Gamma \setminus \Delta) \cup \{A\}$.

The classical validity of *ex-falso quodlibet*, or better, *ex-contradictione quodlibet* — from an inconsistent Γ infer any arbitrary conclusion A — implies that for any natural deduction system that is complete for classical logic there are inferences that admit only of contaminated proofs. However, in *normal* C-intelim this can happen only in the trivial case in which the proof is *essentially a (non-contaminated) refutation* of Γ, that is, immediately obtained from such a refutation by means of a peculiar use of the *ex-falso quodlibet* principle. The paradigmatic example is:

$$\dfrac{\dfrac{A \quad \neg A}{\curlywedge}}{B} \tag{1}$$

where the conclusion is obtained from a non-contaminated refutation of the same assumptions by means of an application of *ex-falso* that is meaningless from the point of view of deductive practice.[9] If this is the case, we call such a proof *improper* and show that it can always be readily transformed into a normal non-contaminated proof of $\curlywedge$ depending on the *same* assumptions, by simply removing the bizarre applications of *ex-falso*. On the other hand, *proper* normal proofs are always non-contaminated and so satisfy the requirement, often called the *variable-sharing* property, that their premises are never syntactically disjoint from their conclusion (except for the special case in which the conclusion is $\curlywedge$ and the proof is, therefore, a non-contaminated refutation of

[7] They are also used in [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2015] in order to characterize various sequences of tractable approximations to classical propositional logic.

[8] Two formulae are syntactically disjoint when they share no atomic subformula.

[9] As Michael Dummett once put it: "Obviously, once a contradiction has been discovered, no one is going to go *through it*: to exploit it to show that the train leaves at 11:52 or that the next Pope will be a woman." [Dummett, 1991, p. 209]. A controversial justification of (1) is provided by the celebrated "Lewis proof". See [Bennett, 1969] for a thorough discussion in which the author provides an interesting example which does not appear to be counterintuitive argues that "the logic of the Lewis argument can be displayed in indefinitely many other examples, whose validity is highlighted by the their being possible — even plausible — slices of real argumentative life" (p. 220).

the assumptions).[10]

Thus, our normalization theorem allows us to show that the restriction of C-intelim to normal proofs enjoys the following

Non-contamination property: if π is a proof of A depending on assumptions Γ, then either π is non-contaminated or π is improper. (NCP)

In the latter case there is a straightforward (linear time) procedure to turn π into a non-contaminated proof of $\curlywedge$ depending on the *same* set Γ of assumptions. To the best of our knowledge, NCP and a notion of normal proof that automatically satisfies it are investigated in this paper for the first time.

NCP is not enforced by the standard notion(s) of normal proof in Gentzen-style natural deduction for classical logic. We maintain that a control discipline on the (automated) generation of proofs that enforces this property is of considerable potential interest for a variety of application areas in that it stops the generation of obviously redundant proofs in which a subset of the assumptions on which the conclusion depends are totally unrelated to the other assumptions and to the conclusion. (This point will be discussed in Section 7.) A prominent research area in which NCP is of crucial importance is, again, that of Formal Argumentation Theory that is now widely regarded as a most promising research program in Artificial Intelligence [Bench-Capon and Dunne, 2007].

Moreover, our normalization theorem leads to a straightforward definition of a measure of *depth* for *normal* natural deduction proofs in such a way that, for every fixed k, if a proof of depth $\leq k$ exists, it can be found in polynomial time. The resulting notion of depth-bounded natural deduction lays the foundations for modelling the deductive practice of resource-bounded agents that reason according to classical logic. We maintain that this approach looks promising in view of practical applications of Argumentation Theory to real-world, resource-bounded agents [D'Agostino and Modgil, 2016, 2018]. From this point of view, this paper is a further articulation of a research program whose philosophical and computational aspects have been investigated in [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2014, 2015]. Here, the focus is on the proof-theoretical presentation, on the normalization theorem — with three different"shades" of normal proofs each of which is interesting in its own right — and on the NCP of normal proofs.

The paper is organized as follows. In Section 2 we discuss the drawbacks of Gentzen-style natural deduction from the point of view of classical logic. In Section 3 we present the C-intelim system and discuss its main features. The normalization theorem is proved in two steps. In Section 4 we discuss the intermediate notion of *quasi-normal* proof and argue that it has a proof-theoretical interest on its own, in that it allows representations of proofs that exclude trivial detours, while not necessarily being "analytic", and that may, in some cases, be significantly shorter than any analytic proof.

[10]This property establishes a surface connection with the proof-theory of relevance logic. It is worth stressing however, that the concerns of relevance logicians are not restricted to banning irrelevant applications of *ex-falso quodlibet*, but also with the beahviour of the conditional operator that, in our approach, is entirely classical.

Then, in Section 6 we define the notion of *normal* proof and show that it enjoys the weak subformula property (every formula occurring in a normal proof of A depending on a set Γ of assumptions is either a subformula of some formula in $\Gamma \cup \{A\}$, or the negation of such a subformula, or is equal to $\curlywedge$). Next, in Sections 7 and 8 we discuss the contamination problem and introduce the distinction between *proper* and *improper* normal proofs, showing that proper normal proofs enjoy the variable-sharing property, are refutation-complete and are also complete with respect to all classical consequences of consistent sets of assumptions. In Section 9 we discuss the correspondence between the C-intelim system analysed in this paper and the tableau-like method used in [D'Agostino, 2015] that here we call *C-intelim Tableaux*. We argue that, in the context of classical logic, the latter could provide a smooth transition to implementable algorithms for the search of natural deduction proofs that may successfully compete with the more popular alternatives based on resolution, sequent calculi or tableaux. This further supports the promise of this approach for use by real-world, resource-bounded agents. (See also the concluding section on this point.) Finally, in Section 10 we report some complexity facts about C-intelim, and in particular that the notion of normal k-depth C-intelim deducibility provides a hierarchy of tractable approximation to classical propositional logic.

In this paper we restrict ourselves to propositional logic. Extending our main results to first-order logic is non-trivial, especially if one attaches a crucial importance to the notion of depth-bounded reasoning . In the first-order case such a notion requires incorporating some of the ideas put forward by Jaakko Hintikka (e.g., in [Hintikka, 1972]) to provide an analogous hierarchy of depth-bounded approximations to first-oder logic. It also involves solving some technical problem related to the transformation of proofs into RB-canonical ones. Since a characterization of depth-bounded reasoning is crucial in view of practical applications in a variety of areas we shall postpone these investigations until a subsequent paper. Throughout this paper we assume that the reader is familiar with the basic notions of Gentzen-style natural deduction.[11]

2 Is Gentzen-style natural deduction really natural for classical logic?

It is well-known that Gentzen-style natural deduction provides a natural formalization of intuitionistic logic, but a quite unnatural formalization of classical logic. Gentzen himself observed that his classical calculus *NK* was obtained by adding to the intuitionistic calculus *NJ* the law of excluded middle in "a purely external manner" that spoiled the harmony between introductions and eliminations [Szabo, 1969, p. 81]. This approach was illuminating in that it clarified the relationship between the two logical systems, but did so from the vantage point of intuitionistic logic. It is, therefore, only to be expected that the *NK* proof of an inference that is classically, but not intuitionistically, valid turns out to be rather unnatural. The same holds true for Prawitz's variant [Prawitz, 1965], that consists in replacing the intuitionistic *ex-falso* rule and the law of excluded middle with *classical reductio*. Prawitz's rules for natural deduction are

[11] For references see footnote 4 above.

shown in Table 1.[12] Recall that in some of the rules, the vertical dots stand for a natural deduction proof of the formula below the dots depending on assumptions that may include the ones enclosed in square brackets. The latter are "discharged" by the application of the rule in the sense that the conclusion no longer *depends* on them, but only on the yet undischarged assumptions that occur in the leaves. We call *discharge rules* those rules that involve the discharge of assumptions and *inference rules* those rules that do not. So, in this system, the inference rules are $\wedge\mathcal{I}$, $\wedge\mathcal{E}$, $\vee\mathcal{I}$, $\rightarrow\mathcal{E}$, $\neg\mathcal{E}$ and $\curlywedge_{\mathsf{I}}$, while the discharge rules are $\vee\mathcal{E}$, $\rightarrow\mathcal{I}$, $\neg\mathcal{I}$. It is important to note that in the discharge rules assumptions may be discharged *vacuously*, i.e., even if they do not occur in the leaves of the proof tree. A system for classical logic is obtained by replacing the inference rule $\curlywedge_{\mathsf{I}}$ with the proof rule $\curlywedge_{\mathsf{C}}$.[13] Note that the intuitionistic falsum rule $\curlywedge_{\mathsf{I}}$ is a special case of the classical one that arises when the assumption $\neg A$ is vacuously discharged.

A proof of A *depending* on Γ is a tree of occurrences of formulae constructed in accordance with the rules[14] such that A occurs at the root and Γ is the set of all the undischarged assumptions occurring at the leaves. The system is complete for classical logic in the sense that for every Δ, A such that A is a classical consequence of Δ, then there is a natural deduction proof of A depending on some $\Gamma \subseteq \Delta$.

Let us consider a proof in Prawitz's system for intuitionistic logic of the intuitionistically valid inference:

$$\neg A \vee \neg B \vdash \neg(A \wedge B) \tag{2}$$

We use numerals to keep track of the assumptions that are discharged by the application of a proof rule. The numerals corresponding to the discharged assumptions are shown beside the inference line.

$$\cfrac{\neg A \vee \neg B \qquad \cfrac{\cfrac{[\neg A]^1 \quad \cfrac{[A \wedge B]^3}{A}}{\curlywedge}}{\neg(A \wedge B)}\,{}^{3} \qquad \cfrac{\cfrac{[\neg B]^2 \quad \cfrac{[A \wedge B]^3}{B}}{\curlywedge}}{\neg(A \wedge B)}\,{}^{3}}{\neg(A \wedge B)}\,{}_{1,2} \tag{3}$$

Let us now consider a standard proof of the reverse inference in Prawitz's system for classical logic:

$$\neg(A \wedge B) \vdash \neg A \vee \neg B \tag{4}$$

[12] For an explanation of the rules the reader can refer to Prawitz's book or to other expositions, for example those cited in footnote 4.

[13] As mentioned above, Gentzen's system *NK* used the law of excluded middle $A \vee \neg A$ as an axiom, i.e. an extra assumption that can be freely introduced in a proof, instead of $\curlywedge_{\mathsf{C}}$. He also remarked that this axiom could have been equivalently replaced by the inference rule $\neg\neg A/A$.

[14] For a formal definition see [Prawitz, 1965].

INTRODUCTION RULES

$$\frac{A \quad B}{A \wedge B}\wedge\mathcal{I} \qquad \frac{A}{A \vee B}\vee\mathcal{I}1 \qquad \frac{B}{A \vee B}\vee\mathcal{I}2$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}\to\mathcal{I} \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \curlywedge\end{array}}{\neg A}\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{A \wedge B}{A}\wedge\mathcal{E}1 \qquad \frac{A \wedge B}{B}\wedge\mathcal{E}2 \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\vee\mathcal{E}$$

$$\frac{A \to B \quad A}{B}\to\mathcal{E} \qquad \frac{\neg A \quad A}{\curlywedge}\neg\mathcal{E}$$

FALSUM RULES

$$\frac{\curlywedge}{A}\curlywedge_{\mathsf{I}} \qquad \frac{\begin{array}{c}[\neg A]\\ \vdots \\ \curlywedge\end{array}}{A}\curlywedge_{\mathsf{C}}$$

Table 1: Prawitz's rules for natural deduction.

which is *not* intuitionistically valid.

$$\dfrac{\dfrac{\neg(A \wedge B) \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^1 \quad \dfrac{[\neg A]^2}{\neg A \vee \neg B}}{\curlywedge}}{A}\,{}^{2} \qquad \dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^1 \quad \dfrac{[\neg B]^3}{\neg A \vee \neg B}}{\curlywedge}}{B}\,{}^{3}}{A \wedge B}}{\curlywedge}}{\neg A \vee \neg B}\,{}^{1} \tag{5}$$

This proof is quite unnatural from the classical point of view in that it does not exploit the inner symmetries of classical logic. Indeed, it is *very different* from the previous one. However, in a classical setting, the two proofs should essentially be the same, modulo the duality of $\vee$ and $\wedge$. By contrast, in both the classical sequent calculus LK and the Tableau method the proofs of (2) and (4) have essentially the same structure.

As these two examples strongly suggest, standard natural deduction may be natural from the point of view of intuitionistic logic, but is not so natural from the point of view of classical logic. If we are interested in a deduction system that is really natural for classical logic, we need introduction and elimination rules that closely reflect the classical meaning of the logical operators and the way in which these are used in classical proofs. In such a system a formula and its negation should be treated symmetrically. Moreover, the conjunction and disjunction operators should be governed by dual rules. This is the case, for instance, with the tableau method, where we have tableau rules for a compound formula of a given logical form and for its negation and where the rules for $\wedge$ and $\vee$ are dual of each other.[15] However, the interplay between introduction and elimination rules as well as the possibility of generating direct proofs, that are typical of natural deduction, are inevitably lost in the tableau method that uses only elimination rules, so that a proof of a conclusion from a set of assumptions is obtained by refuting its negation on the basis of the assumptions.[16]

[15] Smullyan once claimed that tableaux could indeed be presented as a sort of natural deduction for classical logic ([Smullyan, 1965]). Dual rules are also used in the natural deduction system EN^* proposed by Kent Bendall in [1978], who also advocates the use of signed formulae to solve the separation problem for classical logic. (On this point see also Section 3 below.)

[16] As a reviewer pointed out to us, an early use of dual rules for negative conjunctions and disjunctions in a system of natural deduction can be found in [Fitch, 1952, p. 60], some of which, namely the introduction rules, are used also in the system discussed in the next section. However, the elimination rules were not disegned to satisfy the separation property and exploit the De Morgan equivalences. On the other hand, elimination rules for negative disjunctions can be formulated as in the next section, while an elimination rule for negative conjunctions could be introduced as dual of Gentzen's rule of disjunction elimination. The main advantage of the system presented in the next section, is that it minimizes the use of discharge rules, which has important implication from the point of view of computational complexity and depth-bounded reasoning. The interested reader may want to show how Fitch's system can be simulated by C-intelim and viceversa.

3 C-intelim deduction

The C-intelim system is a natural deduction system whose rules, unlike those of standard natural deduction, are faithful to the classical meaning of the logical operators (i.e., to their truth-table interpretation) and not to their intuitionistic meaning. It was introduced in [Mondadori, 1989] and further investigated in [D'Agostino, 2005].[17] The C-intelim rules are shown in Table 2.

For the sake of philosophical analysis, the rules are best presented in terms of *signed formulae* of the form $T\,A$ or $F\,A$, with A an ordinary formula. The standard reading of these signed formulae is "A is true" and "A is false". In the context of classical logic this appears as the most natural way of achieving "separation" in the sense of [Bendall, 1978, p. 250] — each rule deals only with one logical operator and a proof should make use only of intelim rules for the operators that occur in the premises or in the conclusion — as well as the stronger form of separation that is embodied in the subformuila property. This is also the approach followed by Smullyan [1968] in his presentation of the tableau method. For a well-argued philosophical defence of the use of signed formulae in the proof theory of classical logic the reader is referred to [Bendall, 1978]. More recently, the use of signed formulae for philosophical purposes has been central in the discussion on "bilateralism" [Smiley, 1996, Rumfit, 2000, Humberstone, 2000, Ferreira, 2008, Gabbay, 2017] as an inferential approach to the meaning of the classical operators. An alternative, but closely related, way of achieving the same results in terms of ordinary formulae is that of resorting to *multi-conclusion sequents* as the primary components of a deduction system. The connection between multi-conclusion sequents and signed formulae is made apparent when looking at the translation of classical sequent proofs in systems with no structural rules like Kleene's G4 [Kleene, 1967, chapter 6] into closed semantic tableaux.[18]

Note that in C-intelim the introduction and elimination rules for the logical operators, as well as the falsum rules, are all *inference rules*, involving no discharge of assumptions. The only *proof rule* RB is a structural rule that expresses the classical principle of bivalence and is therefore peculiar to classical logic. For each application of RB the formula A that occurs in the discharged assumptions is called the *RB-formula* of that application. A C-intelim proof of A *depending* on Γ is a tree of occurrences of signed formulae constructed in accordance with the C-intelim rules such that $T\,A$ occurs at the root and $\{T\,B \mid B \in \Gamma\}$ is the set of all undischarged assumptions that occur at the leaves.

Observe that the intelim rules for disjunction and conjunction are dual of each other, and that a signed formula and its conjugate are treated symmetrically, as they should be in a classical setting. For each logical operator, we have intelim rules for a signed

[17] In [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2015] similar rules are used in a different format to define a hierarchy of tractable depth-bounded approximations to classical propositional logic.

[18] See [D'Agostino, 1990, Section 2.2] for the details. For another interesting approach, in the context of the Curry-Howard correspondence for classical logic, see [Aschieri et al., 2018]. On the other hand, without essential extensions of the logical language (signed formulae) or of the intuitive notion of inference rule (multi-conclusion sequent calculus), or other non-standard technical devices most authors are skeptical about the possibility of a genuine inferential semantics for classical logic. One notable exception is [Sandqvist, 2009]; see also the analysis in [Makinson, 2013].

formula containing it as main operator as well as intelim rules for its conjugate. This feature is shared by the tableau method and other bilateral systems of deduction, such as Bendall's [Bendall, 1978] or Rumfit's [Rumfit, 2000]. Unlike the tableau method, however, C-intelim contains introduction as well as elimination rules and so can be used for direct proofs as well as for refutations. Moreover, the elimination rules for $T\,A \rightarrow B$, $T\,A \vee B$ and $F\,A \wedge B$ do not generate any branching and are standard inference rules that require an additional minor premise. So, unlike Gentzen-style natural deduction and other "bilateral" natural deduction systems, C-intelim contains no discharge rule for the logical operators.

The rules RNC (Rule of Non-Contradiction) and XFQ (Ex-Falso Quodlibet) are inference rules that express the basic principle of classical semantics according to which no valuation can make the same formula both true and false. We find it convenient to represent this principle by making use of the "falsum" symbol $\curlywedge$. Throughout this paper we make the simplifying assumption that "$\curlywedge$" occurs only in the context of the falsum rules, namely as conclusion or RNC or as premise of XFQ, and nowhere else. From the point of view of classical logic this is a reasonable stipulation and avoids a good deal of tedious details in the proofs and in the definitions. However, there is no loss of generality in making this simplifying assumption. So, for us $\curlywedge$ will be essentially a marker to make it explicit that a contradiction has been reached. Note that each inference rule expresses the fact that every Boolean valuation that satisfies its premise(s) satisfies also its conclusion.[19] Then, RNC and XFQ are classically correct for the simple reason that no Boolean valuation can satisfy their premises. In our view RB, RNC and XFQ are all *structural rules*, in that they reflect the fundamental properties governing the underlying (classical) notions of truth and falsity — governed by the principles of bivalence and non-contradiction — and not the inferential behaviour of some logical operators.[20]

Under an alternative reading of signed formulae, we may interpret them as assertions about the current information state: $T\,A$ would then mean that we possess the information that A is true and $F\,A$ that we possess the information that A is false. In this view all the inference rules are rules that draw straightforward conclusions from information that *we actually possess*. On the other hand the rule RB simulates alternative information states that extend the actual one. Thus, the discharged assumptions in each application of RB are called *virtual assumptions* in that they represent "virtual information", i.e. information that *we do not actually possess*. The notion of proof-tree based on these rules is essentially the same as that of standard natural deduction, except that a proof of A depending on the undischarged assumptions Γ is replaced by a proof of $T\,A$ depending on the undischarged assumptions $\{T\,B \mid B \in \Gamma\}$.

For all practical purposes, however, we may find it convenient to work with *unsigned formulae*, by exploiting the classical meaning of negation. This amounts to simply removing the sign T before a formula and replacing the sign F with the negation operator. The resulting rules for unsigned formulae are displayed in Table 3. In this version, the intelim rules are no longer "pure", the subformula property holds only

[19] That a Boolean valuation v *satisfies* a signed formula is to be intended in the obvious way: v satisfies $T\,A$ when $v(A) = 1$ and $F\,A$ when $v(A) = 0$.

[20] As explained above, we do not treat $\curlywedge$ as a logical operator.

INTRODUCTION RULES

$$\frac{T\,A \quad T\,B}{T\,A \wedge B}\,T\!\wedge\mathcal{I} \qquad \frac{F\,A}{F\,A \wedge B}\,F\!\wedge\mathcal{I}1 \qquad \frac{F B}{F A \wedge B}\,F\!\wedge\mathcal{I}2$$

$$\frac{F\,A \quad F\,B}{F\,A \vee B}\,F\!\vee\mathcal{I} \qquad \frac{T\,A}{T\,A \vee B}\,T\!\vee\mathcal{I}1 \qquad \frac{T\,B}{T\,A \vee B}\,T\!\vee\mathcal{I}2$$

$$\frac{T\,A \quad F\,B}{F\,A \to B}\,F\!\to\mathcal{I} \qquad \frac{F\,A}{T\,A \to B}\,T\!\to\mathcal{I}1 \qquad \frac{T\,B}{T\,A \to B}\,T\!\to\mathcal{I}2$$

$$\frac{T\,A}{F\,\neg A}\,F\neg\mathcal{I} \qquad \frac{F\,A}{T\,\neg A}\,T\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{T\,A \vee B \quad F\,A}{T\,B}\,T\!\vee\mathcal{E}1 \qquad \frac{T\,A \vee B \quad F\,B}{T\,A}\,T\!\vee\mathcal{E}2 \qquad \frac{F\,A \vee B}{F\,A}\,F\!\vee\mathcal{E}1 \qquad \frac{F\,A \vee B}{F\,B}\,F\!\vee\mathcal{E}2$$

$$\frac{F\,A \wedge B \quad T\,A}{F\,B}\,F\!\wedge\mathcal{E}1 \qquad \frac{F\,A \wedge B \quad T\,B}{F\,A}\,F\!\wedge\mathcal{E}2 \qquad \frac{T\,A \wedge B}{T\,A}\,T\!\wedge\mathcal{E}1 \qquad \frac{T\,A \wedge B}{T\,B}\,T\!\wedge\mathcal{E}2$$

$$\frac{T\,A \to B \quad T\,A}{T\,B}\,T\!\to\mathcal{E}1 \qquad \frac{T\,A \to B \quad F\,B}{F\,A}\,F\!\to\mathcal{E}2 \qquad \frac{F\,A \to B}{T\,A}\,F\!\to\mathcal{E}1 \qquad \frac{F\,A \to B}{F\,B}\,F\!\to\mathcal{E}2$$

$$\frac{T\,\neg A}{F\,A}\,T\neg\mathcal{E} \qquad \frac{F\,\neg A}{T\,A}\,F\neg\mathcal{E}$$

FALSUM RULES

$$\frac{T\,A \quad F\,A}{T\,\curlywedge}\ \text{RNC} \qquad \frac{T\,\curlywedge}{S\,A}\ \text{XFQ}^{*}$$

RULE OF BIVALENCE

$$\frac{\begin{array}{cc}[T\,A] & [F\,A]\\ \vdots & \vdots\\ S\,B & S\,B\end{array}}{S\,B}\ \text{RB}^{*}$$

* under any uniform substitution of S with T or F.

Table 2: The C-intelim rules for signed formulae.

INTRODUCTION RULES

$$\frac{A \quad B}{A \wedge B}\,\wedge\mathcal{I} \qquad \frac{\neg A}{\neg(A \wedge B)}\,\neg\wedge\mathcal{I}1 \qquad \frac{\neg B}{\neg(A \wedge B)}\,\neg\wedge\mathcal{I}2$$

$$\frac{\neg A \quad \neg B}{\neg(A \vee B)}\,\neg\vee\mathcal{I} \qquad \frac{A}{A \vee B}\,\vee\mathcal{I}1 \qquad \frac{B}{A \vee B}\,\vee\mathcal{I}2$$

$$\frac{A \quad \neg B}{\neg(A \to B)}\,\neg\!\to\mathcal{I} \qquad \frac{\neg A}{A \to B}\,\to\mathcal{I}1 \qquad \frac{B}{A \to B}\,\to\mathcal{I}2$$

$$\frac{A}{\neg\neg A}\,\neg\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{A \vee B \quad \neg A}{B}\,\vee\mathcal{E}1 \qquad \frac{A \vee B \quad \neg B}{A}\,\vee\mathcal{E}2 \qquad \frac{\neg(A \vee B)}{\neg A}\,\neg\vee\mathcal{E}1 \qquad \frac{\neg(A \vee B)}{\neg B}\,\neg\vee\mathcal{E}2$$

$$\frac{\neg(A \wedge B) \quad A}{\neg B}\,\neg\wedge\mathcal{E}1 \qquad \frac{\neg(A \wedge B) \quad B}{\neg A}\,\neg\wedge\mathcal{E}2 \qquad \frac{A \wedge B}{A}\,\wedge\mathcal{E}1 \qquad \frac{A \wedge B}{B}\,\wedge\mathcal{E}2$$

$$\frac{A \to B \quad A}{B}\,\to\mathcal{E}1 \qquad \frac{A \to B \quad \neg B}{\neg A}\,\to\mathcal{E}2 \qquad \frac{\neg(A \to B)}{A}\,\neg\!\to\mathcal{E}1 \qquad \frac{\neg(A \to B)}{\neg B}\,\neg\!\to\mathcal{E}2$$

$$\frac{\neg\neg A}{A}\,\neg\neg\mathcal{E}$$

FALSUM RULES

$$\frac{A \quad \neg A}{\curlywedge}\ \text{RNC} \qquad \frac{\curlywedge}{A}\ \text{XFQ}$$

RULE OF BIVALENCE

$$\frac{\begin{array}{cc} [A] & [\neg A] \\ \vdots & \vdots \\ B & B \end{array}}{B}\ \text{RB}$$

Table 3: The C-intelim rules for unsigned formulae.

in a weaker form (see Section 6) and the rules RNC and RB no longer appear to be structural, since they do mention the negation operator. On the other hand, we can look at this version of the rules only as a practically convenient "translation" of the rules for signed formulae into an ordinary logical language, and refer to the original version for all philosophical purposes. In particular, we can still regard RNC and RB (as well as XFQ) as structural rules in that their original version for signed formulae does not mention any logical operator.[21] In the sequel we shall present all our results with reference to the rules for unsigned formulae and leave it to the more philosophically-minded readers to translate them into the original version for signed formulae.

In the unsigned version the two-premise elimination rules correspond to time-honoured principles of inference: *modus ponens*, *modus tollens*, *disjunctive syllogism* and its dual.[22] In the elimination rules, the formula containing the logical operator that is to be eliminated is called *major premise* and the other (if any) is called *minor premise*. The less natural rules, from the point of view of ordinary usage, namely the introduction rules for a true conditional and the elimination rules for a false conditional, are related to the Philonian meaning of this operator which is, however, typical of classical logic. The Philonian conditional, also called "material implication", is defined by $A \to B =_{\rm def} \neg A \vee B$ or, equivalently, by $A \to B =_{\rm def} \neg(A \wedge \neg B)$. This is the closest we can get to a "real" conditional operator in the Boolean framework. Clearly, such a conditional can be discarded from the language with no significant loss as far as the structure of arguments is concerned.

In the unsigned calculus, as in Gentzen-Prawitz natural deduction, a C-intelim proof of A *depending* on Γ is a tree of occurrences of formulae constructed in accordance with the C-intelim rules such that A occurs at the root and Γ is the set of all undischarged assumptions that occur at the leaves. The tree in Figure 1 shows a C-intelim deduction of G depending on

$$\{A \to \neg B, B \vee C, \neg(C \wedge \neg B), A \vee E, (E \vee F) \to \neg D, \neg G \to D\}.$$

Note that the last step is an occurrence of RB that discharges the temporary assumptions A (which occurs twice among the leaves of the left subtree) and $\neg A$ (which occurs once among the leaves of the right subtree). This example also shows that the standard format in which proofs are represented, while being very perspicuous, is somewhat inefficient, in that it may involve an unnecessary duplication of assumptions and of identical subproofs. In Section 9 we shall present a more "streamlined" format for C-intelim proofs that partly avoids the redundancy of the standard format and is more suitable for proof-search algorithms. However, the standard format is better suited to proving results on the transformation of proofs. Therefore, se shall stick to in order to provide a clearer presentation of the normalization and non-contamination theorems. The trees in Figure 2 show C-Intelim proofs of (2) and (4). Note that the proof on the right is much closer to the classical way of reasoning than the standard natural

[21] In its version for unsigned formulae RB is sometimes called "Classical Dilemma" and can be used, as in [Tennant, 1990], in addition to the standard intuitionistic rules, to obtain another variant of Gentzen's natural deduction for classical logic.

[22] Chrysippus (III century B.C.) listed these rules among the fundamental indemonstrable principles of reasoning (*anapodeiktoi*), except that he intended disjunction in its exclusive sense.

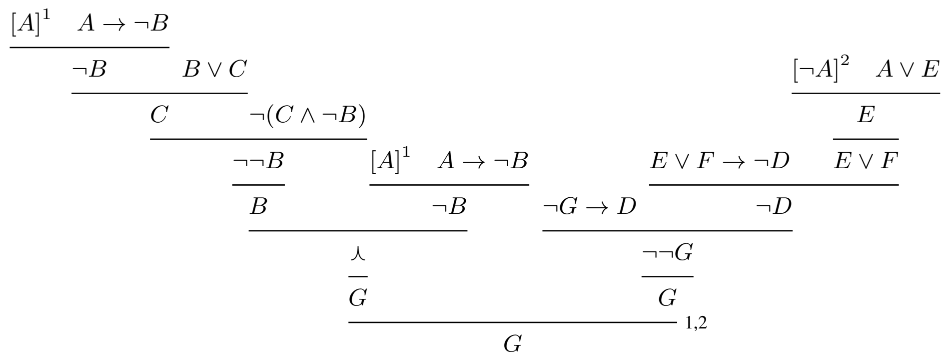

Figure 1: A C-intelim proof.

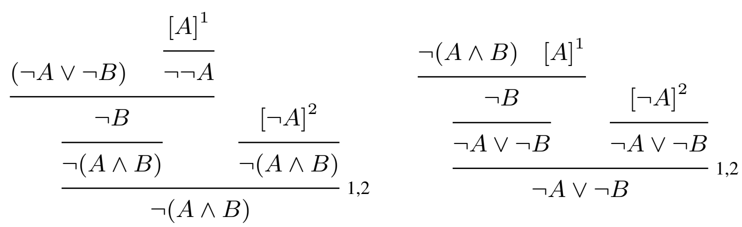

Figure 2: C-intelim proofs of (2) and (4).

deduction proof shown in (5) and, accordingly, exhibits the same structure as the one on the left.

Definition 3.1. *We say that A is* C-intelim deducible from Γ, *and write* $\Gamma \vdash^{\mathrm{IE}} A$, *if there is a C-intelim proof of A depending on $\Delta \subseteq \Gamma$.*

The completeness of C-intelim for classical propositional logic can be easily shown by simulating the rules of Prawitz's system or the truth-table method (this is left to the reader).[23]

Remark 3.1. *RB can be used, together with the elimination rules, to simulate any of the introduction rules. Conversely, RB can be used together with the introduction rules to simulate any of the elimination rules. To see this it is sufficient to look at the two examples in Figure 3. This clearly implies that RB + Introduction rules and RB + Elimination rules are both complete for classical logic and are essentially equivalent to the systems KI and KE (see Section 9 for references). However, using both introduction and elimination rules (i) allows for more natural and shorter proofs (although not* essentially *shorter because the simulation is clearly polynomial); (ii) it makes a substantial difference when the notion of depth-bounded proof is taken into account, in*

[23]To simulate a truth-table proof of A from Γ it is sufficient to express all possible assignments of truth-values to the atomic formulae occurring in $\Gamma \cup \{A\}$ by means of virtual assumptions introduced via RB, and then use the introduction rules to obtain either $T\,A$, or $F\,B$ for some $B \in \Gamma$. In most cases, it is not necessary to represent all possible assignments to *all* the atomic formulae occurring in $\Gamma \cup \{A\}$. (This is essentially Kalmar's strategy for proving completeness.)

$$\dfrac{[A\vee B]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee B)]^2}{\neg A}\qquad A}{\curlywedge}}{A\vee B}}{A\vee B}\;{}_{1,2} \qquad\qquad \dfrac{[B]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg B]^2 \quad \neg A}{\neg(A\vee B)}\qquad A\vee B}{\curlywedge}}{B}}{B}\;{}_{1,2}$$

Figure 3: Simulating introductions via RB + eliminations and simulating eliminations via RB + introductions.

that it reduces the number of applications of the RB rule (the only discharge rule of the system) that, as we shall see, is key to define the depth of a C-intelim deduction.

4 Quasi-normal proofs

In this section we introduce the notion of *quasi-normal* proof as a step towards the notion of normal proof that will be introduced in Section 6. However, quasi-normal proofs are of interest in their own right, in that they allow for the representation of *non-analytic proofs* (i.e., proofs that do not enjoy the subformula property) that, however, contain no "detours" and can, in some cases, be much shorter than any analytic proof.[24]

Definition 4.1. *An application of RB is* canonical *in a C-intelim tree $\mathcal{T}$ if there is no application of an inference (intelim or falsum) rule in $\mathcal{T}$ below its conclusion. A tree $\mathcal{T}$ is* RB-canonical *if all applications of RB in it are canonical.*

The notion of RB-canonical tree is motivated both by technical reasons concerning the proof of normalization and also by the fact that in RB-canonical proofs there is a clear separation between the components consisting only of applications of inference rules and the final applications of the dicharge rule RB. Any C-intelim proof can be turned into an RB-canonical one by applying the following transformations:[25]

$$\dfrac{\dfrac{\begin{matrix}[A]^1\\ \mathcal{T}_1\\ C\end{matrix} \qquad \begin{matrix}[\neg A]^2\\ \mathcal{T}_2\\ C\end{matrix}}{C}\;{}_{1,2}}{D} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}[A]^1\\ \mathcal{T}_1\\ C\end{matrix}}{D} \qquad \dfrac{\begin{matrix}[\neg A]^2\\ \mathcal{T}_2\\ C\end{matrix}}{D}}{D}\;{}_{1,2} \tag{T1}$$

[24]That analytic proofs may be significantly longer than non-analytic ones is well-known in the literature on the relative complexity of proof systems. Proof systems that generate only analytic proofs, such as the Tableau Method or Resolution, can be polynomially simulated by Frege systems (i.e., standard Hilbert-style axiomatic systems) but cannot polynomially simulate Frege systems. For an overview of these results see [Urquhart, 1995].

[25]In the sequel, the transformations must be intended as follows: replace locally a subtree of the form shown on the left of "$\rightsquigarrow$" with a subtree of the form shown on its right.

$$\dfrac{\dfrac{\begin{array}{cc}[A]^1 & [\neg A]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{array}}{C}\,{\scriptstyle 1,2} \qquad \begin{array}{c}\mathcal{T}_3\\ D\end{array}}{E} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{cc}[A]^1 & \\ \mathcal{T}_1 & \mathcal{T}_3\\ C & D\end{array}}{E} \qquad \dfrac{\begin{array}{cc}[\neg A]^2 & \\ \mathcal{T}_2 & \mathcal{T}_3\\ C & D\end{array}}{E}}{E}\,{\scriptstyle 1,2} \tag{T2}$$

The transformations (T1) and (T2) are the analogue, in our system, of Prawitz's "permutative reductions" [Prawitz, 1965, pp. 50–51], and can be applied, respectively, whenever C/D is an instance of a one-premise inference rule, and whenever $C, D/E$ is an instance of a two-premise inference rule. Their repeated application results in pushing downwards all the applications of RB so that, eventually, the conclusion of an application of RB is never used as a premise of an intelim or of a falsum rule and must be identical to the conclusion of the whole proof.[26]

Remark 4.1. *These transformations involve some duplication of occurrences of formulae or of identical subproofs, and therefore can increase in the size of the proof. While in the case of* (T1) *it can be shown that this increase is at most linear in the size of* $\mathcal{T}$*,* (T2) *may lead to exponentially longer proofs.*

Given a subtree $\mathcal{T}'$ of $\mathcal{T}$, let $\mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$ be equal to zero if $\mathcal{T}'$ ends with an application of RB, and equal to the total number of applications of RB in $\mathcal{T}'$ if $\mathcal{T}'$ ends with an application of an inference rule. We introduce the parameter $d_1(\mathcal{T}) = \langle m, n\rangle$ where m is the maximum value of $\mathrm{dnc}_{\mathrm{RB}}$ for a subtree of $\mathcal{T}$ and n is the number of of subtrees for which the value of $\mathrm{dnc}_{\mathrm{RB}}$ is maximum. Consider the lexicographic order on d_1 defined in the usual way: $\langle m, n\rangle < \langle m', n'\rangle$ if and only if $m < m'$ or $m = m'$ and $n < n'$.

The reader can verify that each application of the transformations (T1)–(T2) to a subtree of $\mathcal{T}$ such that its root is not followed by any other application of an inference rule (i.e., it is followed only by appplications of RB, if any) yields a tree for which the value of d_1 is strictly less than that of $\mathcal{T}$, until it eventually drops to $\langle 0, 0\rangle$, yielding an RB-canonical proof of the same conclusion depending exactly on the same set of assumptions.

Remark 4.2. *If the transformations are applied to arbitrary subtrees with no side condition, it is not the case that each transformation step decreases* d_1*. However, it can be shown that: applications of* (T1) *never increase* d_1*; (ii)* d_1 *may be increased by applications of* (T2)*, but reaches a maximum when* (T2) *cannot be further applied, and then starts decreasing until it reaches the value* $\langle 0, 0\rangle$.

Lemma 4.1. *Every C-intelim proof of* A *depending on* Γ *can be transformed into an RB-canonical C-intelim proof of* A *depending on* Γ*, by means of any sufficiently long sequence of applications of* (T1)–(T2).

Definition 4.2. *Let the* depth *of an RB-canonical C-intelim proof* $\mathcal{T}$*, denoted by* $\mathsf{depth}(\mathcal{T})$*, be defined as follows:*

[26] On this reduction to what we call RB-canonical form, see also [Tennant, 1990, p. 94], where it is used to prove that every classically refutable set of assumptions is also intuitionistically refutable (in the propositional case). Tennant uses RB (Classical Dilemma) in addition to the intuitionistic Gentzen rules to characterize classical propositional logic.

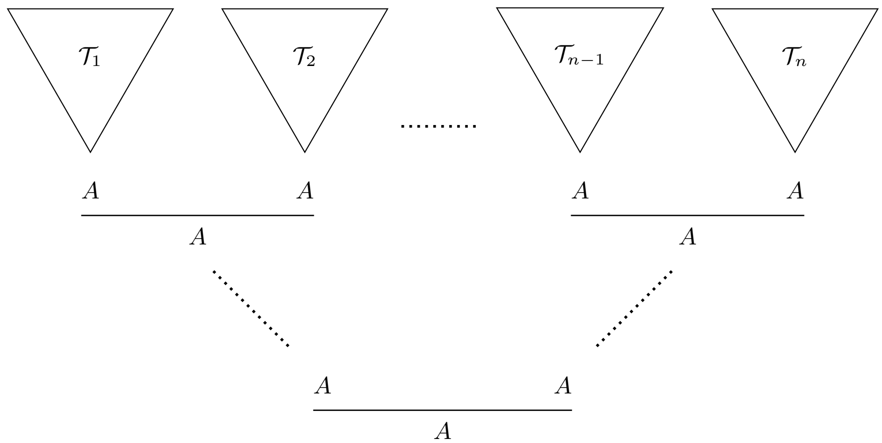

Figure 4: Structure of an RB-canonical C-intelim proof.

1. *If* $\mathcal{T}$ *contains no application of RB, then* $\mathrm{depth}(\mathcal{T}) = 0$*;*
2. *if* $\mathcal{T}$ *has the form*

$$\frac{\begin{matrix}[A] \\ \mathcal{T}_1 \\ B\end{matrix} \qquad \begin{matrix}[\neg A] \\ \mathcal{T}_2 \\ B\end{matrix}}{B}$$

then $\mathrm{depth}(\mathcal{T}) = \max(\mathrm{depth}(\mathcal{T}_1), \mathrm{depth}(\mathcal{T}_2)) + 1$.

The notion of depth for RB-canonical proofs emphasizes the role of RB as the only discharge rule that involves the use of "virtual information". Its significance as a proxy for the *intuitive* notion of depth of the reasoning process represented by the proof[27] depends on how the "virtual space", i.e., the set of formulae that may be used as RB-formulae, is bounded. We shall see in the sequel that the virtual space can be bounded in a variety of ways and, in particular, can be restricted to the subformulae of the assumptions and of the conclusion of the proof (as required for *normal* proofs).

Definition 4.3. *We call* 0-depth component *of an RB-canonical C-intelim proof* $\mathcal{T}$ *any maximal* 0*-depth subtree of* $\mathcal{T}$*, i.e., one that is not a proper subtree of any* 0*-depth subtree of* $\mathcal{T}$.

Note that in an RB-canonical proof $\mathcal{T}$ the conclusion of every 0-depth component is the conclusion of $\mathcal{T}$ itself. The general structure of an RB-canonical k-depth C-intelim proof of A depending on Γ is illustrated in Figure 4. The triangles labelled with $\mathcal{T}_1, \ldots, \mathcal{T}_n$ represent the 0-depth components of the proof. Each 0-depth component $\mathcal{T}_i$ is a proof of A depending on $\Gamma_i \cup \Delta_i$ such that: (i) $\Gamma_i \subseteq \Gamma$, (ii) Δ_i, with $|\Delta_i| \leq k$, is the set of virtual assumptions introduced in $\mathcal{T}_i$ that are subsequently discharged in

[27]By this we mean a notion that can be sensibly associated with the "difficulty" of proving the conclusion from the assumptions both from the computational and the cognitive viewpoint.

$\mathcal{T}$ via applications of RB, (iii) $\mathcal{T}_i$ contains only applications of inference rules (no application of RB). The nodes below the root of each 0-depth component $\mathcal{T}_i$ contain only occurrences of A that result from applications of the structural rule RB discharging the virtual assumptions in Δ_i.

Definition 4.4. *Given a C-intelim proof $\mathcal{T}$, we say that an application of RB in $\mathcal{T}$ is* redundant *if one of the following conditions hold:*

1. *at least one of its virtual assumptions is vacuously discharged,*
2. *its conclusion still depends on one of the discharged virtual assumptions.*

Example 4.1. *Consider a C-intelim proof $\mathcal{T}$ containing the following subproof:*

$$\mathcal{T}' = \dfrac{\dfrac{[A]^1 \quad A \to B}{B} \qquad \dfrac{C \quad C \to B}{B}}{B.}\,{\scriptstyle 1}$$

$\mathcal{T}'$ is a proof of B depending on $\Gamma = \{A \to B, C, C \to B\}$ ending with an application of RB in which the virtual assumption $\neg A$ is vacuously discharged in the righthand subproof. Therefore, the last step of this proof is a redundant application of RB. Such redundant application of RB can be eliminated by replacing $\mathcal{T}'$ with:

$$\dfrac{C \quad C \to B}{B.}$$

Example 4.2. *Consider a proof $\mathcal{T}$ containing the following subproof:*

$$\mathcal{T}' = \dfrac{\dfrac{\dfrac{[A]^1 \quad \neg A}{\curlywedge}}{B} \qquad \dfrac{[\neg A]^2 \quad \neg A \to B}{B}}{B.}\,{\scriptstyle 1,2}$$

Here, the application of RB is redundant because its conclusion still depends on the undischarged occurrence of the assumption $\neg A$ in the lefthand subproof. The latter, however, is discharged as a virtual assumption in the righthand subproof. The redundancy of this application is apparent if we consider that this subproof $\mathcal{T}'$ can be replaced by the following equivalent one (i.e., with the same assumptions and conclusions) that does not contain it:

$$\dfrac{\neg A \quad \neg A \to B}{B}$$

Any C-intelim proof can be turned into one that contains no redundant applications of RB by applying the following transformations[28] that remove the redundant applica-

[28] See Footnote 25.

tions of RB:

$$\dfrac{\begin{matrix}[A] \\ \mathcal{T}_1 \\ B\end{matrix} \qquad \begin{matrix}[\neg A] \\ \mathcal{T}_2 \\ B\end{matrix}}{B} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_i \\ B\end{matrix} \tag{T3}$$

where $i = 1$ if A is vacuously discharged in $\mathcal{T}_1$ or occurs as an undischarged assumption in $\mathcal{T}_2$; $i = 2$ if $\neg A$ is vacuously discharged in $\mathcal{T}_2$ or occurs as an undischarged assumption in $\mathcal{T}_1$.

Note that the result of removing all redundant applications of RB from a C-intelim proof of A depending on Γ will be a C-intelim proof of A depending on $\Delta \subseteq \Gamma$.

Let $d_2(\mathcal{T})$ denote the number of redundant applications of RB in $\mathcal{T}$. Observe that each application of the transformations (T3) yields a tree $\mathcal{T}'$ such that $d_2(\mathcal{T}') < d_2(\mathcal{T})$. Thus:

Lemma 4.2. *Every C-intelim proof of A depending on Γ can be turned into a C-intelim proof of A depending on $\Delta \subseteq \Gamma$ that contains no redundant applications of RB, by means of any sufficiently long sequence of applications of* (T3) *and with no increase in the size or depth of the proof.*

Remark 4.3. *Observe that every application of* (T3) *that decreases $d_2(\mathcal{T})$ does not introduce any new application of RB and so cannot increase $d_1(\mathcal{T})$.*

Definition 4.5. *A* detour *in a C-intelim proof $\mathcal{T}$ is an occurrence of a formula as conclusion of an introduction and, at the same time, as major premise of an elimination.*

The transformations (T4)–(T16) show how detours can be removed from a C-intelim proof.[29] Note that the final proof will have the same conclusion as the original one and will depend on a subset of the original assumptions. (To save space we use the

[29]The notion of "detour" refers to a redundant use of the intelim rules and the removal of such detours is key in the proof of the normalization theorem. In the context of Gentzen-style natural deduction it corresponds to that of "maximum formula" in [Prawitz, 1965].

variable i ranging over $\{1, 2\}$.)

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_i\end{matrix}}{A_{(3-i)}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_i\end{matrix}}{\curlywedge}}{A_{(3-i)}} \tag{T4}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_{(3-i)}\end{matrix}}{A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ A_i\end{matrix} \tag{T5}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_1\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_2\end{matrix}}{\neg(A_1 \vee A_2)}}{\neg A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_i\\ \neg A_i\end{matrix} \tag{T6}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix}}{\neg(A_1 \wedge A_2)} \qquad \begin{matrix}\mathcal{T}_2\\ A_i\end{matrix}}{\neg A_{(3-i)}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ A_i\end{matrix}}{\curlywedge}}{\neg A_{(3-i)}} \tag{T7}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix}}{\neg(A_1 \wedge A_2)} \qquad \begin{matrix}\mathcal{T}_2\\ A_{(3-i)}\end{matrix}}{\neg A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix} \tag{T8}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_1\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ A_2\end{matrix}}{A_1 \wedge A_2}}{A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_i\\ A_i\end{matrix} \tag{T9}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A\end{matrix}}{\neg\neg A}}{A} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ A\end{matrix} \tag{T10}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{A_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{\curlywedge}}{A_2} \tag{T11}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{A_2} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_1\\ A_2\end{array} \tag{T12}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg A_1} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array} \tag{T13}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg A_1} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\curlywedge}}{\neg A_1} \tag{T14}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{A_1} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \tag{T15}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\neg A_2} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array} \tag{T16}$$

Note that (i) the transformations (T4)–(T16) never increase the size of the proof, nor do they increase its depth; (ii) in some cases, their application may introduce new detours; for example in (T4) if A_i or $\neg A_i$ are conclusions of introductions in $\mathcal{T}_1$ or $\mathcal{T}_2$, they are not detours in the original subproof, but become such in the transformed one. However these new detours are always of lower complexity than the one that is removed by the transformation.

So, let $d_3(\mathcal{T})$ be the sum of the complexities (i.e., number of the logical operators) of all detours occurring in $\mathcal{T}$ and equal to 0 when there are no detours. Then, each application of the transformations (T4)–(T16) decreases the value of $d_3(T)$ until it drops to 0, so yielding a proof that is detour-free.

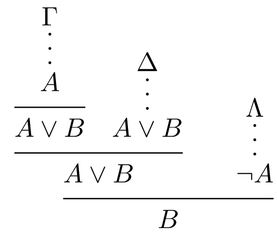

Figure 5: Higher level detours

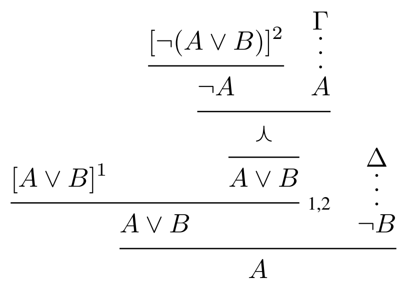

Figure 6: Indirect detours

Lemma 4.3. *Any C-intelim proof of A depending on Γ can be transformed into a C-intelim proof of A depending on $\Delta \subseteq \Gamma$ that contains no detours, by any sufficiently long sequence of applications of* (T4)–(T16) *and with no increase in the size or depth of the proof.*

Remark 4.4. *Note that the transformations* (T4)–(T16) *do not introduce any new application of RB, and therefore cannot increase either $d_1(\mathcal{T})$ or $d_2(\mathcal{T})$.*

Remark 4.5. *We could generalize the notion of "detour" by considering any sequence $A_1, \ldots, A_n$ of occurrences of the* same formula *such that: (i) A_1 is the conclusion of an introduction, (ii) A_n is the major premise of an elimination, (iii) for all i such that $1 < i \leq n$, A_i is an immediate successor of A_{i-1} resulting from an application of RB. Such a sequence is a* detour of level n. *An example of detour of level 2 is given in Figure 5. However, observe that such higher level detours* cannot occur in RB-canonical proofs, *for A_n would be at the same time the conclusion of an application of RB and a premise of an inference rule. Hence, elimination of higher level detours is a side-effect of transforming proofs into RB-canonical ones. Another kind of "indirect" detour is shown in Figure 6 and is related to the possibility of simulating introductions via eliminations and RB, as illustrated in Figure 3. Again, the reduction to RB-canonical proofs has the effect of eliminating this kind of indirect detours.*

Definition 4.6. *Given a C-intelim proof $\mathcal{T}$, an application of RNC is* canonical *in $\mathcal{T}$ if it is not the case that its premises are both conclusions of introductions. A C-intelim proof is* RNC-canonical *if it contains no non-canonical applications of RNC.*

Non-canonical applications of RNC can be removed by means of the the transformations (T17)–(T20) (for $i = 1, 2$).

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ A_i\end{array}}{A_1 \vee A_2} \qquad \dfrac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ \neg A_1 & \neg A_2\end{array}}{\neg(A_1 \vee A_2)}}{\curlywedge} \quad\rightsquigarrow\quad \frac{\begin{array}{cc}\mathcal{T}_0 & \mathcal{T}_i\\ A_i & \neg A_i\end{array}}{\curlywedge} \tag{T17}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ \neg A_i\end{array}}{\neg(A_1 \wedge A_2)} \qquad \dfrac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ A_1 & A_2\end{array}}{A_1 \wedge A_2}}{\curlywedge} \quad\rightsquigarrow\quad \frac{\begin{array}{cc}\mathcal{T}_i & \mathcal{T}_0\\ A_i & \neg A_i\end{array}}{\curlywedge} \tag{T18}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ \neg A_1\end{array}}{A_1 \to A_2} \qquad \dfrac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ A_1 & \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\curlywedge} \quad\rightsquigarrow\quad \frac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_0\\ A_1 & \neg A_1\end{array}}{\curlywedge} \tag{T19}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ A_2\end{array}}{A_1 \to A_2} \qquad \dfrac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ A_1 & \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\curlywedge} \quad\rightsquigarrow\quad \frac{\begin{array}{cc}\mathcal{T}_0 & \mathcal{T}_2\\ A_2 & \neg A_2\end{array}}{\curlywedge} \tag{T20}$$

By the *complexity of an application of RNC* with premises A and $\neg A$, we mean the logical complexity of A. Again, the removal of a non-canonical application of RNC may introduce a new non-canonical application, but the complexity of the latter is always lower. So, let $d_4(\mathcal{T})$ be sum of the complexities of the non-canonical applications of RNC in $\mathcal{T}$ and equal to 0 when all the applications of RNC are canonical. Each of the transformations (T17)–(T20) decreases $d_4(\mathcal{T})$ until its value drops to 0.

Lemma 4.4. *Any C-intelim proof of A depending on Γ can be transformed into an RNC-canonical C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of* (T17)–(T20) *and with no increase in the size or depth of the proof.*

Remark 4.6. *Observe that applications of* (T17)–(T20) *do not introduce new applications of RB, nor do they introduce new detours*[30]*, and so cannot increase any of the parameters $d_1(\mathcal{T})$–$d_3(\mathcal{T})$.*

Definition 4.7. *Given a C-intelim proof $\mathcal{T}$, an application of XFQ is* canonical *in $\mathcal{T}$ if (i) its conclusion is not $\curlywedge$, and (ii) its conclusion is not the premise of an application of an inference rule. A C-intelim proof is* XFQ-canonical *if it contains no non-canonical applications of XFQ.*

[30]Recall that in C-intelim RNC is a structural rule and not an elimination rule; the rules for unsigned formulae are intended as practical proxies for their signed versions. See the discussion in Section 3.

Remark 4.7. *Observe that if a proof of A depending on Γ is XFQ-canonical and RB-canonical: (i) it may contain only applications of XFQ with A itself as conclusion (ii) applications of XFQ may occur only as the last step in one of its 0-depth components.*

The notion of XFQ-canonical proof makes it apparent that, in RB-canonical proofs, the only use of XFQ consists, in fact, in showing that the assumptions are inconsistent, given that this rule can be applied only as the last step of a 0-depth component that immediately follows an application of RNC.

Any C-intelim proof can be turned into one that is XFQ-canonical by repeatedly applying the following transformations:

$$\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{\curlywedge} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array} \tag{T21}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{C}}{D} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{D} \tag{T22}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{C} \quad \begin{array}{c}\mathcal{T}_2\\ D\end{array}}{E} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{E} \tag{T23}$$

As before, the transformations (T22) and (T23) can be applied, respectively, for any instance C/D of a one-premise inference rule, any instance $C, D/E$ of a two-premise inference rule. They eventually yield a proof in which the conclusion of XFQ is never used as premise of an intelim rule or of RNC. This implies that all applications of XFQ are followed only by applications of RB. Note that the final proof will have the same conclusion as the original one and depend on a subset of the original assumptions.

Given a subtree $\mathcal{T}'$ of $\mathcal{T}$, let $\mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ be equal to 0 if $\mathcal{T}'$ does not end with an application of XFQ; otherwise let it be equal to the number of occurrences of formulae below the root of $\mathcal{T}'$ that result from applications of inference rules. Then, we define $d_5(\mathcal{T})$ as follows:

$$d_5(\mathcal{T}) = \sum_{\mathcal{T}' \in \mathrm{Sub}(\mathcal{T})} \mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}') \tag{6}$$

Observe that each application of the transformations (T21)–(T23) yields a tree $\mathcal{T}'$ such that $d_5(\mathcal{T}') < d_5(\mathcal{T})$. Thus:

Lemma 4.5. *Any C-intelim proof of A depending on Γ can be transformed into an XFQ-canonical C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by any sufficiently long sequence of applications of* (T21)–(T23) *and with no increase in the size or depth of the proof.*

DEFINITIONS OF d_1–d_5

- $d_1(\mathcal{T}) = \langle m, n\rangle$, where
  - m is the maximum value of $\mathrm{dnc}_{\mathrm{RB}}$ for a subtree of $\mathcal{T}$ and n is the number of of subtrees for which the value of $\mathrm{dnc}_{\mathrm{RB}}$ is maximum;
  - $\mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$, for any subtree $\mathcal{T}'$ of $\mathcal{T}$, is equal to zero if $\mathcal{T}'$ ends with an application of RB, and equal to the total number of applications of RB in $\mathcal{T}'$ if $\mathcal{T}'$ ends with an application of an inference rule;
  - $\langle m, n\rangle < \langle m', n'\rangle$ iff $m < m'$ or $m = m'$ and $n < n'$;
- $d_2(\mathcal{T})$ = number of redundant applications of RB in $\mathcal{T}$;
- $d_3(\mathcal{T})$ is equal to 0 if there are no detours; otherwise to the sum of the complexities (i.e., number of the logical operators) of all detours occurring in $\mathcal{T}$;
- $d_4(\mathcal{T})$ is equal to 0 if all the applications of RNC in $\mathcal{T}$ are canonical, otherwise to the sum of the complexities of the non-canonical applications of RNC in $\mathcal{T}$;
- $d_5(\mathcal{T}) = \sum_{\mathcal{T}'\in\mathrm{Sub}(\mathcal{T})} \mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$, where
  - $\mathrm{Sub}(\mathcal{T})$ is the set of all subtrees of $\mathcal{T}$;
  - $\mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ is equal to 0 if $\mathcal{T}'$ does not end with an application of XFQ; otherwise to the number of occurrences of formulae below the root of $\mathcal{T}'$ that result from applications of inference rules.

Table 4: Transformation parameters.

Remark 4.8. *Observe that any application of* (T21)–(T23) *does not introduce new applications of RB; nor does it introduce new detours or new non-canonical applications of RNC, and so cannot increase any of the parameters* $d_1(\mathcal{T})$–$d_4(\mathcal{T})$.

Definition 4.8. *We say that a C-intelim proof* $\mathcal{T}$ *is* quasi-normal *if all of the following conditions are satisfied:*

- $\mathcal{T}$ *is RB-canonical, RNC-canonical and XFQ-canonical,*
- $\mathcal{T}$ *contains no redundant applications of RB,*
- $\mathcal{T}$ *contains no detours.*

The definitions of $d_1(\mathcal{T})$–$d_5(\mathcal{T})$ given above are summarized in Table 4. Now, let

$$d(T) = \langle d_1(\mathcal{T}), d_2(T), d_3(\mathcal{T}), d_4(\mathcal{T}), d_5(\mathcal{T})\rangle$$

and consider the usual lexicographic order on $d(\mathcal{T})$ for all C-intelim proofs $\mathcal{T}$. The reader can verify that a transformation that decreases $d_i(\mathcal{T})$ for some $i < 5$, may increase $d_j(\mathcal{T})$ for some $j > i$. However, as observed in Remarks 4.3–4.8, transformations that decrease $d_i(\mathcal{T})$ for $i > 1$, never increase d_j for any $j < i$. So, each of the transformations (T1)–(T23) decreases $d(\mathcal{T})$. Hence the repeated application of these transformations, independently of their order, eventually yields a proof $\mathcal{T}'$ such that $d(\mathcal{T}') = \langle\langle 0, 0\rangle, 0, 0, 0, 0\rangle$, which is therefore quasi-normal.

Theorem 4.1. *Any C-intelim proof of A depending on Γ can be turned into a quasi-normal C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of the transformations* (T1)–(T23).

Moreover, observe that the transformations (T3)–(T23) that decrease d_2–d_5 never increase the size of the proof and do not introduce any new application of RB. Hence,

Theorem 4.2. *Any RB-canonical C-intelim proof of A depending on Γ can be turned into a quasi-normal C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of the transformations* (T3)–(T23) *and with no increase in the size or depth of the proof.*

Remark 4.9. *If $\mathcal{T}$ is a quasi-normal C-intelim proof,*

- *every subproof of $\mathcal{T}$ is also quasi-normal;*
- *every 0-depth component of $\mathcal{T}$ contains at most one application of RNC and at most one application of XFQ (as the last step).*

5 Intermediate conclusions

We have introduced the C-intelim system and argued that it is better suited than Gentzen-style natural deduction to represent naturally the inferences of classical logic. The main technical contribution so far is a set of transformations that can turn, independently of the order in which they are applied, any C-intelim proof into a *quasi-normal* one. What have we achieved through the notion of quasi-normal proof? Not only is this notion a convenient step towards the normalization result that will be presented in the sequel, but it is also interesting in its own right. Quasi-normal proofs avoid trivially redundant applications of RB and trivially redundant applications of inference rules (detours). Moreover, the applications of RB are pushed down at the end of the proof-tree, so that their conclusion is always the conclusion of the whole proof and their role consists in gradually discharging the virtual assumptions made in the properly inferential components that we called "0-depth components" (see Figure 4). In this way we make a clear separation between the applications of inference rules and the application of RB that allows us to define the depth of an argument in a straightforward way, although this notion will show its full significance only in Part II, when the notion of normal proof will be introduced. Furthermore, in each 0-depth component, RNC is applied at most once, and also XFQ is applied at most once as the last step, allowing us to "infer", if only in a pickwickian sense, the conclusion of the whole proof as a result of the inconsistency of the assumptions.

In the Gentzen tradition, it is part of the logical folklore to identify *analytic* proofs — i.e., proofs that enjoy the (weak) subformula property — with normal proofs and the latter with proofs that contain no "detours" – i.e., no obviously redundant sequences of applications of inference rules. However, it is well-known that, for some infinite sets of classically valid inferences, analytic proofs can be exponentially longer than non-analytic ones, e.g., proofs in Frege systems or in the sequent calculus with cut. Here we clearly distinguish between analyticity on the one hand and absence of detours on the other. Quasi-normal proofs are proofs that, albeit being *non-analytic*, can be legitimately regarded as containing no trivial "detour". It is easy to see that such proofs can polynomially simulate Frege system (see Section 10), for which the existence of a polynomial upper bound on proof length has not been disproved yet. Hence, as far as the length of proofs is concerned, the restriction of C-intelim to quasi-normal proofs is among the most powerful proof systems for classical propositional logic. As we shall see, the set of formulae that can be used as RB-formulae, that we have called the *virtual space* in the comment to Definition 4.2, may be bounded in a variety of ways without loss of completeness. The strictest way of bounding it when generating a proof of A from Γ consists in allowing as RB-formulae only atomic formulae that occur in $\Gamma \cup \{A\}$. A more liberal restriction consists in allowing only subformulae of the formulae in $\Gamma \cup \{A\}$. Shorter proofs can be obtained by further liberalizing the composition of the virtual space allowing for proofs that do not enjoy the subformula property, but in which the virtual space is still bounded. In quasi-normal proofs RB is, at the same time, the only rule that may bring about violations of the subformula property and the only rule that increases the depth of the reasoning process. In C-intelim the transition from analytic proofs to non-analytic (possibly shorter) ones depends only on the way in which we bound the virtual space, i.e. on the set of formulae that are allowed as RB-formulae. Once the virtual space is suitably bounded, the transition from easier proofs to more difficult ones depends only on the depth at which applications of RB are required.

6 Normal proofs

As mentioned in the introduction, for Gentzen the importance of the normalization theorem consisted mainly in the fact that normal proofs enjoy the *subformula property* to the effect that in the search for proofs we can restrict our attention to inference steps whose conclusion is a "component" either of the assumptions or of the conclusion. This involves a drastic reduction of the search space that is crucial for the purpose of automated deduction. In the case of propositional logic, this search space is finite for each putative inference, paving the way for decision procedures. We know now that "analytic proofs", i.e., those that enjoy the subformula property are not always shorter than non-analytic ones, and may be exponentially longer for certain infinite classes of inferences. For this reason we attached a special importance to quasi-normal proofs that do not enforce the subformula property while still avoiding trivial detours. We now turn to the notion of *normal* proof, which is simply a quasi-normal proof in which RB is applied only to (weak) subformulae either of the premises or of the conclusion. Given this restriction, the resulting proof will be shown to enjoy the subformula property.

For every formula A, a *subformula* of A is defined inductively as follows: (i) A is a subformula of A, (ii) for every binary operator $\circ$, if $B \circ C$ is a subformula of A, then so are B and C, (iii) if $\neg B$ is a subformula of A, then so is B; (iv) nothing else is a subformula of A. A is a *proper subformula* of B if A is a subformula of B, but B is not a subformula of A. A is an *immediate proper subformula* of B, if A is a proper subformula of B and it is not a proper subformula of any proper subformula of B. We shall often say just *immediate subformula* as an abbreviation for "immediate proper subformula".

We say that B is a *weak subformula* of A if either B is a subformula of A or $B = \neg C$ for some subformula C of A.

Remark 6.1. *Notice that the relation "A is a weak subformula of B" is not, in general, transitive. A simple counterexample is given by the triple $\neg\neg A, \neg A, A$.*

Finally, we say that A is a *proper weak subformula* of B if A is a weak subformula of B, but B is not a weak subformula of A. For example, A is a weak subformula of $\neg A$, but not a proper weak subformula; $\neg A$ is a weak subformula of A and of $\neg\neg A$, but not a proper weak subformula of either. On the other hand, both A and $\neg A$ are proper weak subformulae of $A \wedge B$, of $\neg A \wedge B$, of $\neg(A \wedge B)$, etc.

Remarks 6.2. *Observe that:*

1. *The relation "A is a proper weak subformula of B" is transitive: if A is a proper weak subformula of B and B is a proper weak subformula of C, then A is a proper weak subformula of C;*
2. *the minor premise of an elimination is always a proper weak subformula of its major premise;*
3. *the conclusion of an elimination is always a proper weak subformula of its major premise;*
4. *a premise of an introduction is always a proper weak subformula of its conclusion;*

Recall that C is an *RB-formula of* a C-intelim tree $\mathcal{T}$ if C and $\neg C$ are the virtual assumptions discharged by some application of RB in $\mathcal{T}$.

Definition 6.1. *Given a C-intelim proof $\mathcal{T}$ of A depending on Γ, we say that an application of RB in $\mathcal{T}$ is* analytic *if its RB-formula is a subformula of some formula in $\Gamma \cup \{A\}$ (or, equivalently, both the virtual assumptions discharged by this application are weak subformulae of some formula in $\Gamma \cup \{A\}$).*

Definition 6.2. *We say that an application of RB is* atomic *if its RB-formula is atomic, i.e., the virtual assumptions discharged by it have the form p and $\neg p$ for some atomic p.*

Definition 6.3. *A C-intelim proof $\mathcal{T}$ is* normal *if it is quasi-normal and every application of RB in $\mathcal{T}$ is analytic. It is* atomically normal *if it is normal and every application of RB in $\mathcal{T}$ is atomic.*

Note that a quasi-normal 0-depth C-intelim proof, i.e., one that contains no applications of RB, is by definition (atomically) normal.

Remark 6.3. *If $\mathcal{T}$ is a(n atomically) normal C-intelim proof, every subproof of $\mathcal{T}$ is also (atomically) normal.*

Definition 6.4. *We say that a C-intelim proof $\mathcal{T}$ of A depending on Γ has the* weak subformula property *(WSFP) if every formula occurring in $\mathcal{T}$ is either an occurrence of $\curlywedge$ or a weak subformula of some formula in $\Gamma \cup \{A\}$.*

Theorem 6.1. *If $\mathcal{T}$ is a normal 0-depth proof of A depending on Γ, and B is a formula occurring in $\mathcal{T}$, then either $B \in \Gamma \cup \{A\}$, or $B = \curlywedge$, or B is a proper weak subformula of some formula in $\Gamma \cup \{A\}$.*

Proof. Suppose $\mathcal{T}$ is a normal 0-depth proof of A depending on Γ. Let Δ be the set of all formulae C occurring in $\mathcal{T}$ such that (i) $C \notin \Gamma \cup \{A\}$, (ii) $C \neq \curlywedge$, and (iii) C is not a proper weak subformula of any formula in $\Gamma \cup \{A\}$. Let us assume that $\Delta \neq \emptyset$ and take a formula D in Δ of maximum complexity. Since $D \notin \Gamma$, D occurs in $\mathcal{T}$ as conclusion of an application of an intelim rule or of a falsum rule. It cannot be the conclusion of an application of RNC, otherwise $D = \curlywedge$ and $\curlywedge \notin \Delta$ by definition of Δ. Moreover, since $\mathcal{T}$ is normal, it is XFQ-canonical. Thus, D cannot be the conclusion of an application of XFQ, otherwise it should be equal to the conclusion A of $\mathcal{T}$, which does not belong to Δ by definition of Δ.

Furthermore, D cannot be the conclusion of an elimination. To see this, observe that the major premise of this elimination cannot be in $\Gamma \cup \{A\}$, otherwise, by Remark 6.2.3, D would be a proper weak subformula of a formula in $\Gamma \cup \{A\}$ and therefore would not belong to Δ. Moreover, this major premise cannot be a proper weak subformula of a formula in $\Gamma \cup \{A\}$, because in this case, by Remarks 6.2.1 and 6.2.3, D would also be a proper weak subformula of some formula in $\Gamma \cup \{A\}$ and therefore would not belong to Δ. Hence the major premise of the elimination should be a formula in Δ of greater complexity than D, against the assumption that D is a formula of maximum complexity in Δ.

Hence, D can only be the conclusion of an introduction. Since $D \neq A$, D must be used in $\mathcal{T}$ as premise of some intelim rule or of some of the falsum rules. Since $D \neq \curlywedge$, it cannot be used as premise of XFQ. Moreover, it cannot be used as major premise of an elimination rule, otherwise D would be a detour, against the assumption that $\mathcal{T}$ is normal (and so contains no detours). Furthermore, it cannot be used as premise of RNC, because in this case the complementary premise, call it D', could also be only the conclusion of an introduction, for the same reasons as D; but this is impossible because $\mathcal{T}$ is normal and therefore RNC-canonical (it is not the case that both the premises of an application of RNC are both conclusions of an introduction). Finally, it cannot be used as minor premise of an elimination, otherwise, by Remark 6.2.2, D would be a proper weak subformula of the major premise. So, either this major premise belongs to $\Gamma \cup \{A\}$ and then, by Remark 6.2.1, D would be a proper weak subformula of some formula in $\Gamma \cup \{A\}$, in which case D would not belong to Δ; or the major premise of this elimination would be a formula in Δ of greater complexity than D, against the assumption that D is a formula in Δ of maximum complexity .

Thus, D, must be used as premise of an introduction. But this is impossible, because, by Remark 6.2.4, D would be a proper weak subformula of the conclusion of this introduction. So, either this conclusion belongs to $\Gamma \cup \{A\}$ and, by Remark 6.2.1, D would be a proper weak subformula of some formula in $\Gamma \cup \{A\}$, in which case D would not belong to Δ, or the conclusion of this introduction would be a formula in Δ of greater complexity than D, against the assumption that D is a formula in Δ of maximum complexity . Hence, Δ must be empty. □

The above theorem immediately implies the following:

Corollary 6.1 (WSFP of 0-depth proofs)**.** *Every normal 0-depth C-intelim proof has the WSFP.*

Remark 6.4. *Note that if $\mathcal{T}$ is a quasi-normal proof of A depending on Γ, whose 0-depth components are $\mathcal{T}_1, \ldots, \mathcal{T}_n$, every 0-depth component $\mathcal{T}_i$ is a normal proof of A depending on $\Gamma_i \cup \Delta_i$, where $\Gamma_i \subseteq \Gamma$ and Δ_i are virtual assumptions that are subsequently discharged in $\mathcal{T}$ by applications of RB.*

By virtue of the exclusion of detours, the structure of each 0-depth component of a normal C-intelim proof is determined quite sharply.

Definition 6.5. *Given a 0-depth proof $\mathcal{T}$ of A depending on Γ, an* intelim walk *of $\mathcal{T}$ is a sequence $A_1, \ldots, A_n$ of occurrences of formulae such that (i) $A_1 \in \Gamma$, (ii) for $1 \leq i < n$, A_i is a premise of an application of an intelim rule with A_{i+1} as conclusion; (iii) A_n is either the conclusion A of $\mathcal{T}$, or the minor premise of an elimination or a premise of an application of RNC.*

Definition 6.6. *If $\mathcal{T}$ is a 0-depth proof of A depending on Γ, a* normal intelim walk *of $\mathcal{T}$ is an intelim walk of the form*

$$A_1, \ldots, A_m, \ldots, A_{m+n},$$

with $m \geq 1$ and $n \geq 0$, where

1. *if $m > 1$, for $i < m$, A_i is the major premise of an elimination and the subsequence $A_1, \ldots, A_m$ is called the* E-part *of the intelim walk; if $m = 1$, we say that the E-part is empty;*

2. *if $n > 0$, for $m < i \leq m + n$, A_i is the conclusion of an introduction and the subsequence $A_m, \ldots A_{m+n}$ is called the* I-part *of the intelim walk; if $n = 0$, we say that the I-part is empty;*

3. *A_m is called the* minimum formula *of the normal intelim walk.*

A normal intelim walk is complete *if its last formula is either a premise of RNC or the conclusion A of the proof.*

Examples 6.1. *Both the immediate subproofs of the C-intelim proof. In Figure 1 of Part I are 0-depth proofs. The normal intelim walks of the lefthand subproof are the following sequences: A (two copies, both the E-part and the I-part are empty); $A \to$*

$\neg B, \neg B$ (I-part empty); $B \vee C, C$ (I-part empty); $\neg(C \wedge \neg B), \neg\neg B, B$ (I-part empty, complete); $A \to \neg B, \neg B$ (I-part empty, complete). In the right-hand subproof the normal intelim walks are: $\neg G \to D, \neg\neg G, G$ (I-part empty, complete); $E \vee F \to \neg D, \neg D$ (I-part empty); $\neg A$ (both E-part and I-part empty); $A \vee E, E, E \vee F$. In each of the C-intelim proofs of Figure 1, both immediate subproofs are 0-depth; the left branch of the first subproof is a complete normal intelim walk and its minimum formula is $\neg B$; the only branch of the second subproof is a normal intelim walk whose E-part is empty and its minimum formula is $\neg A$.

The proof of the following lemma is left to the reader.[31]

Lemma 6.1. *If $\mathcal{T}$ is a 0-depth normal proof, then*

- *every branch of $\mathcal{T}$ contains a normal intelim walk;*
- *at least one branch of $\mathcal{T}$ contains a complete normal intelim walk;*
- *the minimum formula of a normal intelim walk is a weak subformula of all the formulae in it; if the E-part is non-empty, the minimum formula is a proper weak subformula of all the formulae preceding it in the walk; if the I-part is non-empty the minimum formula is a proper weak subformula of all the formulae following it in the walk.*

Theorem 6.2 (WSFP of normal proofs)**.** *If $\mathcal{T}$ is a normal C-intelim proof, then $\mathcal{T}$ has the WSFP.*

Proof. Let $\mathcal{T}_1 \ldots, \mathcal{T}_n$ be the 0-depth components of $\mathcal{T}$. By Remark 6.4, every 0-depth component of $\mathcal{T}$ is normal. Recall that in a normal proof, every formula occurring in $\mathcal{T}$ occurs also in some of its 0-depth components, since all the conclusions of applications of RB are equal to the conclusion of all 0-depth components. Thus, for every formula B occurring in $\mathcal{T}$, there is a 0-depth component $\mathcal{T}_i$ of $\mathcal{T}$ such that, by Theorem 6.1, either B is in $\Gamma_i \cup \Delta_i \cup \{A\}$ or $B = \curlywedge$, or B is a proper weak subformula of some formula in $\Gamma_i \cup \Delta_i \cup \{A\}$, where Γ_i are the assumptions of $\mathcal{T}_i$ that are left undischarged in $\mathcal{T}$ and Δ_i are the virtual assumptions subsequently discharged in $\mathcal{T}$. If $\mathcal{T}$ is normal, every formula in Δ_i is a weak subformula of a formula in $\Gamma_i \cup \{A\}$. So, either (i) $B = \curlywedge$, or (ii) $B \in \Gamma_i \cup \Delta_i \cup \{A\}$ and so B is a weak subformula of a formula in $\Gamma_i \cup \{A\}$, or (iii) B is a *proper* weak subformula of some formula in $\Gamma_i \cup \Delta_i \cup \{A\}$. Since all the formulae in Δ_i are weak subformulae of some formula in $\Gamma_i \cup \{A\}$, it is not difficult to verify, that if B is a proper weak subformula of C and C is a weak subformula of D, then B is a weak subformula of D. Hence, B must be a weak subformula of some formula in $\Gamma_i \cup \{A\}$. □

Lemma 6.2. *If $\mathcal{T}$ is a quasi-normal C-intelim proof of A and all the non-atomic applications of RB in $\mathcal{T}$ are analytic, then all the atomic applications of RB in $\mathcal{T}$ are also analytic, i.e. $\mathcal{T}$ is normal.*

[31] Note that what we call "branch" in this paper is called "thread" in [Prawitz, 1965].

Proof. Let $\mathcal{T}$ be a quasi-normal C-intelim proof of A depending on Γ such that all the non-atomic applications of RB in $\mathcal{T}$ are analytic. We now show that $\mathcal{T}$ cannot contain any non-analytic atomic applications of RB and, therefore, $\mathcal{T}$ is normal. For this purpose we prove, by induction on k, that every k-depth subproof of $\mathcal{T}$ is normal.

By Remark 6.4, every 0-depth subproof of $\mathcal{T}$ is normal. For $k > 0$, assume that every subproof of $\mathcal{T}$ of depth $k{-}1$ is normal. We show that under this assumption every k-depth subproof is also normal. Since $k > 0$, any k-depth subproof either ends with a non-atomic application of RB (which is by hypothesis analytic, so that the proof is normal) or has the following form, for some atomic p:

$$\dfrac{\begin{matrix}[p]^1 \\ \mathcal{T}_1 \\ A\end{matrix} \qquad \begin{matrix}[\neg p]^2 \\ \mathcal{T}_2 \\ A\end{matrix}}{A}\,{\scriptstyle 1,2} \tag{7}$$

Suppose, ex absurdo, that this atomic application of RB is non-analytic, i.e. that p does not occur in $\Gamma \cup \{A\}$. By inductive hypothesis, we know that for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$:

- $\mathcal{T}_1$ is a normal proof of A from $\Gamma_1 \cup \Lambda_1 \cup \{p\}$,
- $\mathcal{T}_2$ is a normal proof of A from $\Gamma_2 \cup \Lambda_2 \cup \{\neg p\}$,

where Λ_1 and Λ_2 are the sets of virtual assumptions that are still undischarged in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

Since $\mathcal{T}$ is quasi normal, it contains no redundant applications of RB, and so neither p nor $\neg p$ are vacuously discharged in (7). Thus, p must be used as premise of some application of an inference rule in $\mathcal{T}_1$ and $\neg p$ as premise of some application of an inference rule in $\mathcal{T}_2$. By its logical form, p cannot be used in $\mathcal{T}_1$ as major premise of an elimination. If it is used as minor premise of an elimination, p would occur in the major premise and the latter would not be a weak subformula of some formula in $\Gamma_1 \cup \Lambda_1 \cup \{p\} \cup \{A\}$, for every formula in Λ_1 is either an atomic formula, or the negation of an atomic formula, or is a weak subformula of some formula in $\Gamma_1 \cup \{A_1\}$. But this is impossible, since, by inductive hypothesis, $\mathcal{T}_1$ is normal and, by Theorem 6.2, it has the WSFP.

Moreover, p cannot be used as premise of an introduction, for the conclusion of this introduction would not be a weak subformula of any of the formulae in $\Gamma_1 \cup \Lambda_1 \cup \{p\} \cup \{A\}$, which would again contradict the hypothesis that $\mathcal{T}_1$ is normal and has the WSFP. So, p must be used in $\mathcal{T}_1$ as premise of some application of the falsum rules. It is impossible that p is used as premise of XFQ, for we have stipulated that $\curlywedge$ cannot occur in the assumptions. Hence, p can be used only as premise of an application of RNC in $\mathcal{T}_1$. But, in this case, the other premise $\neg p$ cannot result from the application of an elimination, otherwise $\mathcal{T}_1$ would not have the WSFP, against the hypothesis that it is normal. Nor can it result from an application of XFQ, because normal proofs are XFQ-canonical. So, $\neg p$ should belong to Λ_1 and the application of RB in (7) would be redundant (see Definition 4.4 and Example 4.1) against the hypothesis that $\mathcal{T}$ is quasi normal, which implies that it contains no redundant applications of RB. Thus, it is impossible that p is an atomic formula that does not occur in $\Gamma \cup \{A\}$. Therefore, all applications of RB in $\mathcal{T}$ are analytic and $\mathcal{T}$ is normal. □

Any C-intelim proof can be turned into an (atomically) normal one. by repeatedly applying the following transformations:[32]

$$\dfrac{\begin{matrix}[A\lor B]^1 & [\neg(A\lor B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{\scriptstyle 1,2} \quad\leadsto\quad \dfrac{\begin{matrix}\dfrac{[A]^1}{A\lor B}\\ \mathcal{T}_1\\ C\end{matrix} \quad \dfrac{\begin{matrix}\dfrac{[B]^3}{A\lor B} & \dfrac{[\neg A]^2 \quad [\neg B]^4}{\neg(A\lor B)}\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{\scriptstyle 3,4}}{C}\,{\scriptstyle 1,2} \tag{T24}$$

$$\dfrac{\begin{matrix}[A\land B]^1 & [\neg(A\land B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{\scriptstyle 1,2} \quad\leadsto\quad \dfrac{\dfrac{\begin{matrix}\dfrac{[A]^1 \quad [B]^3}{A\land B} & \dfrac{[\neg B]^4}{\neg(A\land B)}\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{\scriptstyle 3,4} \quad \begin{matrix}\dfrac{[\neg A]^2}{\neg(A\land B)}\\ \mathcal{T}_2\\ C\end{matrix}}{C}\,{\scriptstyle 1,2} \tag{T25}$$

$$\dfrac{\begin{matrix}[A\to B]^1 & [\neg(A\to B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{\scriptstyle 1,2} \quad\leadsto\quad \dfrac{\begin{matrix}\dfrac{[B]^1}{A\to B}\\ \mathcal{T}_1\\ C\end{matrix} \quad \dfrac{\begin{matrix}\dfrac{[A]^3 \quad [\neg B]^2}{\neg(A\to B)} & \dfrac{[\neg A]^4}{A\to B}\\ \mathcal{T}_2 & \mathcal{T}_1\\ C & C\end{matrix}}{C}\,{\scriptstyle 3,4}}{C}\,{\scriptstyle 1,2} \tag{T26}$$

$$\dfrac{\begin{matrix}[\neg A]^1 & [\neg\neg A]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ B & B\end{matrix}}{B}\,{\scriptstyle 1,2} \quad\leadsto\quad \dfrac{\begin{matrix}\dfrac{[A]^1}{\neg\neg A} & [\neg A]^2\\ \mathcal{T}_2 & \mathcal{T}_1\\ B & B\end{matrix}}{B}\,{\scriptstyle 1,2} \tag{T27}$$

For every C-intelim tree $\mathcal{T}$, let $g(\mathcal{T})$ be defined as follows:

$$g(\mathcal{T}) = \begin{cases}\sharp(A) & \text{if } \mathcal{T} \text{ ends with a non-analytic application of RB and } A \text{ is the}\\ & \text{RB-formula of this application}\\ 0 & \text{otherwise,}\end{cases} \tag{8}$$

where $\sharp(A)$ denotes the logical complexity of (the total number of occurrences of logical operators in) A.

Note that, in general, these transformations increase the size of the proof. Moreover, they may introduce new detours; for example in (T24) it may be the case that $A \lor B$ or $\neg(A \lor B)$ or both are used in $\mathcal{T}_1$ or $\mathcal{T}_2$ as major premises of eliminations. They may also introduce new non-canonical applications of RNC.

[32] The reader can compare them to the ones used in [Tennant, 1990], pp. 95–96.

Let

$$d_0(\mathcal{T}) = \langle m, n \rangle \tag{9}$$

where m is the maximum value taken by g for a subtree of $\mathcal{T}$, and n is the number of subtrees for which the value of g is maximum. Consider again the usual lexicographc order on d_0. If the transformations above are appplied to a subtree $\mathcal{T}'$ that ends with a non-analytic application of RB for which the value of g is maximum, each transformation step decreases $d_0(\mathcal{T})$ until its value drops to $\langle 0, 0 \rangle$, which means that all the applications or RB are either analytic (the RB formula is a subformula of the premises or of the conclusion) or non-analytic and atomic.

Remark 6.5. *If the transformations* (T24)–(T27) *are applied to arbitrary subtrees, with no side condtion, the index d_0 may not always decrease. However, it never increases, and it can be shown that eventually it reaches the mimimum value $\langle 0, 0 \rangle$ in a finite number of steps.*

Lemma 6.3. *Any C-intelim proof of A depending on Γ can be transformed into a C-intelim proof of A depending on $\Delta \subseteq \Gamma$ where all the applications of RB are either analytic or atomic by means of any sufficiently long sequence of applications of* (T24)–(T27)

Now, recalling the definitions of d_1–d_5 given in Table 4, let

$$d(\mathcal{T}) = \langle d_0(\mathcal{T}), d_1(\mathcal{T}), d_2(T), d_3(\mathcal{T}), d_4(\mathcal{T}), d_5(\mathcal{T}) \rangle$$

and consider the usual lexicographic order on $d(\mathcal{T})$ for all C-intelim proofs $\mathcal{T}$. By inspection of the transformations (T1)–(T27), the reader can verify that each transformation that decreases $d_i(\mathcal{T})$ for any $i < 4$, may increase $d_j(\mathcal{T})$ for some $j > i$. However, no transformations that decreases $d_i(\mathcal{T})$ for $i > 0$, can ever increase d_j for any $j < i$. So, each of the transformations (T1)–(T27) decreases $d(\mathcal{T})$. Hence, the repeated application of these transformations, independently of their order, eventually yields a proof $\mathcal{T}'$ such that $d(\mathcal{T}') = \langle \langle 0, 0 \rangle, \langle 0, 0 \rangle, 0, 0, 0, 0 \rangle$. Note that such a proof is quasi-normal (because the value of d_1 is equal to $\langle 0, 0 \rangle$ and the values of d_2–d_4 are all equal to 0. Moreover, all the applications of RB in it are either analytic or atomic. Then by Lemma 6.2 above, the proof is normal.

Theorem 6.3. *Any C-intelim proof of A depending on Γ can be transformed into a normal C-intelim proof of A depending on some $\Delta \subseteq \Gamma$ by means of any sufficiently long sequence of applications of the transformations* (T1)–(T27).

If we are interested in *atomically* normal proofs and not just in normal ones, all we need to do is change the definition of $g(\mathcal{T})$ in (8) as follows:

$$g(\mathcal{T}) = \begin{cases} \sharp(A) & \text{if } \mathcal{T} \text{ ends with a non-atomic application of RB and } A \text{ is the} \\ & \text{RB-formula of this application} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then it is not difficult to adapt the previous arguments to show:

Theorem 6.4. *Any C-intelim proof of A depending on Γ can be transformed into an atomically normal C-intelim proof of A depending on some $\Delta \subseteq \Gamma$ by means of any sufficiently long sequence of applications of the transformations* (T1)–(T27).

Restricting to normal proofs has several advantages over restricting to atomically normal proofs. Not only are normal proofs shorter in general, but they also allow for a notion of k-depth normal deducibility from a set of formulae (A is *k-depth normally deducible from* Γ if there is a k-depth normal proof of A depending on $\Delta \subseteq \Gamma$) that is *structural*, i.e., it is invariant under uniform substitutions of atomic formulae with arbitrary ones.

Indeed, we suggest that the *depth of a normal proof* provides a natural measure of the "difficulty" of the reasoning process represented by the proof. It reflects the maximum number of nested uses of "virtual information" (i.e., the assumptions discharged by an application of RB) in a proof. From the computational viewpoint, this idea is confirmed by the fact that, as we shall see in Section 10, k-depth normal deducibility is a *tractable* problem.

7 The contamination problem

Let us say that two formulae are *syntactically disjoint* if they share no atomic formula. Two *sets* Γ and Δ are *syntactically disjoint* if every formula of Γ is syntactically disjoint from every formula of Δ. In what follows we shall write "$\Gamma \parallel \Delta$" for "Γ is syntactically disjoint from Δ".

It is routine to show that the following holds by classical semantics:

Theorem 7.1. *For every Γ and Δ, if*

1. $\Delta \parallel \Gamma \cup \{A\}$, *and*
2. $\Gamma \cup \Delta \vdash_{\mathsf{C}} A$,

then at least one of the following holds true:

- Γ *is consistent and* $\Gamma \vdash_{\mathsf{C}} A$,
- Γ *is inconsistent and* $\Gamma \vdash_{\mathsf{C}} A$;
- Δ *is inconsistent and* $\Delta \vdash_{\mathsf{C}} A$.

This holds for every (possibly empty) Γ, Δ. The special case in which Δ is empty is trivial.

In the special case in which Γ is empty, the theorem implies that:

Corollary 7.1. *If $\Delta \parallel \{A\}$, then $\Delta \vdash_{\mathsf{C}} A$ if and only if either A is a tautology or Δ is inconsistent.*

The *contamination problem* is the problem that arises in classical natural deduction when:

- for some *non-empty* Δ such that $\Delta \parallel \Gamma \cup \{A\}$, with $A \neq \curlywedge$, we have a natural deduction proof of A *depending*[33] on $\Gamma \cup \Delta$, or
- for some *non-empty* Δ, Γ such that $\Delta \parallel \Gamma$, we have a natural deduction proof of $\curlywedge$ depending on $\Gamma \cup \Delta$.

In the first case, it may be that $\Gamma \nvdash_{\mathsf{C}} A$, in which case, by classical semantics, Δ is inconsistent and assumptions that are totally unrelated either to the conclusion or to Γ play an active role in the proof, via a *practically meaningless* use of *ex-falso quodlibet*,[34] to obtain a conclusion that could not have been otherwise obtained from Γ. Or it may be that $\Gamma \vdash_{\mathsf{C}} A$, in which case the assumptions in Δ, besides being unrelated, are also clearly unnecessary to obtain the conclusion. In the second case, either Δ or Γ must be inconsistent on their own and, again, unrelated and redundant assumptions are used to obtain the proof of $\curlywedge$. In any case such a proof violates a basic relevance condition that can indeed be satisfied by a better-behaved natural deduction proof, except for one distinguished case in which all of the following hold true:

1. Γ is empty,
2. Δ is inconsistent and cannot be partitioned into two syntactically disjoint subsets,
3. the conclusion A is not equal to $\curlywedge$ and is obtained from Δ by means of an *ex-falso* (or better *ex-contradictione*) inference.

The simplest example is the inference $p, \neg p/q$ displayed in (1). While such inferences cannot be expunged from any *complete* system for classical logic, we can easily obtain from them proof that Δ is inconsistent, that is a proof of $\curlywedge$ depending on Δ. In view of various applications of natural deduction (see the concluding section on this point) it would be useful to come up with a notion of normal proof that avoids incurring the contamination problem whenever possible, i.e. with the only exception of inferences satisfying the three conditions specified above, in which case, however, a refutation can be promptly obtained.

Definition 7.1 (Contaminated proofs)**.** *Given a natural deduction system S, we say that an S-proof π of A depending on Γ is* contaminated *if one of the following two conditions hold:*

1. $A \neq \curlywedge$ *and for some non-empty* $\Delta \subseteq \Gamma$, $\Delta \parallel (\Gamma \setminus \Delta) \cup \{A\}$*;*
2. $A = \curlywedge$ *and for some non-empty* $\Delta \subset \Gamma$, $\Delta \parallel (\Gamma \setminus \Delta)$.

In both cases we call Δ a contaminating set *for π.*

Definition 7.2 (R-contaminated proofs)**.** *We say that π is* redundantly contaminated *(*R-contaminated *for short) if (i) π is contaminated and (ii) there is a contaminating set Δ for π such that either $\Delta \subset \Gamma$ or $\Delta \nvdash \curlywedge$.*

[33]By this we mean that all the assumptions are actually used in the deduction tree.

[34]This is a Pickwickian way of putting it. See Footnote 9 above.

$$\frac{A \to B \qquad \dfrac{\dfrac{C \quad \neg C}{\curlywedge}}{A}}{B} \qquad\qquad \frac{\dfrac{\dfrac{C \quad \neg C}{\curlywedge}}{B} \qquad \dfrac{\dfrac{\dfrac{D \quad \neg D}{\curlywedge}}{E} \qquad E \to \neg B}{\neg B}}{\curlywedge}$$

Figure 7: R-contaminated normal proofs in Prawitz's natural deduction.

Note that our definition implies that any contaminated *refutation* of Γ (a proof of $\curlywedge$ depending on Γ) is always R-contaminated. Proofs that are contaminated but not R-contaminated correspond to the exceptions discussed above that cannot be expunged by any natural deduction system that is complete for classical logic.[35]

Can we suitably restrict the notion of an acceptable natural deduction proof so as to deliver only proofs that are not R-contaminated? The notion of normal proof put forward in [Prawitz, 1965] is not sufficient for this purpose. The trees shown in Figure 7 represent R-contaminated proofs that are normal in Prawitz's sense (assuming that B and E are atomic to satisfy the restriction on the $\curlywedge_{\mathsf{C}}$ rule, see [Prawitz, 1965, Chapter III]). [36] On the other hand, we shall show, in the next section, that normal C-intelim proofs are never R-contaminated. Let us now focus on the "exceptional" proofs that are contaminated but not R-contaminated. These are proofs of A depending on a non-empty $\Gamma \parallel \{A\}$ in which $A \neq \curlywedge$ and Γ cannot be partitioned into syntactically disjoint subsets. We shall show later on that for a *normal* proof $\mathcal{T}$ this situation obtains only when $\mathcal{T}$ "contains", in a well-defined sense, a proof of $\curlywedge$ depending on the same set Γ of assumptions that can be *easily* extracted from it. We have already commented in the introduction that while such proofs cannot be excluded from a complete system for classical logic, since they express classically valid inferences, they are devoid of any practical value and that their *only* epistemic significance consists in revealing that the assumptions are inconsistent and therefore, from a classical logic perspective, cannot be used to make any reasonable inference at all.[37] This seems a good reason to call such proofs *improper*.

Definition 7.3. *The notion of proper normal proof is defined inductively as follows:*

- *A normal* 0*-depth proof* $\mathcal{T}$ *is* proper *if its last step is not an application of XFQ (so* $\mathcal{T}$ *contains no applications of XFQ);*

[35]They can be expunged if we switch to a paraconsistent system, such as the one that results from restricting to the *proper* normal proofs introduced in Definition 7.3 or from a classical extension of the system discussed in Tennant [1987].

[36]Note that neither can such proofs be turned into proofs of $\curlywedge$ depending on the *same* set of assumptions.

[37]This view is shared, of course, by all the advocates of relevance logic. However, with the notable exceptions of Timothy Smiley and Neil Tennant, most of them argue that disjunctive syllogism should be rejected as well as the ex-falso rule, given the role played by the latter in the proof of the former within the framework of Gentzen-style natural deduction. By contrast, disjunctive syllogism is a primitive rule in the C-intelim system.

- *for $k > 0$, a normal k-depth proof*

$$\mathcal{T} = \quad \cfrac{\begin{matrix}[B] & & [\neg B] \\ \mathcal{T}_1 & & \mathcal{T}_2 \\ A & & A\end{matrix}}{A}$$

is proper *if at least one of its immediate subproofs $\mathcal{T}_1$ and $\mathcal{T}_2$ is proper.*

It follows from the above definition that a normal C-intelim proof $\mathcal{T}$ is *proper* if at least one of its 0-depth components contains no application of XFQ. Observe that every normal refutation of Γ (i.e., a proof of $\curlywedge$ depending on Γ) is proper. This follows from the fact that normal proofs are XFQ-canonical (Definition 4.6). Thus, XFQ can be applied only as the last step of one of its 0-depth components.

Let the *height* of a formula-tree be the maximum length of its branches.

Lemma 7.1. *If $\mathcal{T}$ is an improper normal proof of A depending on Γ, then $\mathcal{T}$ can be transformed into a normal refutation $\mathcal{T}'$ of Γ such that:*

1. $\mathsf{depth}(\mathcal{T}') = \mathsf{depth}(\mathcal{T})$,
2. $h(\mathcal{T}') = h(\mathcal{T}) - 1$.

Moreover, the computational cost of the transformation is linear in the number of applications of XFQ in $\mathcal{T}$.

Proof. If $\mathcal{T}$ improper, then all its 0-depth components (see Definition 4.3) end with an application of XFQ. Hence, each 0-depth component contains, as a strict subproof, a normal proof of $\curlywedge$ depending on the same assumptions. Then all the virtual assumptions occurring in the 0-depth components can be discharged by applying RB with $\curlywedge$, instead of A, as conclusion. □

Figure 8 shows an example of an improper normal proof with its associated normal refutation. Note that a proper normal proof does not need to depend on a consistent set of premises. A trivial example of a proper normal proof depending on inconsistent premises is the following:

$$\frac{A \quad \neg A}{A \wedge \neg A} \tag{11}$$

8 Variable sharing and non-contamination

We shall now show that normal C-intelim proofs are never R-contaminated and that *proper* normal proofs are never contaminated, so that they enjoy the variable-sharing property, except for the case in which their conclusion is $\curlywedge$. We start with a lemma on 0-depth proofs.

Lemma 8.1. *For every non-empty Γ and every A, there is no 0-depth proper normal proof of A depending on Γ such that $A \neq \curlywedge$ and $\Gamma \parallel \{A\}$.*

$$\dfrac{\dfrac{\dfrac{\dfrac{A\to B\quad [A]^1}{B}\qquad \dfrac{A\to\neg B\quad [A]^1}{\neg B}}{\curlywedge}}{D}\qquad \dfrac{\dfrac{\dfrac{\neg A\to C\quad [\neg A]^2}{C}\qquad \dfrac{\neg A\to\neg C\quad [\neg A]^2}{\neg C}}{\curlywedge}}{D}}{D}\;{\scriptstyle 1,2}$$

$$\dfrac{\dfrac{\dfrac{A\to B\quad [A]^1}{B}\qquad \dfrac{A\to\neg B\quad [A]^1}{\neg B}}{\curlywedge}\qquad \dfrac{\dfrac{\neg A\to C\quad [\neg A]^2}{C}\qquad \dfrac{\neg A\to\neg C\quad [\neg A]^2}{\neg C}}{\curlywedge}}{\curlywedge}\;{\scriptstyle 1,2}$$

Figure 8: Transforming an improper normal proof of D into a normal refutation.

Proof. Let $\mathcal{T}$ be a 0-depth proper normal proof of A depending on Γ. Being proper, $\mathcal{T}$ contains no applications of XFQ. Moreover, if $A \neq \curlywedge$, it also contains no application of RNC. By Lemma 6.1, at least one branch of $\mathcal{T}$ is a complete normal intelim walk. So, there are four cases: (i) the E-part and the I-part are both empty; then the minimum formula is the only element of the walk and the whole proof $\mathcal{T}$ is a trivial one-node proof of A depending on A; (ii) the E-part is non-empty and the I-part is empty, in which case the minimum formula is equal to the conclusion A and a proper weak subformula of some assumption in Γ; (iii) the E-part is empty and the I-part is non-empty, in which case the minimum formula is at the same time one of the assumptions in Γ and a proper weak subformula of the conclusion A; (iv) neither the E-part nor the I-part are empty, in which case the minimum formula is a proper weak subformula both of some assumption in Γ and of the conclusion A. In all cases the minimum formula of the complete intelim walk is at the same time a weak subformula of some formula in Γ and of A. Hence, $\Gamma \nparallel \{A\}$. $\square$

It follows that in a normal 0-depth proof $\mathcal{T}$ of A depending on a non-empty Γ, if $\Gamma \parallel \{A\}$, either $A = \curlywedge$ or $\mathcal{T}$ is improper, that is, the proof ends with an application of XFQ. In either case Γ is inconsistent.

Corollary 8.1. *For every non-empty Γ and every A, if there is a normal 0-depth proof of A depending on Γ such that $\Gamma \parallel \{A\}$, then Γ is inconsistent.*

Notice that there are no 0-depth proofs of a tautology depending on the empty set of assumptions. Despite this, 0-depth deducibility is a Tarskian consequence relation. The same absence of tautologies is shared by other logics based on informational notions, such as Kleene's 3-valued and Belnap's 4-valued logics. Interestingly, the 0-depth logic, like intuitionistic logic, admits of no finitely-valued characteristic matrix.[38] However, it admits of an intuitive semantics based on a non-deterministic matrix

[38] See [D'Agostino et al., 2013] where this result is proven for a different, tableau-like, variant of C-intelim.

that (to the best of our knowledge) was first discussed by W.V.O. Quine in order to fix what he called the "primitive meaning of the logical operators" [Quine, 1973, §20].[39]

We are now in a position to show that the variable sharing property applies to proper normal proofs of arbitrary depth.

Theorem 8.1 (Variable-sharing property)**.** *If $\mathcal{T}$ is a proper normal proof of $A \neq \curlywedge$ depending on Γ, then $\Gamma \nparallel \{A\}$.*

Proof. Observe that, every 0-depth component of $\mathcal{T}$ is a 0-depth proper normal proof of A depending on $\Gamma' \cup \Delta$ such that $\Gamma' \subseteq \Gamma$ and Δ is the set of virtual assumptions introduced in this 0-depth component that are subsequently discharged. By Lemma 8.1, each 0-depth component has the variable sharing property. Since$\mathcal{T}$ is normal, all the formulae in Δ are weak subformulae of some formula in $\Gamma \cup \{A\}$. □

Although proper normal proofs are not complete for classical logic, they are complete for the set of valid inferences from consistent sets of assumptions, since if there is no proper normal proof of A depending on Γ, then Γ must be inconsistent, in that any improper proof, by Lemma 7.1, "contains" a refutation of Γ. As shown above, they also enjoy the variable sharing property. Indeed, the system of deduction that accepts as admissible only proper normal proofs has close connections with Tennant-style relevance logic [Tennant, 1984, 1987]. However, it is hard to take it, as it stands, as a well-behaved system of relevance logic. For example, there is a proper normal proof of $(A \vee B) \wedge B$ depending on A and $\neg A$, which is only one breadth away from $A, \neg A/B$. It must be noticed, however, that there is no proper normal proof that is also atomically normal. The connection between our approach and Tennant-style relevance logic will be investigated in a future paper.

Theorem 8.2 (Weak non-contamination)**.** *if $\mathcal{T}$ is a normal proof of A depending on Γ, then $\mathcal{T}$ is not R-contaminated.*

Proof. First, note that if there is a contaminating Δ such that the first disjunct of condition (ii) in Definition 7.2 is false, that is, Γ itself is contaminating for $\mathcal{T}$, then the other disjunct must also be false, for in such a case $\Delta = \Gamma \parallel \{A\}$ and, by Theorem 8.1, $\mathcal{T}$ is improper. Hence, every 0-depth component of $\mathcal{T}$ ends with an application of XFQ. Such an improper proof can be easily turned into a refutation of $\Gamma = \Delta$ (see Lemma 7.1) and so $\Delta \vdash \curlywedge$. Hence, to show the theorem it is sufficient to show that there is no contaminating Δ properly included in Γ.

To spare on parentheses, we shall assume throughout this proof that "$\backslash$" binds more tightly than "$\cup$". The proof is by induction on the height of $\mathcal{T}$ (the maximum length of a branch of $\mathcal{T}$) denoted by $h(\mathcal{T})$.

Base: $h(\mathcal{T}) = 1$. Then $\mathcal{T}$ is a one-node formula tree representing a proper normal proof of A depending on A. Then, trivially, $\mathcal{T}$ is non-contaminated.

Step: $h(\mathcal{T}) = k > 1$. The theorem holds for every normal proof $\mathcal{T}'$ such that $h(\mathcal{T}') < k$. We show that it holds also for $\mathcal{T}$. There are several cases depending on the last step of $\mathcal{T}$.

[39] See [D'Agostino, 2014, 2015] for an in-depth discussion including soundness and completeness results.

Case 1: the last step of $\mathcal{T}$ is an application of XFQ. Then, since in a normal proof XFQ can be applied only as the last step in a 0-depth component, $\mathsf{depth}(\mathcal{T}) = 0$, $\mathcal{T}$ is improper and has the following form:

$$\frac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{A}$$

By inductive hypothesis $\mathcal{T}_1$ is not R-contaminated and so, by Definition 7.1, there is no $\Delta \subset \Gamma$ such that $\Delta \parallel \Gamma \setminus \Delta$. Hence, there is no contaminating Δ for $\mathcal{T}$ properly included in Γ and $\mathcal{T}$ is not R-contaminated.

Case 2: the last step of $\mathcal{T}$ is an application of an introduction rule, then $\mathsf{depth}(\mathcal{T}) = 0$. We discuss only the sub-cases in which the introduction rule is one of those involving $\vee$, the others being similar. So, $\mathcal{T}$ has one of the following forms:

$$\frac{\begin{array}{c}\mathcal{T}_1\\ A\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\mathcal{T}_1\\ B\end{array}}{A \vee B} \qquad \frac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ \neg A & \neg B\end{array}}{\neg(A \vee B)}$$

For the first two sub-cases, let Γ be the set of all the assumptions of $\mathcal{T}$. By inductive hypothesis, $\mathcal{T}_1$ is not R-contaminated. Suppose, *ex-absurdo*, that $\mathcal{T}$ is R-contaminated, i.e., there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A \vee B\}$$

Then

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A\} \text{ and } \Delta \parallel \Gamma \setminus \Delta \cup \{B\},$$

and so in both cases $\mathcal{T}_1$ would be R-contaminated against the inductive hypothesis.

As for the third sub-case, let Γ_1 and Γ_2 be, respectively, the sets of all the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. By inductive hypothesis, neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, given that $\mathcal{T}$ is. Suppose now, *ex-absurdo* that $\mathcal{T}$ is R-contaminated, i.e., $\Delta \parallel \Gamma \setminus \Delta \cup \{\neg(A \vee B)\}$ for some non-empty $\Delta \subset \Gamma$. Let $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{\neg A\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg B\}.$$

Since Δ is a non-empty proper subset of Γ, then either Δ_1 is a non-empty proper subset of Γ_1 or Δ_2 is a non empty proper subset of Γ_2. To see this, first observe that, since $\mathcal{T}_1$ and $\mathcal{T}_2$ are both 0-depth proper normal proofs and their conclusions are different from $\curlywedge$, neither $\Delta_1 = \Gamma_1$, nor $\Delta_2 = \Gamma_2$, otherwise $\Gamma_1 \parallel \{\neg A\}$ or $\Gamma_2 \parallel \{\neg B\}$, against Lemma 8.1. Moreover, suppose Δ_1 (Δ_2) is empty. Then, since $\Delta \subset \Gamma$, Δ_2 (Δ_1) cannot be empty and must be a proper subset of Γ_2 (Γ_1). Therefore, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated against the inductive hypothesis.

Case 3: the last step of $\mathcal{T}$ is an application of an elimination rule, then *depth*$(\mathcal{T}) = 0$. We discuss only the sub-cases in which the elimination rule is one of those involving $\vee$, the others being similar. So, $\mathcal{T}$ has one of the following forms:

$$\frac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ A \vee B & \neg A\end{array}}{B} \qquad \frac{\begin{array}{cc}\mathcal{T}_1 & \mathcal{T}_2\\ A \vee B & \neg B\end{array}}{A} \qquad \frac{\begin{array}{c}\mathcal{T}_1\\ \neg(A \vee B)\end{array}}{\neg A} \qquad \frac{\begin{array}{c}\mathcal{T}_1\\ \neg(A \vee B)\end{array}}{\neg B}$$

Consider the first sub-case. By inductive hypothesis neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, since $\mathcal{T}$ is. Suppose *ex absurdo* that $\mathcal{T}$ is R-contaminated, i.e, there is some non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{B\}.$$

Let Γ_1 and Γ_2 be, respectively, the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let also $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{B\}.$$

By Corollary 6.1, $A \vee B$ is a subformula of some formula $C \in \Gamma_1 \cup \{B\}$. Clearly it is not a subformula of B; moreover C cannot occur in Δ, otherwise Δ would not be syntactically disjoint from $\{B\}$. Hence, C occurs in $\Gamma \setminus \Delta$ and $\Delta \parallel \{C\}$; so both $\Delta_1 \parallel \{C\}$ and $\Delta_2 \parallel \{C\}$. Given that $A \vee B$ is a subformula of C, $\Delta_1 \parallel \{A \vee B\}$ and $\Delta_2 \parallel \{\neg A\}$. Then,

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{A \vee B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg A\}.$$

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are both proper and their conclusions are different from $\curlywedge$, neither $\Delta_1 = \Gamma_1$, nor $\Delta_2 = \Gamma_2$, otherwise $\Gamma_1 \parallel \{\neg A \vee B\}$ or $\Gamma_2 \parallel \{\neg A\}$, against Lemma 8.1. As in the previous case, it can be easily shown that, either Δ_1 is a non-empty proper subset of Γ_1 or Δ_2 is a non-empty proper subset of Γ_2, and so at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis. The proof is similar for the second sub-case.

As for the third sub-case, suppose there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{\neg A\}.$$

By Corollary 6.1, $\neg(A \vee B)$ is a weak subformula of some formula $C \in \Gamma \cup \{\neg A\}$. Clearly it is not a weak subformula of $\neg A$. Again, C cannot occur in Δ, otherwise Δ would not be syntactically disjoint from $\{\neg A\}$. Thus C occurs in $\Gamma \setminus \Delta$. Since $\Delta \parallel \Gamma \setminus \Delta$, then $\Delta \parallel \{C\}$ and, therefore, $\Delta \parallel \{\neg(A \vee B)\}$. Hence, $\Delta \parallel \Gamma \setminus \Delta \cup \{\neg(A \vee B)\}$ and so $\mathcal{T}_1$ is R-contaminated, against the inductive hypothesis. The fourth sub-case is similar to the third.

Case 4: the last step of $\mathcal{T}$ is an application of RNC. Then $\mathsf{depth}(\mathcal{T}) = 0$ and $\mathcal{T}$ has the following form:

$$\frac{\begin{matrix}\mathcal{T}_1 \\ B\end{matrix} \qquad \begin{matrix}\mathcal{T}_2 \\ \neg B\end{matrix}}{\curlywedge}$$

where, by inductive hypothesis, neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, given that $\mathcal{T}$ is, and $B \neq \curlywedge$. Suppose, *ex absurdo*, that $\mathcal{T}$ is R-contaminated. This means that there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta.$$

Let Γ_1 and Γ_2 be, respectively, the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let also $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then,

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2.$$

By Theorem 6.1, either $B \in \Gamma$ or B is a proper weak subformula of some formula $C \in \Gamma$. In either case, B is a weak subformula of some $C \in \Gamma$. Now, either $C \in \Delta$ or $C \in \Gamma \setminus \Delta$.

In the first case, it follows that $\{B\} \parallel \Gamma \setminus \Delta$. Therefore:

$$\{B\} \parallel \Gamma_1 \setminus \Delta_1 \text{ and } \{\neg B\} \parallel \Gamma_2 \setminus \Delta_2.$$

Therefore:

$$\Gamma_1 \setminus \Delta_1 \parallel \Delta_1 \cup \{B\} \text{ and } \Gamma_2 \setminus \Delta_2 \parallel \Delta_2 \cup \{\neg B\},$$

Observing that $\Delta_1 = \Gamma_1 \setminus (\Gamma_1 \setminus \Delta_1)$ and $\Delta_2 = \Gamma_2 \setminus (\Gamma_2 \setminus \Delta_2)$,

$$\Gamma_1 \setminus \Delta_1 \parallel \Gamma_1 \setminus (\Gamma_1 \setminus \Delta_1) \cup \{B\} \text{ and } \Gamma_2 \setminus \Delta_2 \parallel \Gamma_2 \setminus (\Gamma_2 \setminus \Delta_2) \cup \{\neg B\},$$

As in the previous cases, given that $\mathcal{T}_1$ and $\mathcal{T}_2$ are both 0-depth proper normal proofs, it can be easily shown, using Lemma 8.1, that either $\Gamma_1 \setminus \Delta_1$ is a non-empty proper subset of Γ_1 or $\Gamma_2 \setminus \Delta_2$ is a non-empty proper subset of Γ_2, and so at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis.

In the second case, it follows that $\{B\} \parallel \Delta$. Therefore:

$$\{B\} \parallel \Delta_1 \text{ and } \{\neg B\} \parallel \Delta_2.$$

Hence:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg B\}.$$

Again, this implies that at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is either improper (by Lemma 8.1) or R-contaminated, against the inductive hypothesis.

Case 5: The last step of $\mathcal{T}$ is an application or RB. Then $\mathcal{T}$ is a k-depth proof of A depending on Γ with $k > 0$ and has the following form:

$$\frac{\begin{matrix}[B] \\ \mathcal{T}_1 \\ A\end{matrix} \qquad \begin{matrix}[\neg B] \\ \mathcal{T}_2 \\ A\end{matrix}}{A}$$

By inductive hypothesis neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Let $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be defined as usual. Suppose that $\mathcal{T}$ is R-contaminated, that is, for some non-empty $\Delta \subset \Gamma$ we have that:

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A\}, \tag{12}$$

and therefore:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{A\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{A\}. \tag{13}$$

Since both $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal, and so all the applications of RB are analytic, B is a subformula of some $C \in \Gamma \cup \{A\}$.

Now, either $C \in \Delta$ or $C \in \Gamma \setminus \Delta \cup \{A\}$. In the first case, by (12),

$$\{B\} \parallel \Gamma_1 \setminus \Delta_1 \cup \{A\} \text{ and } \{\neg B\} \parallel \Gamma_2 \setminus \Delta_2 \cup \{A\}.$$

It follows that:

$$\Delta_1 \cup \{B\} \parallel (\Gamma_1 \cup \{B\}) \setminus (\Delta_1 \cup \{B\}) \cup \{A\} \tag{14}$$

and

$$\Delta_2 \cup \{\neg B\} \parallel (\Gamma_2 \cup \{\neg B\}) \setminus (\Delta_2 \cup \{\neg B\}) \cup \{A\}. \tag{15}$$

Observe that both $\Delta_1 \cup \{B\}$ and $\Delta_2 \cup \{\neg B\}$ are non-empty. Moreover, since Δ is a *proper* subset of Γ it cannot be the case that both $\Delta_1 = \Gamma_1$ and $\Delta_2 = \Gamma_2$, otherwise it should be $\Delta = \Gamma$. Thus, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated against the inductive hypothesis.

In the second case, it follows from (12) that $\Delta \parallel \{B\}$, and so:

$$\Delta_1 \parallel \{B\} \text{ and } \Delta_2 \parallel \{\neg B\}. \tag{16}$$

Therefore, by (13),

$$\Delta_1 \parallel (\Gamma_1 \cup \{B\}) \setminus \Delta_1 \cup \{A\} \text{ and } \Delta_2 \parallel (\Gamma_2 \cup \{\neg B\}) \setminus \Delta_2 \cup \{A\}. \tag{17}$$

Now, at least one of Δ_1 and Δ_2 is non-empty. Moreover Δ_1 (Δ_2) is by definition a subset of Γ_1 (Γ_2) and, by (16), does not contain B ($\neg B$). Therefore, either Δ_1 is a non-empty proper subset of $(\Gamma_1 \cup \{B\})$ or Δ_2 is a non-empty proper subset of $(\Gamma_2 \cup \{\neg B\})$. Hence, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis. □

Putting together Lemma 7.1, Theorem 8.1 and Theorem 8.2 we have shown that:

Corollary 8.2 (Non-Contamination Property)**.** *If $\mathcal{T}$ is a normal proof of A depending on Γ, either $\mathcal{T}$ is non-contaminated or $\mathcal{T}$ is improper and can be turned in linear time into a non contaminated proof of $\curlywedge$ depending on Γ.*

9 C-intelim tableaux

The format of C-intelim proofs that we have presented so far, where proofs are trees with the conclusion as root and the assumptions as leaves, allows for easy comparison with Gentzen-style natural deduction, which exhibits the same formal structure. Although very perspicuous, this format involves a good deal of redundancy in the representation of proofs. Whenever a formula, which can be inferred from the assumptions, is used more than once as a premise of further inferences, its proof tree has to be replicated, as in Figure 1, where the derivation of $\neg B$ from A and $A \to \neg B$ is repeated twice. We shall therefore shift to a different format, that we call here *C-intelim tableaux*, that provides a more concise representation of arguments and an easier implementation of the RB-rule, bringing the format of C-intelim proofs closer to that of Smullyan's [Smullyan, 1968] or KE tableaux [D'Agostino and Mondadori, 1994], with the notable difference that C-intelim tableaux can represent direct proofs as well as refutations. A C-intelim tableau can be easily turned into a C-intelim natural deduction proof in the standard format. Readers who find the latter more perspicuous as a presentation of proofs, could look at the content of this section as a step towards the

1	$(A \vee B) \rightarrow \neg(C \vee D)$	Assumption
2	A	Assumption
3	$(A \wedge E) \rightarrow C$	Assumption
4	$A \vee B$	$\vee \mathcal{I}1$ (2)
5	$\neg(C \vee D)$	$\rightarrow \mathcal{E}1$ (1,4)
6	$\neg C$	$\neg\vee \mathcal{E}1$ (5)
7	$\neg D$	$\neg\vee \mathcal{E}2$ (5)
8	$\neg(A \wedge E)$	$\rightarrow \mathcal{E}2$ (3,6)
9	$\neg E$	$\neg\wedge \mathcal{E}1$ (8,2).

Figure 9: A C-intelim sequence.

development of efficient *automated proof search procedures* for the natural deduction system of the previous sections. On the other hand, readers who are more familiar with Smullyan's tableaux and KE may look at C-intelim tableaux as an extension of KE that includes also introduction rules and allows for direct proofs as well as for refutations.

In this new format the application of the intelim rules is *sequential*: their premises do not occur on adjacent branches, but on the same branch as the conclusion and anywhere above it. Moreover, as will be seen below, there is no need for explicit falsum rules such as RNC and XFQ. As a result, 0-depth proofs, i.e., the ones involving no application of the discharge rule RB, can be represented as *intelim sequences*.

Definitions 9.1 (C-intelim sequence)**.** *A* C-intelim sequence *is a sequence of formulae such that each formula is either (i) an assumption, or (ii) the conclusion of the application of an intelim rule to preceding formulae.*

A C-intelim sequence based on Γ *is a C-intelim sequence such that all its assumptions belong to* Γ.

A C-intelim sequence is closed *if it contains both* A *and* $\neg A$ *for some formula* A*, otherwise it is* open.

The array of formulae in Figure 9 is a C-intelim sequence based on the set $\{(A \vee B) \rightarrow \neg(C \vee D), A, (A \wedge E) \rightarrow C\}$. The sequence starts by listing the assumptions and each subsequent formula is obtained by an application of an intelim rule to preceding formulae. In this example, for the reader's convenience, the full justification of each formula in the sequence is specified on its right. In the sequel the justification will be omitted and left to the reader. (Therefore numbering the formulae will no longer be required.) Note that the change of format implies that an intelim sequence may well contain *idle* occurrences of formulae, namely occurrences of formulae other than the last one that are not used as premises of a rule application, such as the one in step 7. This is not possible when 0-depth proofs are represented as formula-trees as in the previous sections, where all the formula occurrences in the tree, except for the end formula occurring at the root, are used as premises of some rule application. When a proof is completed, such idle formulae can be simply removed without affecting soundness. It may well be that removing an idle formula makes idle some of the premises used to obtain it, which can be, in turn, removed until no idle formula is left in the tree.

Observe also that each intelim sequence based on Γ can be easily turned into a 0-depth C-intelim proof of its last formula depending on some subset Δ of Γ, provided that all idle occurrences of formulae are first removed from the sequence.

In this new format, the Rule of Bivalence (RB) is a *branching rule* that splits an intelim sequence into two branches as follows (depending on whether it is used for signed or unsigned formulae):

$$\begin{array}{ccc} & \vdots & \\ & / \quad \backslash & \\ T\,A & & F\,A \end{array} \qquad\qquad \begin{array}{ccc} & \vdots & \\ & / \quad \backslash & \\ A & & \neg A \end{array}$$

So, deductions are again represented as trees, except that these trees now grow upside-down, like analytic tableaux. When a deduction tree is expanded in this way we say that RB has been applied to the formula A and that A is *the RB-formula* of this application of RB. Each application of this rule introduces, on each of the two branches, an extra assumption that we call *virtual assumption* to distinguish it from the *actual* assumptions displayed at the beginning, in such a way that exactly one of these two virtual assumptions must be true as a consequence of the classical Principle of Bivalence. This is the only branching rule of the system.

In the following definition we use the expression "tree of formulae" as an abbreviation of "tree whose nodes, except possibly the root, are labelled with formulae". (The special case when the root is unlabelled will be used to represent proofs from the empty set of assumptions, as will be explained below.)

Definition 9.1. *A* C-intelim tableau *is a tree of formulae such that each formula occurrence is either (i) an actual assumption, or (ii) results from previous formula occurrences in the same branch by an application of an intelim rule, or (iii) is one of the complementary virtual assumptions introduced by an application of the branching rule RB.*

A C-intelim tableau based on a set Γ *of formulae is a C-intelim tableau such that all its actual assumptions belong to* Γ.

Observe that each branch of a C-intelim tableau based on Γ is an intelim sequence based on $\Gamma \cup \Delta$ where Δ is the set of virtual assumptions introduced by the applications of RB in that branch. We say that a branch of a C-intelim tableau is *closed* when it contains both A and $\neg A$ for some formula A, otherwise we say that it is *open*. A C-intelim *tableau* is *closed* when all its branches are closed, otherwise it is *open*. Then the notion of *C-intelim tableau proof* can be defined as follows:

Definition 9.2. *A* C-intelim tableau proof *of A from* Γ *is an intelim tableau $\mathcal{T}$ based on* Γ *such that A occurs in each open branch of $\mathcal{T}$.*

Observe that this definition allows us to dispense with XFQ and implies that, if there are no open branches, $\mathcal{T}$ is a C-intelim proof of any formula A from Γ.

Definition 9.3. *A* C-intelim tableau refutation *of* Γ *is a closed C-intelim tableau based on* Γ.

Let us write $\Gamma \vdash^{IET} A$ if there is C-intelim tableau proof of A from Γ. Being essentially an alternative way of representing C-intelim proofs, C-intelim tableaux are complete for classical logic:

$$\text{If } A \text{ is a classical consequence of } \Gamma \text{, then } \Gamma \vdash^{IET} A.$$

Examples of C-intelim tableau proofs based on non-empty sets of assumptions are shown in Figure 10. The first one is a proof of G that contains a closed branch, and an open branch ending in G that correspond, respectively, to the left and right subproofs of the proof in Figure 1. Closed branches are marked with the symbol $\times$. It is customary to list all the actual assumptions at the beginning starting from the root. Proofs from the empty set of assumptions are represented by trees with an unlabelled root as illustrated in Figure 11. The first example shows how the introduction rules can be used to simulate the truth-table method. The second example shows how to represent a typical pattern of proof ex-absurdo. C-intelim tableaux can be naturally used as a *refutation system*, like resolution or semantic tableaux, as well as a system of direct proof. In fact, if we disallow the introduction rules, we obtain the system KE (Mondadori [1988a], D'Agostino and Mondadori [1994]), which is a variant of analytic tableaux (but essentially more efficient).[40] On the other hand if we disallow the elimination rules we obtain the system KI (Mondadori [1988b, 1995], D'Agostino [1999]), which can be regarded as a proof-theoretical version of the truth-table method (but essentially more efficient).[41] Using both introduction and elimination rules allows for shorter[42] and more natural deductions that require fewer applications of the discharge rule RB.

Note that C-intelim tableaux directly correspond to proofs in the conventional format of the previous sections that are both RB and XFQ canonical, i.e., such that all the applications of RB and XFQ have been pushed downwards. If one wants to allow for proofs that are not RB-canonical, then the RB rule could take the following format, which is closer, in spirit, to the Jaśkoswski-Fitch style of natural deduction:[43]

$$
\begin{array}{c}
\vdots \\
\boxed{\begin{array}{l} A \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{l} \neg A \\ \vdots \\ B \end{array}} \\
B \\
\vdots
\end{array}
$$

In the tree format, on the other hand, a branch corresponds to a 0-depth component in an RB-canonical proof. In general an RB-canonical and XFQ canonical C-intelim

[40] As shown in D'Agostino and Mondadori [1994], KE can p-simulate analytic tableaux but analytic tableaux cannot p-simulate KE. In fact, analytic tableaux cannot even p-simulate the truth-tables (D'Agostino [1992]).

[41] The truth-table method cannot p-simulate KI (Mondadori [1995]).

[42] But not *essentially* shorter, for both KI and KE can p-simulate C-intelim (D'Agostino [1999]).

[43] A similar way of using RB in the context of KE-style natural deduction is explored in Indrzejczak [2010].

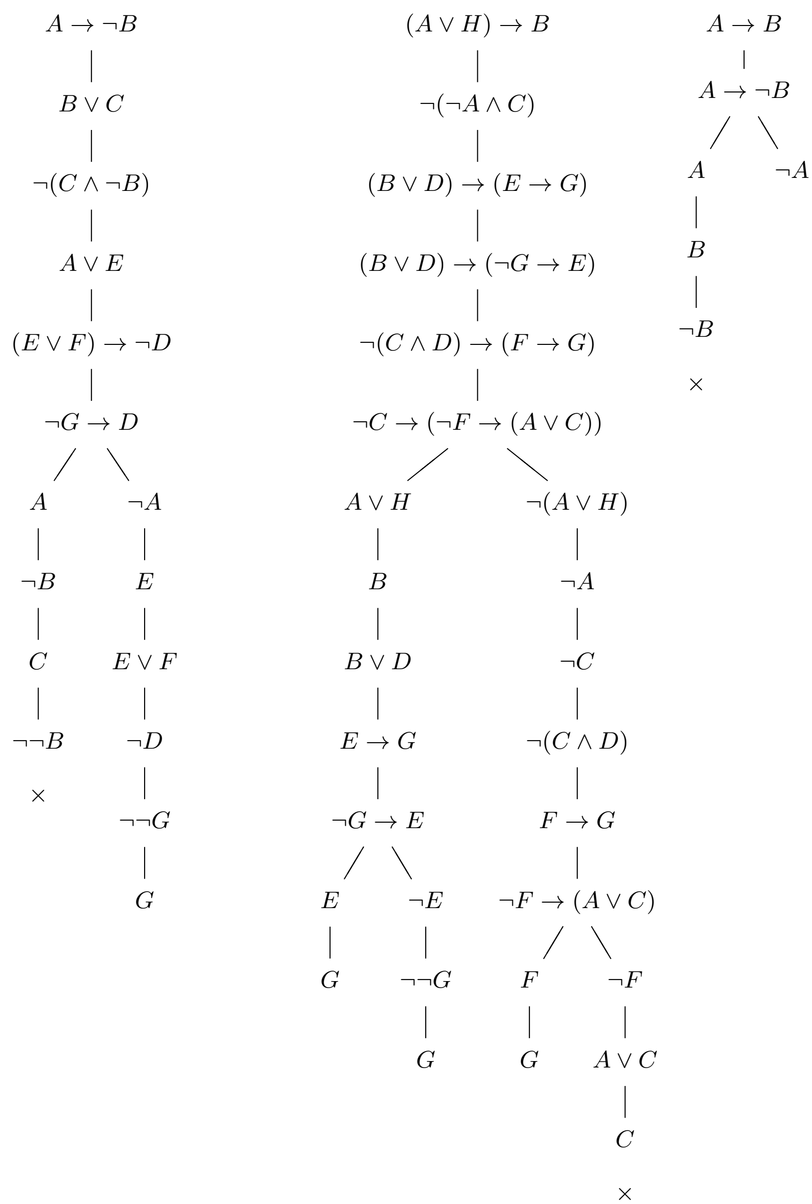

Figure 10: C-intelim tableaux. Each branch is an intelim sequence.

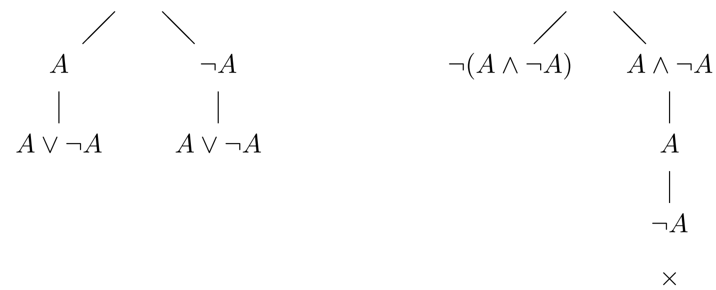

Figure 11: C-intelim tableau proofs from the empty set of assumptions.

proof of depth k corresponds to a C-intelim tableau that contains at most k nested applications of the RB rule (i.e., such that the maximum number of virtual assumptions in a branch is k). However, the applications of RNC and XFQ are no longer necessary, since they are absorbed by the notion of closed branch and by the definition of C-intelim tableau proof. A *path* in a C-intelim tableau $\mathcal{T}$ is a finite sequence of nodes such that the first node is the root of $\mathcal{T}$ and each subsequent node occurs immediately below the previous one (so, a branch is a maximal path). A path is *closed* if it contains occurrence of both B and $\neg B$ for some formula B.

The notion of quasi-normal C-intelim proof is replaced by a much simpler (and somewhat stronger) notion of *non-redundant* C-intelim tableaux.

Definition 9.4. *A C-intelim tableau proof $\mathcal{T}$ of A from Γ (refutation of Γ) is* non-redundant *if the following conditions are satisfied:*

1. *$\mathcal{T}$ contains no idle occurrences of formulae;*
2. *no branch of $\mathcal{T}$ contains more than one occurrence of the same formula;*
3. *no branch of $\mathcal{T}$ properly contains a closed path.*

The reader can verify that the two conditions 2 and 3 are sufficient to ensure that a non-redundant C-intelim tableau contains no detours, where detours are defined as before (see Definition 1) and that every non-redundant C-intelim tableau can be easily represented as a quasi-normal C-intelim proof, adding applications of RNC and, possibly, of XFQ at the end of each 0-depth component resulting from a closed branch. The notion of non-redundant C-intelim tableau, however, is stronger than the notion of quasi-normal C-intelim proof, in that it removes redundancies that are not purged by the notion of quasi-normal proof. For example, the following:

$$\frac{A \quad A \to A}{A} \tag{18}$$

is a quasi-normal C-intelim proof (since it contains no application of RB it is also trivially normal). However, its translation into a (one-branch) C-intelim tableau:

$$\begin{array}{c} A \\ A \to A \\ A \end{array} \tag{19}$$

is redundant because it violates condition 2 in Definition 9.4. The only non-redundant version of the above C-intelim proof is the trivial one containing only one occurrence of A.

The procedure to turn a C-intelim tableau into a non-redundant one is quite trivial:

1. if a branch properly contains a closed path, remove all the nodes following the closed path;

2. remove, one by one, all idle formulae and all repetitions of the same formula in a branch; if the idle or repeated formula is a virtual assumption introduced by an application of RB, we must remove also the whole subtree below the sibling node.

The elimination of idle or repeated occurrences of a formula in a branch may turn some previously used occurrences of formulae into idle ones; but at each reduction step the size of the tree decreases, and so the procedure terminates in a number of steps that is linear in the size of the initial tableau.

Example 9.1. *An example of the procedure described above is illustrated in Figure 12. This is an expansion of the example in Figure 9. The leftmost tree is a proof of the same conclusion from the same assumptions. However, here the occurrence of $\neg D$ is not idle, because it is used as premise of modus ponens to obtain $\neg E$ at the end of the left branch. On the other hand the virtual assumption on the right branch is idle, so we have to remove it together with all the subree below the sibling node. So, we obtain the intelim sequence in the center. Now, the occurrence of $\neg D$ has turned idle as in Figure 9, so we can just remove it to obtain the rightmost non-redundant C-intelim tableau.*

Definition 9.5. *A C-intelim tableau proof or refutation is* normal *if it is non-redundant and all applications of RB in it are analytic. It is* atomically normal *if it is normal and all applications or RB are atomic.*

Note that a *normal* C-intelim tableaux proof of A from Γ can be easily turned to a normal C-intelim proof of A depending on Γ which is essentially identical, except for the format and for the applications of RNC and, possibly, XFQ at the end of its 0-depth components (whenever the latter correspond to a closed branch). However, the opposite is not true because some normal C-intelim proofs may fail to fully satisfy the non-redundancy condition.

The proofs of Theorem 6.2, Theorem 6.3 and Theorem 8.2 can be easily adapted to yield the following:

Theorem 9.1. *If $\mathcal{T}$ is a normal C-intelim tableaux, then $\mathcal{T}$ has the WSFP.*

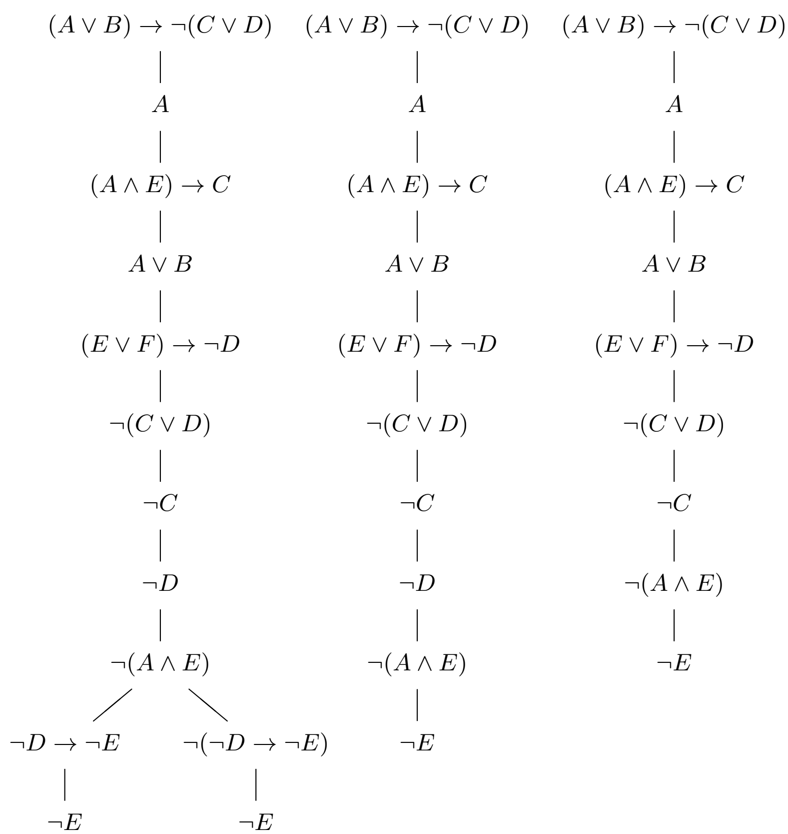

Figure 12: Turning a redundant C-intelim tableau into a non-redundant one.

Theorem 9.2. *Every C-intelim tableau proof of A from Γ (refutation of Γ) can be transformed into a(n atomically) normal one of A from some $\Delta \subseteq \Gamma$.*

Theorem 9.3. *If $\mathcal{T}$ is a normal C-intelim tableaux, then $\mathcal{T}$ is not R-contaminated.*

The *depth* of a C-intelim tableau $\mathcal{T}$ is simply the maximum number of virtual assumptions occurring in one of its branches.

Definition 9.6. *$\mathcal{T}$ is a* proper *normal C-intelim tableau proof of A based on Γ if $\mathcal{T}$ is normal C-intelim tableau proof of A based on Γ and $\mathcal{T}$ is open (at least one of its branches is not closed).*

Again, the proof of the next theorems is parasitic on the proof of their analogues for C-intelim proofs (namely, Theorem 8.1 and Corollary 8.2).

Theorem 9.4. *If $\mathcal{T}$ is a proper normal C-intelim tableau proof of A from Γ and $A \neq \curlywedge$, then $\Gamma \nVdash \{A\}$.*

Theorem 9.5 (Non-contamination property)**.** *If $\mathcal{T}$ is a normal C-intelim tableau proof of A from Γ, then $\mathcal{T}$ is non-contaminated or improper (i.e., a closed tableau for Γ).*

One can consider a restricted version of the C-intelim rules that automatically generates normal tableaux (except at most for the presence of idle occurrences of formulae), by requiring that, in the attempt to prove A from Γ:

- an inference rule can be applied in a branch only if its conclusion does not already occur in the branch;
- RB can be applied in a branch only (i) if neither of the virtual assumptions introduced by it already occurs in the branch, and (ii) the RB formula is a subformula either of one of the premises in Γ or of the conclusion A;
- a closed branch cannot be further expanded.

A tableau $\mathcal{T}$ constructed in accordance with these restrictions can be easily turned into a normal one by simply removing all idle occurrences of formulae (if any) and then, if required, into a normal C-intelim proof in the upward tree format of Section 1.

10 The complexity of C-intelim proofs

Let QNC-intelim be the restriction of C-intelim to quasi-normal proofs.

Theorem 10.1. *QNC-intelim can p-simulate Frege systems.*

Hint. As the reader can verify, the negation of any instance A of any axiom scheme of a typical complete axiomatic systems for classical propositional logic admits of a 0-depth C-intelim refutation. So, by means of an application of RB one can obtain a proof of A as follows:

$$
\dfrac{[A]^1 \qquad \dfrac{\begin{matrix}[\neg A]^2 \\ \mathcal{T} \\ \curlywedge\end{matrix}}{A}}{A}\ {\scriptstyle 1,2}
$$

Moreover, Modus Ponens is a rule of C-intelim. □

Consider now normal C-intelim tableaux. Since the introduction rules can be easily simulated by means of eliminations and applications of RB, it is not difficult to show that the subsystem of C-intelim tableaux consisting only of RB and the elimination rules, which amounts to the KE system of D'Agostino and Mondadori [1994], can linearly simulate C-intelim tableaux and vice versa. Since KE can p-simulate the cut-free Gentzen sequent calculus, but not vice versa [D'Agostino and Mondadori, 1994], it immediately follows that:

Theorem 10.2. *C-intelim tableaux can p-simulate cut-free Gentzen systems, but cut-free Gentzen systems cannot p-simulate C-intelim tableaux.*

Let us write $\Gamma \vdash_k^{\mathrm{IET}} A$ to mean that there is a normal C-intelim tableau proof of A from Γ of depth $\leq k$, and $\Gamma \vdash_k^{\mathrm{IET}} \curlywedge$ to mean that there is a normal refutation of Γ of depth $\leq k$. In [D'Agostino et al., 2013, D'Agostino, 2015] it is shown that each $\vdash_k^{\mathrm{IET}}$ is a tractable approximation to classical propositional logic that converges to it for $k \to \infty$.

Theorem 10.3. *For each $k \in \mathbb{N}$, whether or not $\Gamma \vdash_k^{\mathrm{IET}} A$ can be decided in time $O(n^{k+2})$, where n is the total number of occurrences of symbols in $\Gamma \cup \{A\}$.*

Let *NC-intelim* be the restriction of C-intelim to normal proofs and *NC-intelim*$_k$ be the restriction of NC-intelim in which only proofs of depth $\leq k$ are allowed. Given the correspondence between NC-intelim proofs and C-intelim tableaux discussed in Section 9, it follows from the results above that proof search in NC-intelim$_k$ is feasible.

11 Conclusions

We have carried out a detailed proof-theoretical study of a system of natural deduction for classical propositional logic, the C-intelim system, that considerably departs from the standard Gentzen-style approach in that classical logic is not characterized by a system of intelim rules that extend the rules that represent the intuitionistic meaning of the logical operators, but directly by means of rules that are faithful to their classical interpretation. As a result, the typical symmetries of classical logic are not lost, no intelim rule involves the discharge of assumptions and the only discharge rule is the one that expresses the classical Principle of Bivalence (often called "Classical Dilemma" Tennant [1990]). We have shown sets of transformations that, independently of the order in which they are applied, yield respectively *quasi-normal proofs* — i.e. non-analytic proofs that, yet, contain no trivial detours — and *normal proofs* that are fully analytic and enjoy the (weak) subformula property. We have also introduced *proper* *normal proofs*, that avoid uses of the ex-falso quodlibet principle that are inferentially meaningless, and enjoy the variable sharing property. We have also shown that our notion of normal proof paves the way for the proof of a *non-contamination* theorem that does not hold if the standard notion of normal proof in classical Gentzen-style natural deduction is adopted. This result shows that a weak relevance property that holds for classical logic, i.e., the existence of non-contaminated proofs except for trivial cases in which the proof is more aptly described as a refutation of the assumptions, is automatically enforced by the restriction to normal proofs. Next, we have presented a different format for C-intelim proofs that is more suitable to provide a more concise representation of proofs and to implement proof-search algorithms, highlighting the connection between C-intelim and certain variants of the method of semantic tableaux. Finally we have stated some complexity facts about C-intelim and in particular that the notion k-depth C-intelim deducibility provides a hierarchy of tractable approximation to classical propositional logic. This approach to tractable depth-bounded reasoning will be extended to first-order logic in a subsequent paper, by incorporating some ideas from Hintikka [1972].

We maintain that our results are useful in a variety of application areas. In particular, a key application area of increasing prominence is the use of argumentation theory to formalise individual agent, and distributed, non-monotonic reasoning. Typically, arguments are classical logic proofs possibly augmented by defeasible inference rules [Modgil and Prakken, 2013]. Evaluation of the interacting (attacking and counter-attacking) arguments determines those that are justified, and the conclusions of these justified arguments identify the inferences from the theory that supplies the assumptions for constructing the arguments. Two key reasons for the increasing prominence of argumentation theory are that: 1) its characterisation of non-monotonic reasoning, in terms of argument and counter-argument, is intuitively understandable to human users familiar with everyday principles of debate and discussion; 2) it paves the way for practical applications of individual and distributed non-monotonic reasoning accommodating both computational and human agents.

The perspicuity of natural deduction makes it an obvious proof-theoretic choice for constructing the deductive parts of arguments, especially in light of argumentation theory's aim to make computational reasoning transparent for human inspection and interaction. We suggest that our more natural representation of classical proofs (as compared with standard Gentzen-style classical natural deduction) further supports this rationale for natural deduction representations of the deductive parts of arguments.

Moreover, as explored in D'Agostino and Modgil [2016, 2018], our formulation of NC-intelim$_k$ proofs has important implications for practical applications of argumentation, and more generally for applications of logic to modelling the inferential behaviour of non-ideal agents. Firstly, the computational tractability of constructing k-depth proofs can be exploited for use by real-world agents with limited inferential capabilities; each increase in depth naturally equates with the inferential resources that agents deploy in constructing proofs. Indeed, D'Agostino and Modgil [2018] show that key rationality postulates for argumentation [Caminada and Amgoud, 2007, Caminada et al., 2012], previously shown only under the assumption that agents have unbounded resources, are in fact satisfied for agents reasoning to any given depth k. Secondly, evaluation of arguments yields counterintuitive results, and violates rationality postulates, if arguments include assumptions that are redundantly used in deriving the conclusion, i.e., arguments that are contaminated. For example, a redundant assumption may inappropriately be accounted for in determining the weight/strength of an argument and hence its evaluation. Now, the standard approach to solving this problem is to ensure that arguments are not contaminated by verifying that an argument is valid only if it satisfies the stronger property that no proper subset of an argument's assumptions suffice to entail the conclusion. Clearly this is computationally impractical. Hence, there are good practical and theoretical reasons for ensuring that only non-contaminated arguments are delivered by a natural deduction proof theory. Again, D'Agostino and Modgil [2018] show that all rationality postulates hold when arguments are formalised as NC-intelim$_k$ proofs, and without requiring that one check that the arguments' assumptions are subset minimal.

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