A Dialectical Formalisation of Preferred Subtheories Reasoning Under Resource Bounds

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Abstract

Dialectical Classical Argumentation (Dialectical *Cl-Arg*) has been shown to satisfy rationality postulates under resource bounds. In particular, the consistency and non-contamination postulates are satisfied despite dropping the assumption of logical omniscience and the consistency and subset minimality checks on arguments' premises that are deployed by standard approaches to *Cl-Arg*. This paper studies Dialectical *Cl-Arg*'s formalisation of Preferred Subtheories (*PS*) non-monotonic reasoning under resource bounds. The contribution of this paper is twofold. First, we establish soundness and completeness for Dialectical *Cl-Arg*'s credulous consequence relation under the *preferred* semantics and credulous *PS* consequences. This result paves the way for the use of argument game proof theories and dialogues that establish membership of arguments in admissible (and so preferred) extensions, and hence the credulous *PS* consequences of a belief base. Second, we refine the non-standard characteristic function for Dialectical *Cl-Arg*, and use this refined function to show soundness for Dialectical *Cl-Arg* consequences under the grounded semantics and resource-bounded sceptical *PS* consequence. We provide a counterexample that shows that completeness does not hold. However, we also show that the grounded consequences defined by Dialectical *Cl-Arg* strictly subsume the grounded consequences defined by standard *Cl-Arg* formalisations of *PS*, so that we recover sceptical *PS* consequences that one would intuitively expect to hold.

1. Introduction

Background

Dung's argumentation theory [15] has proven to be a unifying framework for characterizing a large class of non-monotonic (*nm*) logics, in terms of evaluation of interacting arguments. A Dung framework (DF) consists of defeats amongst arguments

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Preprint submitted to Elsevier *September 11, 2024*

constructed from a belief base $\mathcal{B}$, where arguments consist of a conclusion derivable from premises in $\mathcal{B}$. Sets of jointly justifiable arguments (extensions) are then identified under various *argumentation semantics* [6, 15]. The conclusions of credulously or sceptically justified arguments under a semantics (s), yield argumentation defined credulous, respectively sceptical, *nm* consequences ϕ. These can be shown (e.g., see [4, 30]) to equate with the credulous, respectively sceptical, consequences of various non-monotonic logics $\mathcal{L}$, defined directly over $\mathcal{B}$:

> ϕ is an argumentation defined credulous (sceptical) consequence of $\mathcal{B}$ under a given semantics s *if* (completeness) *and only if* (soundness) ϕ is a credulous (sceptical) $\mathcal{L}$ consequence of $\mathcal{B}$. (1)

For example, [2, 29, 32] show that given a totally ordered belief base $\mathcal{B}$ of classical logical formulae, the conclusions of classical logic arguments in stable extensions of the DF constructed from $\mathcal{B}$ (i.e., the argumentation defined credulous consequences under the *stable* semantics) equate with the credulous consequences defined over $\mathcal{B}$ by the well known non-monotonic *Preferred Subtheories* formalism [8].

To accommodate the dynamics of *distributed nm* reasoning, various dialogical generalisations of argumentation have been developed (e.g., [18, 27, 31]). One can then aim at regulating exchange of agent locutions that conform to protocols, such that:

> the status of a communicated claim ϕ is 'winning' in the resultant dialogue graph of locutions *if and only if* ϕ is an argumentation defined credulous (sceptical) consequence of $\mathcal{B}$ (2)

where $\mathcal{B}$ are the beliefs *incrementally defined* by the declarative contents of exchanged locutions (rather than by a given static belief base $\mathcal{B}$ as in single agent reasoning).

Given argumentative characterizations of *nm* logics (1), establishing these dialogical results (2) thereby yields dialogical accounts of distributed *nm* reasoning, via protocols providing normative guidance for rational joint deliberation and decision making amongst human agents and human and AI agents. Indeed, enabling human-AI dialogue may be of particular importance if AI reasoning and decision making is to be aligned with human values (see [19, 21, 26]).

Enabling practical accounts of single agent and distributed *nm* reasoning implies desiderata for argumentative formalisations. In particular that they: D1) yield rational outcomes under resource bounds; D2) accommodate uses of argument typical of dialectical practice; D3) enable generalisation of argument game proof theories for single agent *nm* reasoning (e.g., [28]) to obtain dialogical accounts of joint reasoning.

Regarding D1), in order to satisfy consistency, closure and non-contamination rationality postulates [9, 10], standard approaches to Classical Argumentation (*Cl-Arg*, for short) [1, 20, 29] tacitly assume 'logical omniscience' in the sense that *all* arguments defined by a belief base $\mathcal{B}$ (of first-order formulae) are assumed to instantiate a DF. That is to say, it is assumed that ($\vdash_{\mathsf{CL}}$ denotes the entailment relation of classical logic):

C1) if $\Delta \subseteq \mathcal{B}, \Delta \vdash_{\mathsf{CL}} \alpha$, then resources suffice to construct the argument (Δ, ϕ).[2]

This assumption is clearly not practically feasible for agents with limited resources, given the undecidability of first order classical logic, and that even in the propositional case, deciding whether $\Delta \vdash_{\mathsf{CL}} \alpha$ is in general NP-hard, and therefore most likely intractable. Moreover, the intractability of deductive closure is further exacerbated by *Cl-Arg*, enforcing that arguments' premises are checked for consistency and subset-minimality. Namely, for each argument (Δ, ϕ):

C2) $\Delta \nvdash_{\mathsf{CL}} \bot$ and **C3**) there is no $\Delta' \subset \Delta$, $\Delta' \vdash_{\mathsf{CL}} \alpha$.

In fact, checking for subset minimlaity is a problem in the second level of the polynomial hierarchy [17]. However, **C1**, **C2** and **C3**, and so called 'reasonable' preference relations [29], are required to show that *Cl-Arg* satisfies the consistency, closure and non-contamination postulates.

Regarding D2), enforcing **C2** and **C3** additionally does not reflect real-world dialectical practice. Firstly, the inconsistency of arguments' premises is typically demonstrated dialectically – "you've contradicted yourself!" – as illustrated in the classic Socratic move [33]. Also, checking every argument for subset minimality, which enforces relevance of an argument's premises with respect to the conclusion (thus avoiding violation of non-contamination), is not what one expects of agents in practice.

Regarding D3), argument game proof theories for the preferred, as opposed to the stable semantics, more naturally lend themselves to dialogical generalisation. This is because membership of a stable extension requires a global accounting of all arguments in a DF. However, as stated above, *Cl-Arg*'s formalisation of credulous Preferred Subtheories (*PS*) consequence is shown under the stable semantics. Moreover, *Cl-Arg*'s formalisation of sceptical *PS* consequence can only be approximated under the grounded semantics. Completeness fails, *in part* because of limitations imposed on attacks in *Cl-Arg*. This then means that argument games and dialogues for the grounded semantics may fail to identify intuitively desirable sceptical *PS* consequences.

Dialectical Classical Logic Argumentation (Dialectical *Cl-Arg*) [12] has been developed in order to satisfy desiderata D1 and D2. Arguments adopt a distinction ubiquitous in dialectical practice: the *epistemic* distinction between premises that an agent commits to, and premises that can be supposed in virtue of their commitment by a (possibly imaginary in the case of single agent reasoning) interlocutor, thus anticipating dialogical formalisations of *nm* reasoning[3]. Dialectical *Cl-Arg* drops the omniscience assumption **C1**; only minimal assumptions are made as to the resources available for constructing arguments. Also, **C2** and **C3** are not enforced, and if arguments commit to inconsistent premises, they can be defeated by an argument that dialectically ('Socratically') demonstrates the inconsistency: "supposing only the premises you've committed to, you've contradicted yourself!" However, full rationality is preserved. The consistency and closure postulates are satisfied, and non-contamination is satisfied if

[2]Recall that *Cl-Arg* arguments consist of tuples referencing an argument's premises Δ and conclusion ϕ.

[3]Satisfaction of D1 and D2 by ASPIC+ [29] has also prompted adoption of this dialectical distinction for ASPIC+ arguments [14].

one either deploys proof theories that exclude use of irrelevant premises (e.g., the natural deduction proof systems formalised in [24, 25]), or else one enforces that arguments are not strengthened by inclusion of syntactically disjoint premises. The upshot of this novel approach is that an argumentative account of maxiconsistent *nm* consequence[4], and in particular *Preferred Subtheories* consequence [8], can be obtained that is *fully rational under resource bounds and accommodates the dialectical use of suppositions*, and is thus more suitable for real world use [12].

Contributions. This paper is concerned with desideratum D3. Our focus is on Dialectical *Cl-Arg*'s formalisation of *Preferred Subtheories* (*PS*) [8], which is amongst the most influential approaches to non-monotonic reasoning, as well as belief revision, and has also been used for reasoning about time, reasoning by analogy, reasoning with compactly represented preferences, judgement aggregation, and voting [23]. Section 2 recapitulates Dialectical *Cl-Arg* and its formalisation of resource bounded Preferred Subtheories reasoning.

In Section 3, we present this paper's first contribution. We show soundness and completeness – i.e., (1) above – for Dialectical *Cl-Arg*'s credulous consequence relation under the *preferred* semantics and credulous *PS* consequences [5]. This result paves the way for deploying argument game proof theories and their generalisation to multi-agent dialogues for establishing credulous *PS* consequences.

In Section 4, we present our second tranche of contributions. Specifically, Dialectical *Cl-Arg* employs a non-standard characteristic function, in the sense that it does not suffice that the function returns all arguments E' acceptable with respect to a given set E of dialectical arguments as in *Cl-Arg*. Rather, [12] shows that one must incorporate an additional step when defining the characteristic function, in order to preserve monotonicity of the function, and so obtain an iterative procedure for constructing the grounded extension. In this paper, we define an alternative characteristic function that simplifies the one defined in [12], and that, moreover, only requires evaluation of the acceptability of arguments with single premises. We deploy this newly defined characteristic function to show soundness for sceptical *PS* consequence: if ϕ is a Dialectical *Cl-Arg* consequence under the grounded semantics, then it is a sceptical *PS* consequence. We also provide a counterexample that shows that completeness does not hold: some sceptical *PS* consequences are not grounded consequences. Still, we demonstrate that the grounded consequences defined by Dialectical *Cl-Arg* strictly subsume the consequence relation defined by standard *Cl-Arg*'s formalisation of *PS*, hence recovering sceptical *PS* consequences that one would intuitively expect to hold. We show that this is because Dialectical *Cl-Arg* can simulate attacks on sets of premises; attacks that ordinarily (in *Cl-Arg*) result in violation of consistency [20]. We conclude with a conjecture as to how one might obtain completeness for Dialectical *Cl-Arg*'s formalisation of sceptical *PS* consequences. Finally, we discuss related work and conclude in Section

[4]That is to say, *nm* consequence relations defined over possibly inconsistent sets of classical formulae that arbitrate amongst conflicting classical consequences by appeal to a priority ordering over the formulae.

[5]This result was first presented in [13], in which the equivalence of preferred and stable extensions for both standard *Cl-Arg* and Dialectical *Cl-Arg* formalisations of *PS* is shown. The current paper extends [13] with the results in Section 4 for the grounded semantics and resource-bounded *PS* sceptical consequence.

5.

2. Dialectical Classical Logic Argumentation and Preferred Subtheories

Dialectical *Cl-Arg* assumes instantiation of a Dung framework (DF) *by some subset* of the classical logic arguments defined by a finite set of classical (propositional or first order) formulae (Definition 1 below). Attacks are defined in the usual way, but note that arguments are *not* checked for premise consistency and subset minimality. Then, the *epistemic* distinction between committed and supposed premises is only adopted when evaluating the acceptability of arguments (Definition 2 below).

Notation 1. In what follows, $\vdash_{\mathsf{CL}}$ denotes the classical consequence relation over a propositional or first order language $\mathcal{L}$. We let a belief base $\mathcal{B} \subseteq \mathcal{L}$ denote a finite set of classical wff. We use upper and lower case Greek letters as metavariables ranging over individual, respectively sets of formulas in $\mathcal{L}$, and write $\Delta \| \Gamma$ to indicate that $\Delta \subseteq \mathcal{L}$ is syntactically disjoint from $\Gamma \subseteq \mathcal{L}$ (see [12, p.20] for an exact definition). Furthermore, if ϕ is of the form $\neg\alpha$, then $-\phi = \alpha$, else $-\phi = \neg\phi$.

Given a (classical) argument $X = (\Delta, \alpha)$, we write $\mathtt{Conc}(X) = \alpha, \mathtt{Prem}(X) = \Delta$, to denote $X's$ conclusion, respectively premises. Uppercase Roman letters $\ldots, X, Y, Z$ are reserved to denote such arguments and we write $\ldots, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ to denote their respective 'epistemic variants' (specified in Definition 2).

Definition 1. *Let $\mathcal{B}$ be a finite belief base such that $\perp \notin \mathcal{B}$. Let $DF_{\mathcal{B}} = (\mathcal{A}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}})$ be a $\mathcal{B}$-based Dung Framework, where $\mathcal{A}_{\mathcal{B}} \subseteq \{(\Delta, \alpha) \mid \Delta \subseteq \mathcal{B}, \Delta \vdash_{\mathsf{CL}} \alpha\}$ is a set of $\mathcal{B}$-based arguments, and $\mathcal{C}_{\mathcal{B}} = \{(X,Y) | X, Y \in \mathcal{A}_{\mathcal{B}}, \mathtt{Conc}(X) = \phi, -\phi \in \mathtt{Prem}(Y)\}$ is an attack relation over $\mathcal{A}_{\mathcal{B}}$.*[6]

Definition 2. *Let $X = (\Delta, \alpha) \in \mathcal{A}_{\mathcal{B}}$. Then, $\mathbb{X} = (\Sigma, \Gamma, \alpha)$ is an* epistemic variant *of X iff $\mathtt{Prem}(X) = \Delta = \Sigma \cup \Gamma$ and $\Sigma \cap \Gamma = \emptyset$. In general, we refer to $\mathbb{X}$ as a* dialectical argument*.*

- $\mathtt{Conc}(\mathbb{X}) = \mathtt{Conc}(X)$ *and* $\mathtt{Com}(\mathbb{X})$ *denotes the committed premises (commitments) Σ of $\mathbb{X}$,* $\mathtt{Sup}(\mathbb{X})$ *the supposed premises (suppositions) Γ of $\mathbb{X}$.*
- $\|X\|$ *denotes the set of all epistemic variants of X, $\|E\|$ denotes $\bigcup_{X \in E} \|X\|$. Let $\mathcal{E}$ be a set of dialectical arguments. Then* $\mathtt{Com}(\mathcal{E})$, $\mathtt{Sup}(\mathcal{E})$ *and* $\mathtt{Conc}(\mathcal{E})$ *respectively denote* $\bigcup_{\mathbb{X} \in \mathcal{E}} \mathtt{Com}(\mathbb{X})$, $\bigcup_{\mathbb{X} \in \mathcal{E}} \mathtt{Sup}(\mathbb{X})$ *and* $\bigcup_{\mathbb{X} \in \mathcal{E}} \mathtt{Conc}(\mathbb{X})$.

We assume a strict preference relation $\prec$ over $\mathcal{A}_{\mathcal{B}}$ (which may or may not be defined by an ordering over $\mathcal{B}$). Then, when establishing whether $\mathbb{X} = (\Sigma, \Pi, \alpha)$ is defended by (acceptable w.r.t.) a set $\mathcal{E}$ of dialectical arguments, it is only the committed premises Σ that can be targeted. Furthermore, an attack by $\mathbb{Y} = (\Delta, \Gamma, \alpha)$ on $\beta \in \Sigma$ is contingent on the suppositions Γ of $\mathbb{Y}$ being commitments in $\mathbb{X}$ *and* $\mathcal{E}$, i.e., $\Gamma \subseteq \mathtt{Com}(\mathcal{E} \cup \{\mathbb{X}\})$. Intuitively:

[6] Other notions of attack for *Cl-Arg* can be defined, but alternatives result in violation of rationality postulates [20] and/or practical guidelines for argumentation [16].

"Given that I commit to Δ and supposing for the sake of argument your[7] commitments Γ in $\mathcal{E}$ and $\mathbb{X}$, I can construct an argument $\mathbb{Y}$ that challenges your premise $\beta \in \Sigma$."

Such an attack succeeds as a defeat only if $(\{\beta\}, \beta)$ is not strictly preferred to Y (recall that $\prec$ is defined over $\mathcal{A}_\mathcal{B}$). An argument of the form $\mathbb{Y} = (\Delta, \Gamma, \bot)$ can challenge $\mathbb{X}$ by arguing that the premises Γ committed in $\mathcal{E} \cup \{\mathbb{X}\}$, together with Δ, are inconsistent. $\mathbb{X}$ should only then be targeted if at least one of its committed premises β is in Γ and so is 'culpable' in contributing to the inconsistency. Again, $\mathbb{Y}$ defeats $\mathbb{X}$ if $Y \not\prec (\{\beta\}, \beta)$. However, if $\Delta = \emptyset$ then $\mathbb{Y}$ dialectically demonstrates a commitment to inconsistent premises Γ in $\mathcal{E} \cup \{\mathbb{X}\}$. To prefer that one commits to inconsistent premises is clearly incoherent. Therefore such an attack succeeds as a defeat independently of preferences. Finally, if $\mathbb{Y}$ challenges the acceptability of $\mathbb{X}$ w.r.t. $\mathcal{E}$, it is not required that $\mathbb{Y}$ itself be acceptable w.r.t. some set of dialectical arguments. Hence, $\mathbb{Z} \in \mathcal{E}$ can defend $\mathbb{X}$ by defeating $\mathbb{Y}$, while supposing only $\mathbb{Y}$'s commitments, i.e., $\texttt{Sup}(\mathbb{Z}) \subseteq \texttt{Com}(\mathbb{Y})$. The following definition makes the above formally precise.

Definition 3. *Let $DF_\mathcal{B} = (\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$, and $\prec$ a strict partial ordering over $\mathcal{A}_\mathcal{B}$. Let $\mathcal{E} \subseteq \|\mathcal{A}_\mathcal{B}\|$, $\mathbb{Y} = (\Delta, \Gamma, \phi) \in \|Y\|$, $\mathbb{X} = (\Pi, \Sigma, \psi) \in \|X\|$, $X, Y \in \mathcal{A}_\mathcal{B}$.*

1. *if $\phi \neq \bot$, then* $\mathbb{Y}$ defeats $\mathbb{X}$ *w.r.t. $\mathcal{E}$, denoted $\mathbb{Y} \Rightarrow_\mathcal{E} \mathbb{X}$, iff:*
   (a) $(Y, X) \in \mathcal{C}_\mathcal{B}$ *on* $X' = (\{-\phi\}, -\phi)$, $-\phi \in \texttt{Com}(\mathbb{X})$ *and* $Y \not\prec X'$*;*
   (b) $\Gamma \subseteq \texttt{Com}(\mathcal{E} \cup \{\mathbb{X}\})$.

   We say that $\mathbb{Y}$ defeats $\mathbb{X}$ on $\mathbb{X}' = (\{-\phi\}, \emptyset - \phi)$ or $\mathbb{Y}$ defeats $\mathbb{X}$ on $-\phi$.
2. *if $\phi = \bot$, then* $\mathbb{Y}$ defeats $\mathbb{X}$ *w.r.t. $\mathcal{E}$, denoted $\mathbb{Y} \Rightarrow_\mathcal{E} \mathbb{X}$, iff:*
   (a) $\Gamma \cap \texttt{Com}(\mathbb{X}) \neq \emptyset$ *and* $\Gamma \subseteq \texttt{Com}(\mathcal{E} \cup \{\mathbb{X}\})$*;*
   (b) *either $\Delta = \emptyset$ or $\forall \beta \in \Gamma \cap \texttt{Com}(\mathbb{X})$, $Y \not\prec (\{\beta\}, \beta)$.*

   We say $\mathbb{Y}$ defeats $\mathbb{X}$ on $\mathbb{X}' = (\{\beta\}, \emptyset, \beta)$ or $\mathbb{Y}$ defeats $\mathbb{X}$ on β, where $\beta \in \Gamma \cap \Pi$.

One may then adopt the usual definition of argumentation semantics, where acceptability is evaluated in terms of dialectical defeats [12].

Definition 4. *Let $\mathcal{E} \subseteq \|\mathcal{A}_\mathcal{B}\|$ and $X \in \mathcal{A}_\mathcal{B}$.*

- *$\mathbb{X} \in \|X\|$ is acceptable w.r.t. $\mathcal{E}$ iff $\forall \mathbb{Y} \in \|\mathcal{A}_\mathcal{B}\|$ s.t. $\mathbb{Y} \Rightarrow_\mathcal{E} \mathbb{X}$, $\exists \mathbb{Z} \in \mathcal{E}$ s.t. $\mathbb{Z} \Rightarrow_\emptyset \mathbb{Y}$*[8].
- *$\mathcal{E} \subseteq \|\mathcal{A}_\mathcal{B}\|$ is dialectical-conflict free iff $\neg\exists \mathbb{X}, \mathbb{Y} \in \mathcal{E}$ s.t. $\mathbb{Y} \Rightarrow_\mathcal{E} \mathbb{X}$.*
- *Let $\mathcal{E} \subseteq \|\mathcal{A}_\mathcal{B}\|$ be conflict free. Then $\mathcal{E}$ is: a dialectical-admissible extension iff $\forall \mathbb{X} \in \mathcal{E}$, $\mathbb{X}$ is acceptable w.r.t. $\mathcal{E}$; a dialectical-complete extension iff $\mathcal{E}$ is admissible and $\forall \mathbb{X} \in \|\mathcal{A}_\mathcal{B}\|$, $\mathbb{X}$ is acceptable w.r.t. $\mathcal{E}$ implies $\mathbb{X} \in \mathcal{E}$; the dialectical-grounded extension iff $\mathcal{E}$ is the minimal under set inclusion dialectical-complete*

[7] Recall that in the case of single agent reasoning, the 'your' can be thought of as referring to an imaginary interlocutor.

[8] By Definition 3, $\mathbb{Z} \Rightarrow_\emptyset \mathbb{Y}$ is equivalent to $\mathbb{Z} \Rightarrow_{\{\mathbb{Y}\}} \mathbb{Y}$ (since both require that $\texttt{Sup}(\mathbb{Z}) \subseteq \texttt{Com}(\mathbb{Y})$).

extension; a dialectical-preferred extension iff $\mathcal{E}$ is a maximal under set inclusion dialectical-complete extension; a dialectical-stable extension iff $\forall \mathbb{Y} \in \|\mathcal{A}_{\mathcal{B}}\| \setminus \mathcal{E}$, $\exists \mathbb{X} \in \mathcal{E}$ s.t. $\mathbb{X} \Rightarrow_{\emptyset} \mathbb{Y}$.

Remark 1. *Notice that no dialectical argument of the form $\mathbb{X} = (\emptyset, \Gamma, \phi)$ can be defeated, given the empty commitments, and so $\mathbb{X}$ is clearly acceptable w.r.t. any set of arguments, and hence is said to be '*unassailable*'.*

Example 1. *Let $\mathcal{B} = \{a, b, c, (a \wedge b) \rightarrow \neg c, g, g \rightarrow \neg a \vee \neg b\}$ where a, b, c are atoms respectively denoting 'attend conference A,B,C' and g denotes 'the conference budget = £2000.' In other words, $\mathcal{B}$ illustrates a scenario in which a university's limited conference budget precludes joint attendance at conferences A and B, but leaves open the possibility to attend conferences C and A, or C and B. Suppose the total ordering $\leq$ over $\mathcal{B}$ (with $<$ and $\approx$ defined in the usual way):*

$$c < a \approx b \approx (a \wedge b) \rightarrow \neg c < g \approx g \rightarrow \neg a \vee \neg b$$

Let $\mathcal{A}_{\mathcal{B}}$ be the arguments:

$A = (\{a\}, a)$	$G_1 = (\{g\}, g)$
$B = (\{b\}, b)$	$G_2 = (\{g \rightarrow \neg a \vee \neg b\}, g \rightarrow \neg a \vee \neg b)$
$C = (\{c\}, c)$	$G_3 = (\{g, g \rightarrow \neg a \vee \neg b\}, \neg a \vee \neg b)$
$D_1 = (\{(a \wedge b) \rightarrow \neg c\}, (a \wedge b) \rightarrow \neg c)$	$G_4 = (\{g, a, g \rightarrow \neg a \vee \neg b\}, \neg b)$
$D_2 = (\{a, b, (a \wedge b) \rightarrow \neg c\}, \neg c)$	$G_5 = (\{g, b, g \rightarrow \neg a \vee \neg b\}, \neg a)$
$H = (\{a, b, g \rightarrow \neg a \vee \neg b\}, \neg g)$	$F_1 = (\{a, b, g \rightarrow \neg a \vee \neg b, g\}, \bot)$
$F_2 = (\{a, b, (a \wedge b) \rightarrow \neg c, c\}, \bot)$	

We have attacks (G_4, B), (G_5, A), (D_2, C) and (H, G_1) and $G_4 \not\prec B, G_5 \not\prec A, D_2 \not\prec C, H \prec G_1, F_1 \prec G_1$ given the Elitist *preference relation [29] defined by an ordering over $\mathcal{B}$:*

$$(\Gamma, \phi) \prec (\Delta, \theta) \text{ iff } \exists \alpha \in \Gamma \text{ such that } \forall \beta \in \Delta, \alpha < \beta \qquad (\text{Eli})$$

As shown in Fig. 1, $\mathbb{D}_2 \Rightarrow_{\mathcal{E}} \mathbb{C}$ given $D_2 \not\prec C$, and $\mathbb{D}_2 \in \|D_2\|$, $\mathbb{C} \in \|C\|$, and (trivially) $\texttt{Sup}(\mathbb{D}_2) = \emptyset \subseteq \texttt{Com}(\mathcal{E} \cup \mathbb{C})$. Note the epistemic variants $\mathbb{G}'_4 \in \|G_4\|$ and $\mathbb{G}'_5 \in \|G_5\|$:

$$\mathbb{G}'_4 = (\{g, g \rightarrow \neg a \vee \neg b\}, \{a\}, \neg b), \mathbb{G}'_5 = (\{g, g \rightarrow \neg a \vee \neg b\}, \{b\}, \neg a)$$

are both in $\mathcal{E}$ given that $\mathbb{G}'_4$ and $\mathbb{G}'_5$ are undefeated (since $H \prec G_1$ and $F_1 \prec G_1$). Both $\mathbb{G}'_4$ and $\mathbb{G}'_5$ defeat $\mathbb{D}_2$, given that $\texttt{Sup}(\mathbb{G}'_4) = \{a\} \subseteq \texttt{Com}(\mathbb{D}_2)$ and $\texttt{Sup}(\mathbb{G}'_5) = \{b\} \subseteq \texttt{Com}(\mathbb{D}_2)$.
Notice also that $\mathbb{F}'_1 = (\{a, b, g \rightarrow \neg a \vee \neg b\}, \{g\}, \bot)$ but $\mathbb{F}'_1 \not\Rightarrow_{\mathcal{E}} \mathbb{G}_1$ since $F_1 \prec G_1$, and $\mathbb{F}''_1 = (\{a, b, g\}, \{g \rightarrow \neg a \vee \neg b\}, \bot)$ but $\mathbb{F}''_1 \not\Rightarrow_{\mathcal{E}} \mathbb{G}_2$ since $F_1 \prec G_2$.

To recap, we assume a DF instantiated by a *subset* of the classical logic arguments defined by $\mathcal{B}$. The arguments' premises *need not* be checked for subset minimality or consistency. Furthermore, only when determining acceptability do epistemic variants of these arguments deploy the commitment/supposition distinction characteristic of dialectical practice. Once the dialectical extensions are defined, only the conclusions of *unconditional* arguments – those committing to *all* their premises – identify the

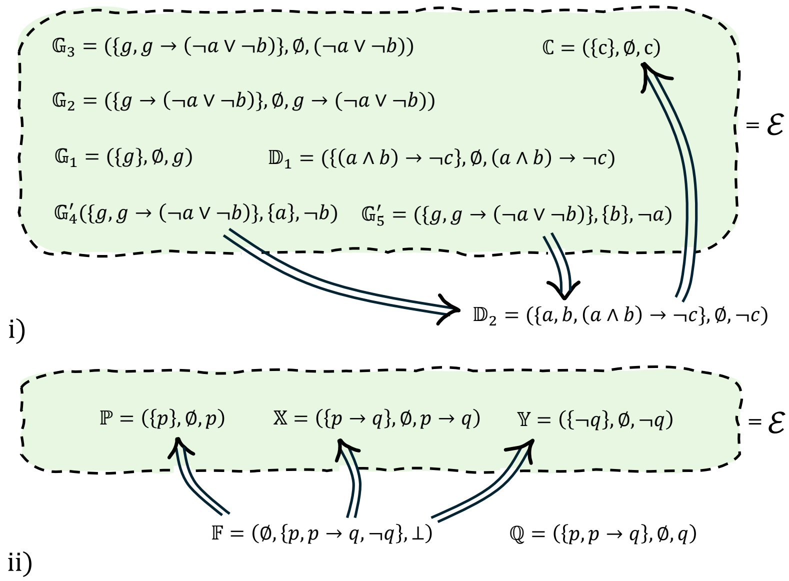

Figure 1: Figure i) shows a subset of the *dialectical*-grounded extension $\mathcal{E}$ for the conference attendance Example 1. Figure ii) illustrates the role of unassailable defeaters. $\mathcal{E}$ cannot be dialectical admissible, since $\mathbb{P}$, $\mathbb{X}$, and $\mathbb{Y}$ jointly commit to inconsistent premises, as demonstrated by the unassailable $\mathbb{F}$.

conclusions supported by the extensions. Hence, we identify a DF's s extensions ($s \in \{$admissible, complete, grounded, preferred, stable$\}$) by reference to the *unconditional* arguments in the DF's *dialectical* s extensions. Defintion 5 stipulates the resulting argumentation defined *nm* consequence relations.

Definition 5. *Let $\mathcal{E}$ be a dialectical-s extension of $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$. Then $E = \{(\Delta, \alpha) | (\Delta, \emptyset, \alpha) \in \mathcal{E}\}$ is an s extension of $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$. For $s \in \{grounded, preferred, stable\}$:*

- $\mathcal{B} \mid\!\sim^{cr}_{s} \phi$ *iff* $\exists$ *s extension E of $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$, $X \in E$, and $\texttt{Conc}(X) = \phi$.*
- $\mathcal{B} \mid\!\sim^{sc}_{s} \phi$ *iff* $\exists X$ *s.t.* $\forall$ *s extensions E of $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$, $X \in E$ and $\texttt{Conc}(X) = \phi$.*

In Example 1, $\mathcal{E}$ is the *dialectical*-grounded extension shown in Fig. 1i, and so the grounded extension is $\{G_1, G_2, G_3, D_1, C\}$. Intuitively, since the budget g precludes attendance at both conferences a and b (and neither A nor B are in the grounded extension since the ordering over $\mathcal{B}$ does not decide between the two), then attendance at c is justified since only if one attends both a and b is attendance at c precluded.

Consider Fig. 1-ii. As shown in [29] for standard *Cl-Arg*, ensuring no complete extension E contains $P = (\{p\}, p), X = (\{p \to q\}, p \to q), Y = (\{\neg q\}, \neg q)$, resources must suffice to construct $R = (\{p, \neg q\}, \neg(p \to q))$ and $Z = (\{\neg q, p \to q\}, \neg p)$ (i.e., arguments are 'closed under contraposition'), and one must assume a preference relation such that either $Z \not\prec P$, $R \not\prec X$ or $Q \not\prec Y$. Given such a preference relation, then one of the attacks from Z to P or R to X or Q to Y must succeed as a defeat.

But then defending against any one of these defeats requires an argument in E that defeats Z or R or Q. But then such an argument would defeat P or X or Y and thus E would not be conflict free, contradicting E is complete. However, suppose $Q \prec Y$ and resources are insufficient to construct R or Z. Then one cannot exclude E being a complete extension.

On the other hand, Dialectical *Cl-Arg* only requires that if resources suffice to recognise the joint inconsistency of a set of premises, by stint of constructing two arguments from these premises with conflicting conclusions, and resources suffice to combine the premises of these two arguments to obtain an argument concluding $\perp$, then no *admissible* extension can contain arguments committing to mutually inconsistent premises (or indeed arguments with conflicting conclusions). Thus, given that construction of Q and Y signals the inconsistency of $\{p, p \rightarrow \neg q, q\}$, it suffices to combine their premises to construct $F = (\{p, p \rightarrow \neg q, q\}, \perp)$. Then, $\mathcal{E}$ in Fig. 1ii cannot be *dialectical*-admissible (hence $\{P, X, Y\}$ cannot be admissible), since the unassailable epistemic variant $\mathbb{F}$ of F demonstrates that $\mathbb{P}, \mathbb{X}, \mathbb{Y}$ commit to inconsistency.

The non-contamination postulates [10] state that adding syntactically disjoint premises to $\mathcal{B}$ should not invalidate any of $\mathcal{B}$'s argumentation defined consequences. We illustrate what is meant by 'contamination'. Firstly, suppose $\mathcal{B} = \{s\}$. Hence $\mathcal{B} \mid\!\sim^{sc}_{gd} s$ ('*gd*' is short for 'grounded'). If upon adding $\{p, \neg p\}$ to $\mathcal{B}$ one were to license (in standard *Cl-Arg*) construction of $X = (\{p, \neg p\}, \neg s)$ which then defeats $(\{s\}, s)$, then $\mathcal{B} \cup \{p, \neg p\} \not\mid\!\sim^{sc}_{gd} s$. Secondly, suppose $\mathcal{B} = \{s, \neg s\}$ and $(\{\neg s\}, \neg s) \prec (\{s\}, s)$, so that $\mathcal{B} \mid\!\sim^{sc}_{gd} s$. If upon adding $\{r\}$ to $\mathcal{B}$, one licences construction of $Y = (\{r, \neg s\}, \neg s)$, and $(\{r, \neg s\}, \neg s) \not\prec (\{s\}, s)$, then $\mathcal{B} \cup \{r\} \not\mid\!\sim^{sc}_{gd} s$. Now of course, the consistency and subset minimality checks on standard *Cl-Arg* arguments precludes construction of X and Y, and so non-contamination is satisfied.

However, for the aforementioned reasons, Dialectical *Cl-Arg* does not enforce these premise checks. Rather if the 'explosively contaminating' $\mathbb{X} = (\{p, \neg p\}, \emptyset, \neg s)$ defeats $(\{s\}, \emptyset, s)$, the latter is then defended by $\mathbb{F} = (\emptyset, \{p, \neg p\}, \perp)$ defeating $\mathbb{X}$, and since $\mathbb{F}$ is unassailable, $\mathbb{F}$ is in the grounded extension. Moreover, [12] shows that non-contamination is satisfied if one deploys proof theories for classical logic that do not generate arguments – such as the 'redundantly contaminated' $Y = (\{r, \neg s\}, \neg s)$ – that incorporate syntactically disjoint redundant premises (e.g., the natural deduction proof theories in [24, 25]). On the other hand, if a proof theory is used that does generate such arguments, then non-contamination is satisfied if given an argument concluding α with syntactically disjoint subsets of premises Δ and Γ (denoted $\Delta \| \Gamma$), resources suffice to construct an argument concluding $\perp$ from Δ, or an argument concluding α from Γ[9], and adding syntactically disjoint redundant premises (e.g., r) to an argument (e.g., $(\{s\}s)$) does not strengthen arguments (i.e., $(\{\neg s\}, \neg s) \prec (\{s\}, s)$ implies $(\{r, \neg s\}, \neg s) \prec (\{s\}, s)$):

$$\forall X, Y, Y' \text{ s.t } Y = (\Gamma, \alpha), Y' = (\Delta \cup \Gamma, \alpha), \Delta \| \Gamma \cup \{\alpha\} : \text{ if } Y \prec X \text{ then } Y' \prec X \qquad \textbf{\textit{(RPref)}}$$

Finally, observe that [12] assumes that preference relations are *dialectically coher-*

[9][12, Proposition 30, p. 34] shows that if $\Delta \cup \Gamma \vdash_{\mathsf{CL}} \alpha$, and $\Delta \| \Gamma \cup \{\alpha\}$, then $\Delta \vdash_{\mathsf{CL}} \perp$ or $\Gamma \vdash_{\mathsf{CL}} \alpha$

ent. The idea is that $(\Delta, -\phi) \prec (\{\phi\}, \phi)$ can be interpreted as:
Given that it is rationally incoherent to commit to the inconsistent $\Delta \cup \{\phi\}$*, one would retain a commitment to* ϕ *in preference to retaining a commitment to all premises in* Δ.

Thus, $\prec$ is assumed to satisfy:

$$\forall(\Delta, \bot) : \exists\alpha \in \Delta \text{ s.t. } (\Delta, \bot) \not\prec (\{\alpha\}, \alpha) \qquad \textbf{\textit{(DCPref 1)}}$$

If $\prec$ does *not* satisfy *DCPref 1* (and assuming $\Delta \subseteq \texttt{Com}(\mathcal{E})$) then

$$\forall\alpha \in \Delta : (\Delta \setminus \{\alpha\}, \{\alpha\}, \bot) \not\Rightarrow_{\mathcal{E}} (\{\alpha\}, \emptyset, \alpha)$$

which constitutes an irrational preference for committing to an inconsistent set of premises Δ[10]. Moreover, our notion of dialectical coherence implies that one would expect that $\prec$ also satisfy:

$$\forall X = (\{\alpha\}, \alpha), \forall Y = (\Delta, -\alpha), \forall Y' = (\Delta \cup \{\alpha\}, -\phi)\ (\phi = \bot \text{ or } \phi = -\alpha):$$
$$Y \prec X \text{ implies } Y' \prec X \qquad \textbf{\textit{(DCPref 2)}}$$

since $Y \prec X$ implies a preferential commitment to α from amongst the inconsistent $\Delta \cup \{\alpha\}$, which would then be contradicted by $Y' \not\prec X$.

In summary, if $\prec$ satisfies *RPref*, *DCPref 1* and *DCPref 2*, then full rationality is satisfied by a DF $(\mathcal{A}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}})$ where $\mathcal{A}_{\mathcal{B}}$ is any subset of arguments defined by $\mathcal{B}$ that satisfy the conditionals P1, P2 and P3 below, each of which express that if the antecedent holds, then it is assumed that resources suffice to construct the arguments in the consequent.

Definition 6. *Let* $\mathcal{B}$ *be a set of classical wff, and* $(\mathcal{A}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}})$ *be a* DF *such that:*

P1 $\alpha \in \mathcal{B}$ *implies* $(\{\alpha\}, \alpha) \in \mathcal{A}_{\mathcal{B}}$.
P2 (Δ, α) *and* $(\Gamma, -\alpha) \in \mathcal{A}_{\mathcal{B}}$ *implies* $(\Delta \cup \Gamma, \bot) \in \mathcal{A}_{\mathcal{B}}$.
P3 $(\Delta \cup \Gamma, \alpha) \in \mathcal{A}_{\mathcal{B}}$ *and* $\Delta \| \Gamma \cup \{\alpha\}$*, implies* $(\Delta, \bot) \in \mathcal{A}_{\mathcal{B}}$ *or* $(\Gamma, \alpha) \in \mathcal{A}_{\mathcal{B}}$.[11]

Remark 2. *For the remainder of this article, we assume (unless stated otherwise) that given a belief base* $\mathcal{B}$*, the* DF *defined by* $\mathcal{B}$ *consists of arguments* $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{B}}$ *that satisfy P1-P3, in which case we say that the* DF *yields rational outcomes*[12].

Given an arbitrary set $\mathcal{A}_0$ *of classical logic arguments one can also define the closure of* $\mathcal{A}_0$ *under P1-P3 through the following stepwise procedure:*

- $\mathcal{A}_1 = \mathcal{A}_0 \cup \{(\{\alpha\}, \alpha) \mid \alpha \in \texttt{Prem}(\mathcal{A}_0)\}$;
- $\mathcal{A}_2 = \mathcal{A}_1 \cup \{(\Gamma, \bot) \mid (\Pi, \phi) \in \mathcal{A}_1, \Pi = \Delta \cup \Gamma,\ \Gamma \| \Delta \cup \{\phi\}$ *and* $\Gamma \vdash_{\mathsf{CL}} \bot\} \cup \{(\Delta, \phi) \mid (\Pi, \phi) \in \mathcal{A}_1, \Pi = \Delta \cup \Gamma,\ \Gamma \| \Delta \cup \{\phi\}$ *and* $\Delta \vdash_{\mathsf{CL}} \phi\}$;

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[10]As is also signified by the un-defendable defeats $(\emptyset, \Delta, \bot) \Rightarrow_{\mathcal{E}} \mathbb{X}$ (for any $\mathbb{X}$ s.t. $\Delta \cap \texttt{Com}(\mathbb{X}) \neq \emptyset$) which precludes $\mathcal{E}$ committing to Δ.

[11]Note that it is straightforward to formulate an algorithm that checks that a set of premises consists of syntactically disjoint subsets; one that runs in quadratic time c.f. checking for subset minimality.

[12]In [12, p. 27-28], Thms. 11 & 12 show mutual consistency of the conclusions and premises of arguments in *admissible* extensions, and Thms. 10 & 13 show sub-argument closure and closure under strict rules for complete extensions. Thms. 44 & 46 show satisfaction of the non-contamination postulates [12, p. 39&40].

- $\mathcal{A}_3 = \mathcal{A}_2 \cup \{(\Delta \cup \Gamma, \bot) \mid (\Delta, \phi) \in \mathcal{A}_1, (\Gamma, -\phi) \in \mathcal{A}_1\}$;
- $\mathcal{A}_4 = \mathcal{A}_3 \cup \{(\Gamma, \bot) \mid (\Pi, \phi) \in \mathcal{A}_3, \Pi = \Delta \cup \Gamma$, $\Gamma \| \Delta \cup \{\phi\}$ *and* $\Gamma \vdash_{\mathsf{CL}} \bot\} \cup \{(\Delta, \phi) \mid (\Pi, \phi) \in \mathcal{A}_3, \Pi = \Delta \cup \Gamma$, $\Gamma \| \Delta \cup \{\phi\}$ *and* $\Delta \vdash_{\mathsf{CL}} \phi\}$.

The resulting set is defined $\mathcal{A} = \bigcup_{0 \le i \le 4} \mathcal{A}_i$*. It can be straightforwardly checked that this procedures is exhaustive. Furthermore, it is easy to see that:*

Fact 1. *Closing a finite set of arguments* $\mathcal{A}$ *under P1-P3 generates a finite set* $\mathcal{A}'$*.*

We conclude this section by reviewing Dialectical *Cl-Arg*'s formalisation of Preferred Subtheories (*PS*) [8], which is amongst the most widely studied maxiconsistent approaches to *nm* reasoning. *PS* defines both sceptical and credulous *nm* consequences over a set $\mathcal{B}$ of classical formulae, stratified into equivalence classes $\mathcal{B}_1, \ldots, \mathcal{B}_n$ induced by a total ordering $\le$ over $\mathcal{B}$. A preferred subtheory (*ps*) is obtained by taking a maximal under set inclusion consistent subset (*mcs*) of $\mathcal{B}_1$, extending this to a *mcs* of $\mathcal{B}_1 \cup \mathcal{B}_2$, and so on. One can then define *PS*-based *nm* consequence by reference to any *resource-bounded* approximation $\vdash_r \subseteq \vdash_{\mathsf{CL}}$ of classical consequence.

Definition 7. *Let* $\le$ *be a total ordering over* $\mathcal{B}$*, and* $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ *the stratification of* $\mathcal{B}$ *such that* $\forall \alpha \in \mathcal{B}_i, \forall \beta \in \mathcal{B}_j : i < j$ *iff* $\beta < \alpha$*. Hence* $\alpha, \beta \in \mathcal{B}_i$ *iff* $\alpha \approx \beta$*.*

- *Let* $\vdash_r \subseteq \vdash_{\mathsf{CL}}$ *be any resource-bounded approximation of classical consequence s.t.:* $\forall \mathcal{B}$ *: 1) if* $\beta \in \mathcal{B}$ *then* $\mathcal{B} \vdash_r \beta$*; 2) if* $\mathcal{B} \vdash_r \beta$ *and* $\mathcal{B} \vdash_r \neg\beta$ *then* $\mathcal{B} \vdash_r \bot$*. We may also say that* $\vdash_r$ *is 'well-behaved'.* $\mathcal{B}$ *is said to be* r-inconsistent *iff* $\mathcal{B} \vdash_r \bot$*; else* r-consistent.
- *For any well-behaved* $\vdash_r \subseteq \vdash_{\mathsf{CL}}$*, a* r-preferred subtheory *(*rps*) defined by* $(\mathcal{B}, \le)$ *is a set* $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ *such that for* $i = 1, \ldots, n$*,* $\Sigma_1 \cup \cdots \cup \Sigma_i$ *is a maximal (under set inclusion)* r-consistent *subset of* $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_i$*.*
- *Let* $\Delta_1, \ldots, \Delta_m$ *be the r-preferred subtheories of* $(\mathcal{B}, \le)$*. Then* ϕ *is a credulous (sceptical)* rps*-consequence from* $(\mathcal{B}, \le)$*, denoted* $(\mathcal{B}, \le) \mid\!\sim^{cr}_{rps} \phi$ *(*$(\mathcal{B}, \le) \mid\!\sim^{sc}_{rps} \phi$*) iff* $\exists \Delta_i$*,* $\Delta_i \vdash_r \phi$ *(for* $1 \le i \le m$*:* $\Delta_i \vdash_r \phi$*).*

Remark 3. *It can then be shown that [12, Theorem 54] given a* DF *consisting of classical logic arguments* $\mathcal{A}$ *and attacks* $\mathcal{C}$ *as defined in Definition 1, with* $\vdash_r$ *substituting for* $\vdash_{\mathsf{CL}}$*, assuming the elitist principle lifting* $\le$ *to* $\prec$ *(as defined in Example 1), and the stable extensions of the* DF *as defined in Definition 5 (i.e., by reference to the unconditional arguments in the dialectical-stable extensions):*

$$\text{if } \Sigma \text{ is an r-preferred subtheory, } E = \{(\Delta, \alpha) | \Delta \subseteq \Sigma\} \text{ is a stable extension,} \quad (1)$$

$$\text{if } E \text{ is a stable extension, } \bigcup_{X \in E} \mathtt{Prem}(X) \text{ is an r-preferred subtheory,} \quad (2)$$

and so:

$$\mathcal{B} \mid\!\sim^{cr}_{\text{stable}} \phi \text{ iff } (\mathcal{B}, \le) \mid\!\sim^{cr}_{rps} \phi \quad (3)$$

Example 2. *(*Example 1 continued*) Suppose $\vdash_r$ such that for the base $\mathcal{B}$ in Example 1, $\mathcal{B} \vdash_r \alpha$, $\alpha \in \mathcal{B} \cup \{\neg a \vee \neg b, \neg b, \neg a, \neg c, \neg g, \bot\}$, and $\{g, g \rightarrow \neg a \vee \neg b, a, b\}$ and $\{a, b, (a \wedge b) \rightarrow \neg c, c\}$ are r-inconsistent. Given the ordering $\leq$ over $\mathcal{B}$ in Example 1:*

$\mathcal{B}_1$	g	$g \rightarrow \neg a \vee \neg b$	
$\mathcal{B}_2$	$(a \wedge b) \rightarrow \neg c$	a	b
$\mathcal{B}_3$		c	

one obtains the rps $\Delta_1 = \{g, g \rightarrow \neg a \vee \neg b, (a \wedge b) \rightarrow \neg c, a, c\}$ *and* $\Delta_2 = \{g, g \rightarrow \neg a \vee \neg b, (a \wedge b) \rightarrow \neg c, b, c\}$*. Hence:*

$$(\mathcal{B}, \leq) \mid\!\sim^{cr}_{rps} a \ , \ (\mathcal{B}, \leq) \mid\!\sim^{cr}_{rps} b, \text{ and}$$

$$(\mathcal{B}, \leq) \mid\!\sim^{sc}_{rps} c \ , \ (\mathcal{B}, \leq) \not\mid\!\sim^{sc}_{rps} a \ , \ (\mathcal{B}, \leq) \not\mid\!\sim^{sc}_{rps} b.$$

These consequences correspond to the conclusions of arguments in the stable extensions and grounded extension of the DF instantiated by the arguments in Example 1. Notice that the arguments in this DF satisfy P1, P2 and P3 in Definition 6. There are two stable extensions $\{G_1, G_2, G_3, G_4, D_1, A, C\}$ and $\{G_1, G_2, G_3, G_5, D_1, B, C\}$ and the grounded extension $\{G_1, G_2, G_3, D_1, C\}$.

3. Soundness and Completeness for Resource Bounded Preferred Subtheories Consequence and the Preferred Semantics

Section 2 recapitulated work on dialectical characterisations of maxiconsistent *nm* reasoning that yield rational outcomes under resource bounds. In particular, we recalled the correspondence between the conclusions of arguments in stable extensions of a DF and the credulous consequences of Preferred Subtheories defined by reference to resource bounded approximations $\vdash_r$ of classical logic. Indeed, for $\vdash_r = \vdash_{\mathsf{CL}}$, this result is also shown [1, 29, 32] for standard approaches to *Cl-Arg* that assume omniscience and enforce subset minimality and consistency checks on arguments' premises.

However, one would ideally identify credulous *PS* consequences with the conclusions of arguments in *preferred* extensions. This is because: 1) *any admissible extension is a subset of a preferred extension*[13], whereas the stable semantics requires a global accounting of *all* attacks/defeats in a DF. Hence, an admissible extension may not be a subset of a stable extension. For example, suppose a DF consisting of X and Y_1, Y_2, Y_3 where Y_1, Y_2 and Y_3 comprise an odd cycle of defeats. Then $\{X\}$ is admissible (and preferred), but the DF has no stable extension; 2) numerous works establish argument game proof theories and algorithms for deciding membership of admissible extensions (e.g., [28]); 3) argumentation based dialogues typically show membership of an argument in an admissible extension of the DF incrementally defined by the contents of locutions, (e.g., [31]), where variations on the dialogue protocol rules establish that the admissible extension is a subset of the grounded/a preferred extension.

Hence, it would be useful to show that credulous *PS* consequences correspond to the claims of arguments in preferred extensions so that one can then utilise the above

[13]This result is shown for Dung AFs in [15], and for dialectical DFs in [12, Proposition 22].

argument game proof theories and dialogues for individual and distributed (dialogical) credulous *PS* reasoning. We now establish such a correspondence.

We show soundness and completeness for the argumentation defined credulous preferred consequences ($\mid\!\sim^{cr}_{preferred}$) and credulous resource bounded *PS* consequences ($\mid\!\sim^{cr}_{rps}$, where given a totally ordered $\mathcal{B}$, the former is defined as in Remark 3, and the latter is defined as in Definition 7.

Note that in the following proofs we may write 'd-conflict free' and 'd-s extension' instead of 'dialectical-conflict free' and 'dialectical-s extension' for $s \in \{$*admissible, complete, grounded, preferred, stable*$\}$.

We start by proving some useful lemmas.

Lemma 1. *For any dialectical complete extension $\mathcal{E}$:*

$$\texttt{Com}(\mathcal{E}) = \bigcup_{(\Delta,\emptyset,\phi)\in\mathcal{E}} \Delta \tag{4}$$

Proof. It is easy to see that the committed premises in a dialectical-complete extension $\mathcal{E}$ are exactly those committed premises in the unconditional arguments. The result follows immediately from satisfaction of the sub-argument closure postulate ([12, Theorem 10]), which states that for any $(\Pi, \Sigma, \alpha) \in \mathcal{E}$, if $\beta \in \Pi$ then $(\{\beta\}, \emptyset, \beta) \in \mathcal{E}$. □

Corollary 1. *If $\mathcal{E}$ is a dialectical-complete extension as defined in Definition 3 and E is the complete extension defined by the unconditional arguments (Definition 5), then* $\texttt{Prem}(E) = \texttt{Com}(\mathcal{E})$.

Remark 4. *Observe that:*

1. *If $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$, then the suppositions $\texttt{Sup}(\mathbb{X})$ are a subset of the commitments $\texttt{Com}(\mathbb{Y})$ of $\mathbb{Y}$. Hence, for all sets of dialectical arguments $\mathcal{E}$, $\texttt{Sup}(\mathbb{X}) \subseteq \texttt{Com}(\mathcal{E} \cup \mathbb{Y})$, and so (by Definition 3-1.b and 3-2.a)), $\mathbb{X} \Rightarrow_\mathcal{E} \mathbb{Y}$.*
2. *By [12, Lemma 9], for any d-complete $\mathcal{E}$: $\forall \mathbb{Y}$ s.t. $\texttt{Com}(\mathbb{Y}) \subseteq \texttt{Com}(\mathcal{E})$, $\mathbb{Y} \in \mathcal{E}$.*

Lemma 2. *Let $(\mathcal{A}, \mathcal{C})$ be a DF and $E \subseteq \mathcal{A}$ be a complete extension: $\forall X \in \mathcal{A}$ such that $\texttt{Prem}(X) \subseteq \texttt{Prem}(E)$, $X \in E$.*

Proof. By Definition 3, $E = \{(\Delta, \alpha) | (\Delta, \emptyset, \alpha) \in \mathcal{E}\}$ and $\mathcal{E}$ is d-complete. Let $X = (\Gamma, \beta) \in \mathcal{A}$, $\Gamma \subseteq \texttt{Prem}(E)$. By Corollary 1, $\texttt{Prem}(E) = \texttt{Com}(\mathcal{E})$. Hence $\Gamma \subseteq \texttt{Com}(\mathcal{E})$. By Remark 4-2, $\mathbb{X} = (\Gamma, \emptyset, \beta) \in \mathcal{E}$. Hence $X \in E$. □

Lemma 3. *If E is a stable extension of $(\mathcal{A}, \mathcal{C})$ then E is a preferred extension of $(\mathcal{A}, \mathcal{C})$.*

Proof. $E = \{(\Delta, \phi) | (\Delta, \emptyset, \phi) \in \mathcal{E}\}$ where $\mathcal{E}$ is d-stable. Hence $\forall \mathbb{Y} \notin \mathcal{E}$, $\exists \mathbb{X} \in \mathcal{E}$ s.t. $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$. Suppose for contradiction that $\mathcal{E}$ is not d-preferred. Hence, $\exists \mathcal{E}' \supset \mathcal{E}$ s.t. $\mathcal{E}'$ is d-admissible (hence d-conflict free). Hence, $\exists \mathbb{Y} \notin \mathcal{E}$, $\mathbb{Y} \in \mathcal{E}'$. But then since $\mathcal{E}$ is d-stable, $\exists \mathbb{X} \in \mathcal{E}$ and so $\mathbb{X} \in \mathcal{E}'$ s.t. $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$ and so (by Remark 4-1), $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$, contradicting $\mathcal{E}'$ is d-conflict free. □

The above lemmas are used in proving our central claim: the credulous preferred semantics are sound and complete w.r.t. credulous *PS* consequences. This result paves the way for using argument games and dialogues (for the admissible semantics) in order to establish credulous *PS* consequences (recall desideratum D3 from Section 1).

Theorem 2. *Let $\mathcal{B}$ be a belief base of classical wff, $\leq$ a total ordering over $\mathcal{B}$. Let $\vdash_r \subseteq \vdash_{CL}$ be well-behaved. Then:*

$$\mathcal{B} \mid\!\sim^{cr}_{\text{preferred}} \phi \textit{ iff } (\mathcal{B}, \leq) \mid\!\sim^{cr}_{rps} \phi \tag{5}$$

Proof. Given Eq.3 ($\mathcal{B} \mid\!\sim^{cr}_{stable} \phi$ iff $(\mathcal{B}, \leq) \mid\!\sim^{cr}_{rps} \phi$) it suffices to show that

E is a stable extension iff E is a preferred extension.

Left-to-Right. By Lemma 3.

Right-to-Left. Let $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ be the stratification of $\mathcal{B}$ (see Definition 7). Let E be a preferred extension of the DF $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$. Let $\Sigma = \texttt{Prem}(E)$. By Lemma 2, $E = \{(\Delta, \alpha) | \Delta \subseteq \Sigma\}$. Hence, by Eq. 2 in Remark 3 (i.e., Σ is a r-preferred subtheory (*rps*) implies $E = \{(\Delta, \alpha) | \Delta \subseteq \Sigma\}$ is a stable extension), it suffices to show that Σ is a *rps*. That is to say, given

$$E = \{(\Delta, \phi) | (\Delta, \emptyset, \phi) \in \mathcal{E}\} \text{ where } \mathcal{E} \text{ is } d\text{-preferred} \tag{6}$$

and given (Corollary 1) $\Sigma = \texttt{Prem}(E) = \texttt{Com}(\mathcal{E})$, then it suffices to show that

$$\Sigma = \texttt{Com}(\mathcal{E}) \text{ is a r-preferred subtheory.} \tag{7}$$

We show Σ is r-consistent. Suppose otherwise. Then $\Sigma \vdash_r \bot$ and $\exists \Sigma' \subseteq \Sigma$ such that (by [12, Lemma 9]) $(\Sigma', \emptyset, \bot) \in \mathcal{E}$, contradicting *direct consistency* ([12, Theorem 11]). **We show that Σ is *maximally* r-consistent**. Let $\Sigma = \Sigma_1, \ldots, \Sigma_n$ where for $i = 1, \ldots, n$, $\Sigma_i = \texttt{Com}(\mathcal{E}) \cap \mathcal{B}_i$. Suppose Σ is not *maximally* r-consistent. We show that this contradicts $\mathcal{E}$ is d-preferred. Without loss of generality assume that for $j = 1 \ldots i-1$:

$$\Sigma_1 \cup \cdots \cup \Sigma_{i-1} \; is \; a \; maximal \; r\text{-}consistent \; subset \; of \; \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{i-1} \tag{8}$$

$$\big(\bigcup_{j=1}^{i-1} \Sigma_j\big) \cup \Sigma_i \; is \; r\text{-}consistent \; but \; not \; maximally \; r\text{-}consistent \tag{9}$$

Hence let $\{\delta_1, \ldots, \delta_m\}$ be any non-empty maximal subset of $\mathcal{B}_i$ not in Σ_i but that is r-consistent with $\Sigma_1 \cup \cdots \cup \Sigma_i$. That is, let

$$\{\delta_1, \ldots, \delta_m\} \text{ be any maximal subset of } \mathcal{B}_i \setminus \Sigma_i \tag{10}$$

such that:

$$\bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\} \nvdash_r \bot. \tag{11}$$

We can then show that expanding $\mathcal{E}$ with arguments whose committed premises include formulae in $\{\delta_1, \ldots, \delta_m\}$, we obtain a $\mathcal{E}' \supset \mathcal{E}$ that is d-admissible, contradicting $\mathcal{E}$ is d-preferred (i.e., maximally admissible). Let:

$$\mathcal{E}' = \{(\Delta, \Phi, \phi) | \Delta \subseteq \texttt{Com}(\mathcal{E}) \cup \{\delta_1, \ldots, \delta_m\}, \Delta \cup \Phi \vdash_r \phi\} \tag{12}$$

We show that $\mathcal{E}'$ is d-admissible:

1. $\mathcal{E}'$ **is** d**-conflict free**. Suppose for contradiction that $\mathbb{X}, \mathbb{Y} \in \mathcal{E}'$, $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$ on $\mathbb{B} = (\{\beta\}, \emptyset, \beta)$ and so $X \not\prec B$ and $\mathtt{Com}(\mathbb{X}) \cup \mathtt{Sup}(\mathbb{X}) \cup \{\beta\} \subseteq \mathtt{Com}(\mathcal{E}')$.

1.1 Suppose $\beta \in \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$ Then, given $X \not\prec B$, it must be that $\mathtt{Com}(\mathbb{X}) \cup \mathtt{Sup}(\mathbb{X}) \subseteq \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$[14]. If $\mathtt{Conc}(\mathbb{X}) = \bot$ then this contradicts Eq. 11. If $\mathtt{Conc}(\mathbb{X}) = -\beta$, then $\beta, -\beta$ and so $\bot$ are all $\vdash_r$ entailed from $\bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$, contradicting Eq. 11.

1.2 Suppose $\beta \in \bigcup_{j=i+1}^{n} \Sigma_i$ and so $\beta \in \mathtt{Com}(\mathcal{E})$. Let $\mathbb{X} = (\Pi, \Omega, \phi)$.

1.2.1 Suppose $(\Pi \cup \Omega) \cap \{\delta_1, \ldots, \delta_m\} = \emptyset$.
By Eq. 12 and $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$: $\Pi \cup \Omega \subseteq \mathtt{Com}(\mathcal{E})$. Let $\mathbb{X}' = (\Pi \cup \Omega, \emptyset, \phi)$. By [12, Lemma 9], $\mathbb{X}' \in \mathcal{E}$ and $\mathbb{B} \in \mathcal{E}$. But then $\mathbb{X}' \Rightarrow_{\mathcal{E}} \mathbb{B}$, contradicting $\mathcal{E}$ is d-conflict free. Hence:

1.2.2 Suppose $(\Pi \cup \Omega) \cap \{\delta_1, \ldots, \delta_m\}) = \{\delta_i, \ldots, \delta_k\}$.
We have $\mathbb{X}' = (\{\delta_i, \ldots, \delta_k\}, ((\Pi \cup \Omega) \setminus \{\delta_i, \ldots, \delta_k\}), \phi) \in \|X\|$.
Since $(\Pi \cup \Omega) \setminus \{\delta_i, \ldots, \delta_k\} \subseteq \mathtt{Com}(\mathcal{E})$ and $\beta \in \mathtt{Com}(\mathcal{E})$, we have that $\mathbb{X}' \Rightarrow_{\mathcal{E}} \mathbb{Y}'$ on β, for some $\mathbb{Y}' \in \mathcal{E}$.
Since $\mathcal{E}$ is d-admissible, $\exists \mathbb{Z} \in \mathcal{E}$ s.t. $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{X}'$ on some $\delta \in \{\delta_i, \ldots, \delta_k\}$. Hence $Z \not\prec (\{\delta\}, \delta)$ and so $\mathtt{Com}(\mathbb{Z}) \cup \mathtt{Sup}(\mathbb{Z}) \subseteq \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$ (again, by the definition of $\prec$; recall Footnote 14).
If $\mathtt{Conc}(\mathbb{Z}) = \bot$ this contradicts Eq. 11. If $\mathtt{Conc}(\mathbb{Z}) = -\delta$, then $-\delta, \delta$ and so $\bot$ are $\vdash_r$ entailed by $\bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$, contradicting Eq. 11.

2. If $\mathbb{Y} \in \mathcal{E}'$ **then** $\mathbb{Y}$ **is acceptable w.r.t.** $\mathcal{E}'$. Suppose $\mathbb{X} = (\Delta, \Psi, \phi)$ defeats $\mathbb{Y}$. There are two cases: $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$ on (1) $\gamma \in \mathtt{Com}(\mathcal{E})$ or (2) $\gamma \notin \mathtt{Com}(\mathcal{E})$.

2.1) $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$ on $\gamma \in \mathtt{Com}(\mathcal{E})$. By [12, Lemma 9]: $\mathbb{Y}' = (\{\gamma\}, \emptyset, \gamma) \in \mathcal{E}$. Consider two cases:

2.1.1) $\mathbb{X} \Rightarrow_{\mathcal{E}} \mathbb{Y}'$. By admissibility of $\mathcal{E}$, $\exists \mathbb{Z} \in \mathcal{E}$ (hence $\mathbb{Z} \in \mathcal{E}'$) such that $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{X}$, and so $\mathbb{Y}$ is acceptable w.r.t. $\mathcal{E}'$.

2.1.2) $\mathbb{X} \not\Rightarrow_{\mathcal{E}} \mathbb{Y}'$. Since $\{\delta_1, \ldots, \delta_m\} = \mathtt{Com}(\mathcal{E}' \setminus \mathcal{E})$ (recall Eq. 10 and Eq. 12) and by assumption $\mathbb{X} \Rightarrow_{\mathcal{E}'} \mathbb{Y}$, there must be some $\delta_i \in (\mathtt{Sup}(\mathbb{X}) = \Psi)$. Let:

$$\{\delta_i, \ldots, \delta_k\} = \Psi \cap \{\delta_1, \ldots, \delta_m\} \tag{13}$$

and:

$$\mathbb{X}' = (\Delta \cup \{\delta_i, \ldots, \delta_k\}, \Psi \setminus \{\delta_i, \ldots, \delta_k\}, \phi) \text{ where } \mathbb{X}' \Rightarrow_{\mathcal{E}} \mathbb{Y}' \tag{14}$$

By admissibility of $\mathcal{E}$:

$$\exists \mathbb{Z} = (\Pi, \Phi, \gamma) \in \mathcal{E}, \mathbb{Z} \Rightarrow_{\emptyset} \mathbb{X}' \textit{ and so } (\mathtt{Sup}(\mathbb{Z}) = \Phi) \subseteq (\Delta \cup \{\delta_i, \ldots, \delta_k\}) \tag{15}$$

There are two possibilities, $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{X}'$ on $\delta \in \{\delta_i, \ldots, \delta_k\}$ or $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{X}'$ on $\alpha \in \Delta$:

[14] Recall the definition of the Elitist $\prec$ in Example 1. Given $\beta \in \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$, then $\exists \gamma \in \mathtt{Com}(\mathbb{X}) \cup \sup(\mathbb{X}) \notin \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\}$ implies $X \prec B$.

2.1.2.1) Suppose $\mathbb{Z} \Rightarrow_\emptyset \mathbb{X}'$ on $\delta \in \{\delta_i, \ldots, \delta_k\}$. Hence $Z \not\prec (\{\delta\}, \delta)$, and so given $\delta \in \mathcal{B}_i$ it must be that $\Pi \cup \Phi \subseteq \bigcup_{j=1}^{i} \Sigma_j$. If $\texttt{Conc}(\mathbb{Z}) = \gamma = \bot$, then $\bigcup_{j=1}^{i} \Sigma_i \cup \{\delta_1, \ldots, \delta_m\} \vdash_r \bot$, contradicting Eq. 11. If $\gamma = -\delta$, then $-\delta, \delta$ and so $\bot$ are $\vdash_r$ entailed by $\bigcup_{j=1}^{i} \Sigma_i \cup \{\delta_1, \ldots, \delta_m\}$, contradicting Eq. 11. Hence:

$$\exists \beta \in (\texttt{Sup}(\mathbb{Z}) = \Phi) \text{ s.t. } \beta \in \bigcup_{j=1}^{i} (\mathcal{B}_j \setminus (\Sigma_j \cup \{\delta_1, \ldots, \delta_m\})) \qquad (16)$$

(since if $\beta \in \mathcal{B}_{j>i}$ this would mean $Z \prec (\{\delta\}, \delta)$, invalidating $\mathbb{Z} \Rightarrow_\emptyset \mathbb{X}'$). Moreover, since $\beta \notin \{\delta_1, \ldots, \delta_m\}$, then $\beta \notin \{\delta_i, \ldots, \delta_k\}$, and so recalling Eq. 15:

$$\exists \beta \in (\texttt{Sup}(\mathbb{Z}) = \Phi) \text{ s.t. } \beta \in \Delta \qquad (17)$$

If **(i)** $\beta \notin \mathcal{B}_i$, then (recalling Eq. 16) by Eq. 8, $\bigcup_{j=1}^{i-1} \Sigma_j \cup \{\beta\} \vdash_r \bot$, else;
If **(ii)** $\beta \in \mathcal{B}_i$, then by Eqs. 16, 10 & 11: $\bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\} \cup \{\beta\} \vdash_r \bot$.
In either case:

$$\exists \Omega \subseteq \bigcup_{j=1}^{i} \Sigma_j \cup \{\delta_1, \ldots, \delta_m\} \subseteq \texttt{Com}(\mathcal{E}') \text{ s.t. } \Omega \cup \{\beta\} \vdash_r \bot \qquad (18)$$

Recall from Eq. 12 that $\texttt{Com}(\mathcal{E}') = \texttt{Com}(\mathcal{E}) \cup \{\delta_1, \ldots, \delta_m\})$. Hence, by construction of $\mathcal{E}'$, $(\Omega, \{\beta\}, \bot) \in \mathcal{E}'$. Moreover, $(\Omega, \{\beta\}, \bot) \Rightarrow_\emptyset \mathbb{X}'$ on $\beta \in \Delta$ (recall that $\beta \in \Delta$ by Eq. 17, and given Eq. 16, $(\Omega, \{\beta\}, \bot) \not\prec (\{\beta\}, \emptyset, \beta)$). Hence $(\Omega, \{\beta\}, \bot) \Rightarrow_\emptyset \mathbb{X}$ on $\beta \in (\texttt{Com}(\mathbb{X}) = \Delta)$. Hence $\mathbb{Y}$ is acceptable w.r.t. $\mathcal{E}'$.

2.1.2.2) Suppose $\mathbb{Z} \Rightarrow_\emptyset \mathbb{X}'$ on $\alpha \in \Delta$. By construction of $\mathcal{E}'$, $\mathbb{Z}' = (\Pi \cup \{\delta_i, \ldots, \delta_k\}, \Phi \backslash \{\delta_i, \ldots, \delta_k\}, \gamma) \in \mathcal{E}'$. Since $(\Phi \backslash \{\delta_i, \ldots, \delta_k\}) \subseteq \Delta$ (by Eq. 15), $\mathbb{Z}' \Rightarrow_\emptyset \mathbb{X}$ on α. Hence $\mathbb{Y}$ is acceptable w.r.t. $\mathcal{E}'$.

2.2) Suppose $\mathbb{X} = (\Delta, \Psi, \phi) \Rightarrow_{\mathcal{E}'} \mathbb{Y} \in \mathcal{E}'$, on $\gamma \notin \texttt{Com}(\mathcal{E})$. Hence γ is some $\delta \in \{\delta_1, \ldots, \delta_m\}$, and so it must be that $\Delta \cup \Psi \subseteq \bigcup_{j=1}^{i} \mathcal{B}_j$ (else $X \prec (\{\delta\}, \delta)$).
Then either $\Delta \cup \Psi \cup \{\delta\} \vdash_r \bot$ (if $\phi = \bot$) or $\Delta \cup \Psi \cup \{\delta\} \vdash_r -\delta, \delta$ and so $\bot$ (if $\phi = -\delta$). Then, as in **(i)** and **(ii)** above, $\exists (\Omega, \{\beta\}, \bot) \in \mathcal{E}'$ such that $(\Omega, \{\beta\}, \bot) \Rightarrow_\emptyset \mathbb{X}$ on $\beta \in \texttt{Com}(\mathbb{X}) = \Delta$, and $\mathbb{Y}$ is acceptable w.r.t. $\mathcal{E}'$.

We have shown that $\mathcal{E}'$ is d-admissible. This contradicts the assumption that $\mathcal{E}$ is d-preferred. Hence, Σ is maximally r-consistent. Therefore Σ is a r-preferred subtheory. We have shown Equation 7 and the theorem is shown. □

4. Results for the Grounded Semantics and Sceptical Preferred Subtheories Consequence

This section presents three contributions to the study of grounded semantics for Dialectical Classical Argumentation. The first defines a novel characteristic function based on the notion of *defensible premises* that is then used in constructing the dialectical grounded extension. We then establish soundness for the grounded semantics. That is to say, if ϕ is an argumentation defined grounded consequence over $\mathcal{B}$, then ϕ is a sceptical consequence of *PS* over $\mathcal{B}$. We then discuss why the right to left direction (completeness) does not obtain for both standard *Cl-Arg* and Dialectical *Cl-Arg* formalisations of *PS* consequence. Nevertheless, we show that Dialectical *Cl-Arg*'s use of suppositions means that one can simulate attacks targeting sets of premises; the upshot being that the Dialectical *Cl-Arg* grounded consequence recovers sceptical *PS* consequences that are not obtained using standard *Cl-Arg* grounded consequence.

4.1. Defining a Characteristic Function and Construction of the Grounded Extension

The notion of 'epistemic maximality' plays a key role in showing results for Dialectical Cl-Arg [12]. Epistemic maximality recognises that given the epistemic distinction between committed and supposed premises, then if $\mathbb{A} = (\Delta, \Gamma, \alpha) \in \mathcal{E}$ and $\mathcal{E}$ also includes some $\mathbb{B}$ that commits to a subset $\Gamma' \subseteq (\texttt{Sup}(\mathbb{A}) = \Gamma)$ then an agent having committed to Γ' in $\mathcal{E}$, would be expected to include in $\mathcal{E}$ the epistemic variant $\mathbb{A}' = (\Delta \cup \Gamma', \Gamma \setminus \Gamma', \alpha)$. For example, if $\{(\{p\}, \{p \rightarrow q\}, (\{p \rightarrow q\}, \emptyset, p \rightarrow q)\} \subseteq \mathcal{E}$, then epistemic maximality requires that $\mathcal{E}$ also include $(\{p, p \rightarrow q\}, \emptyset, q)$ (assuming resources suffice to construct).

Definition 8. *Let* $(\mathcal{A}, \mathcal{C})$ *be a* DF. *Let the* epistemically maximal closure *be a function* $\mathcal{C}l_{em}$*:* $\mathcal{P}(\|\mathcal{A}\|) \mapsto \mathcal{P}(\|\mathcal{A}\|)$*:*

$$\mathcal{C}l_{em}(\mathcal{E} \subseteq \|\mathcal{A}\|)=\{\mathbb{X}'\in\|X\| \mid X \in \mathcal{A}, \mathbb{X}\in\|X\|, \mathbb{X} \in \mathcal{E}, \texttt{Com}(\mathbb{X}) \subseteq \texttt{Com}(\mathbb{X}') \subseteq \texttt{Com}(\mathcal{E})\}$$

Let $\mathcal{E} = \mathcal{C}l_{em}(\mathcal{E})$*. Then* $\mathcal{E}$ *is epistemically maximal.*

Dung's Fundamental Lemma [15, Lemma 10] – if X, X' are acceptable w.r.t. an admissible E then $E \cup \{X\}$ is admissible and X' is acceptable w.r.t. E' – is then shown to hold only for *epistemically maximal* d-admissible extensions in[15].

Epistemic maximality also plays an important role in defining a DF's characteristic function. Recall that in [15], a DF's characteristic function – F_{DF} – returns the set of arguments acceptable w.r.t. a set E of arguments, and so the fixed points of F_{DF} are the DF's complete extensions. F_{DF} is monotonic, and so the grounded extension of a *finitary* DF $(\mathcal{A}, \mathcal{C})$ (in which each $X \in \mathcal{A}$ is attacked by a finite number of arguments) can be constructively defined using F_{DF}. Starting with the empty set, the iteration of F_{DF} yields the least fixed point (*lfp*) of F_{DF} which is the grounded extension $G(DF)$.

Analogously, the least fixed points of the characteristic function $\mathcal{F}_{DF}$ (henceforth, we will omit the subscript DF) defined over sets of dialectical arguments

$$\mathcal{F}(\mathcal{E} \subseteq \|\mathcal{A}\|) = \{\mathbb{X} \in \|\mathcal{A}\| \mid \mathbb{X} \text{ is acceptable w.r.t. } \mathcal{E}\} \tag{19}$$

identify the DF's d-complete extensions. However, $\mathcal{F}$ is *not monotonic* for arbitrary sets of dialectical arguments, but only for *epistemically maximal* d*-admissible* sets. Hence, the composed function

$$\mathcal{F}_p = \mathcal{C}l_{em} \circ \mathcal{F}$$

[15] Hence, the results implied by the Fundamental Lemma are also shown to hold for Dialectical *Cl-Arg*; in particular that the epistemically maximal d-admissible extensions form a complete partial order w.r.t. set inclusion, and so any (d-)admissible extension is a subset of a (d-)preferred extension.

applies $\mathcal{F}$ and then closes under epistemic maximality. The fixed points (including of course the *lfp* / d-grounded extension) of $\mathcal{F}_p$ and $\mathcal{F}$ coincide ([12, Eq.4, Proposition 25]):

$$\mathcal{E}\ is\ a\ fixed\ point\ of\ \mathcal{F}_p \text{ iff } \mathcal{E}\ is\ a\ fixed\ point\ of\ \mathcal{F} \tag{20}$$

Moreover, (unlike $\mathcal{F}$) iterative application of $\mathcal{F}_p$, starting with $\emptyset$, *does* yield a monotonically increasing sequence of epistemically maximal d-admissible extensions [12, Corollary 24], resulting in construction of the d-grounded extension. For all intents and purposes one can therefore adopt $\mathcal{F}_p$ as a DF's dialectical characteristic function.

In this section we propose an alternative dialectical characteristic function $\mathcal{F}_+$, which returns only the set of acceptable arguments that commit to a single premise, i.e., *elementary dialectical arguments*[16], and then additionally includes any dialectical arguments whose committed premises are committed by the returned set of elementary arguments. One can then show that the fixed points of $\mathcal{F}_+$ are the fixed points of $\mathcal{F}$, and that $\mathcal{F}_+$ is monotonic when applied to d-admissible sets, and so yields the d-grounded extension when iterating from $\emptyset$.

Definition 9. *An* elementary dialectical argument $\mathbb{X}$ *is of the form* $(\{\alpha\}, \emptyset, \alpha)$. *If any such* $\mathbb{X}$ *is acceptable w.r.t. a set* $\mathcal{E}$ *of dialectical arguments, we say that* $\mathbb{X}$ *is* $\mathcal{E}$-defensible*, and define:*

$$\text{Def}(\mathcal{E}) = \{(\{\alpha\}, \emptyset, \alpha) \mid (\{\alpha\}, \emptyset, \alpha) \textit{ is } \mathcal{E}\textit{-defensible}\}$$

Henceforth, we may write α_p *as an abbreviation for* $(\{\alpha\}, \emptyset, \alpha)$ *and may refer to* $\mathcal{E}$*-defensible elementary arguments as* $\mathcal{E}$*-defensible 'premises'.*

Remark 5. *Let* $(\mathcal{A}, \mathcal{C})$*,* $\mathcal{E} \subseteq \|\mathcal{A}\|$*, and* $\phi_p \in \text{Def}(\mathcal{E})$*. It is obviously the case that* $\forall \mathbb{X} \in \|\mathcal{A}\|$ *s.t.* $\texttt{Com}(\mathbb{X}) = \{\phi\}$*,* $\mathbb{X}$ *is acceptable w.r.t.* $\mathcal{E}$*.*

Definition 10. *Let* $(\mathcal{A}, \mathcal{C})$ *be a* DF *and* $\mathcal{E} \subseteq \|\mathcal{A}\|$*. The characteristic function* $\mathcal{F}_+ : \mathcal{P}(\|\mathcal{A}\|) \mapsto \mathcal{P}(\|\mathcal{A}\|)$ *is defined as follows:*

$$\mathcal{F}_+(\mathcal{E}) = \{\mathbb{X} \in \|\mathcal{A}\| \mid \forall \phi \in \texttt{Com}(\mathbb{X}), (\{\phi\}, \emptyset, \phi) \in \text{Def}(\mathcal{E})\ \}$$

$\mathcal{F}_+$ closes a set under all dialectical arguments built entirely from premises whose corresponding elementary arguments are defended by that set. We show that the fixed points of $\mathcal{F}_+$ and $\mathcal{F}_p$ (and hence $\mathcal{F}$; recall Equations 19 and 20) coincide, and that $\mathcal{F}_+$ can be used to construct the d-grounded extension.

Proposition 1. *Let* $(\mathcal{A}, \mathcal{C})$ *be a* DF *and* $\mathcal{E} \subseteq \|\mathcal{A}\|$ *be* d*-admissible. Then* $\mathcal{F}_+(\mathcal{E}) = \mathcal{F}_p(\mathcal{E})$*.*

Proof. Left-to-Right. Let $\mathbb{A} \in \mathcal{F}_+(\mathcal{E})$. Hence, $\forall \phi \in \texttt{Com}(\mathbb{A})$: $\phi_p \in$ *Def*$(\mathcal{E})$ and so $\phi \in \texttt{Com}(\mathcal{F}(\mathcal{E}))$. Trivially, the unassailable $\mathbb{A}' = (\emptyset, \texttt{Com}(\mathbb{A}) \cup \texttt{Sup}(\mathbb{A}), \texttt{Conc}(\mathbb{A})) \in \|\mathcal{A}\|$, $\mathbb{A}' \in \mathcal{F}(\mathcal{E})$. Since $\texttt{Com}(\mathbb{A}) \subseteq \texttt{Com}(\mathcal{F}(\mathcal{E}))$ and $\mathbb{A}' \in \mathcal{F}(\mathcal{E})$, then by Definition 8, $\mathbb{A} \in \mathcal{C}l_{em}(\mathcal{F}(\mathcal{E}))$ (recall that $\mathcal{F}_p = \mathcal{C}l_{em} \circ \mathcal{F}$).

[16] Recall (Definition 6) that we assume DFs that satisfy P1: for any ϕ in a belief base, resources suffice to construct the elementary argument $(\{\phi\}, \phi)$.

Right-to-Left. Let $\mathbb{A} \in \mathcal{C}l_{em}(\mathcal{F}(\mathcal{E}))$. By Definition 8, $\forall \phi \in \mathtt{Com}(\mathbb{A})$, $\phi \in \mathtt{Com}(\mathcal{F}(\mathcal{E}))$ and, clearly, $\mathtt{Com}(\mathcal{F}(\mathcal{E})) = \mathtt{Com}(\mathcal{C}l_{em}(\mathcal{F}(\mathcal{E})))$. Suppose towards a contradiction that $\mathbb{A} \notin \mathcal{F}_{+}(\mathcal{E})$. Then, $\exists \phi \in \mathtt{Com}(\mathbb{A})$ s.t. $\exists \mathbb{B} \in \|\mathcal{A}\|$ with $\mathbb{B} \Rightarrow_{\mathcal{E}} \phi_p$ on ϕ ($\mathtt{Sup}(\mathbb{B}) \subseteq \mathtt{Com}(\mathcal{E} \cup \{\phi\})$) and $\neg\exists \mathbb{C} \in \mathcal{E}$ s.t. $\mathbb{C} \Rightarrow_{\emptyset} \mathbb{B}$. Since $\phi \in \mathtt{Com}(\mathcal{F}(\mathcal{E}))$, $\exists \mathbb{Q} \in \mathcal{F}(\mathcal{E})$ s.t. $\phi \in \mathtt{Com}(\mathbb{Q})$. Hence, $\mathbb{B} \Rightarrow_{\mathcal{E}} \mathbb{Q}$ on ϕ. Since $\mathbb{Q} \in \mathcal{F}(\mathcal{E})$, $\exists \mathbb{C} \in \mathcal{E}$ s.t. $\mathbb{C} \Rightarrow_{\emptyset} B$. Contradiction. □

Proposition 2. *Let $(\mathcal{A}, \mathcal{C})$ be a DF, and $\mathcal{E} \subseteq \mathcal{E}' \subseteq \|\mathcal{A}\|$ be d-admissible. Then:*

1. *$\mathcal{E}$ is a d-complete extension iff $\mathcal{E} = \mathcal{F}_{+}(\mathcal{E})$ iff $\mathcal{E} = \mathcal{F}(\mathcal{E})$.*
2. *$\mathcal{E}$ is the d-grounded extension iff $\mathcal{E}$ is the least fixed point of $\mathcal{F}_{+}$.*
3. *$\mathcal{F}_{+}(\mathcal{E})$ is admissible.*
4. *$\mathcal{F}_{+}(\mathcal{E}) \subseteq \mathcal{F}_{+}(\mathcal{E}')$.*

Proof. 1 and 2 follow from Proposition 1, the fact ([12, Eq.4, Proposition 25]) that the fixed points of $\mathcal{F}$ and $\mathcal{F}_p$ coincide, and the fact ([12, Propositions 25 and 26]) that the *lfp* of $\mathcal{F}$ is the d-grounded extension. 3 follows from Proposition 1 and the fact ([12, Lemma 21]) that $\mathcal{F}_p(\mathcal{E})$ is d-admissible. 4 follows from Proposition 1 and the fact ([12, Corollary 24]) that $\mathcal{F}_p(\mathcal{E}))$ is monotonic. □

We therefore adopt $\mathcal{F}_{+}$ as an alternative characteristic function and show that iterating $\mathcal{F}_{+}$ from $\emptyset$ yields the d-grounded extension. Henceforth, we assume DFs are finitary.[17]

Theorem 3. *Let $(\mathcal{A}, \mathcal{C})$ be a* finitary *DF. The fixed point of $\mathcal{F}_{+}$ constructed from $\emptyset$ is the d-grounded extension $\mathcal{E}_G$.*

Proof. Same reasoning as in [12, Proposition 26]: Let $\mathcal{E}_0 = \emptyset$ and let $\mathcal{E}_{i+1} = \mathcal{F}_{+}(\mathcal{E}_i)$. Since $\mathcal{E}_0$ is d-admissible, by Proposition 2-3 and Proposition 2-4, $\mathcal{E}_0, \mathcal{E}_1, \ldots$ is a monotonically increasing sequence of d-admissible extensions. By Proposition 2-2, there is a $\mathcal{E}^* \subseteq \|\mathcal{A}\|$ which is the *lfp* of $\mathcal{F}_{+}$ and, hence, the infinite set $\mathcal{E}_\infty = \bigcup_{i=0}^{\infty}(\mathcal{F}_{+}(\mathcal{E}_i))$ is a monotonic expansion of $\mathcal{E}^*$, i.e., $\mathcal{E}^* \subseteq \mathcal{E}_\infty$. Let $\mathbb{X} \in \mathcal{F}_{+}(\mathcal{E}_\infty)$. Since $(\mathcal{A}, \mathcal{C})$ is finitary, there are finitely many defeats on $\mathbb{X}$, hence there is an $\mathcal{E}_m$ occurring in the expanding sequence s.t. $\mathbb{X} \in \mathcal{F}_{+}(\mathcal{E}_m)$. Hence, $\mathcal{E}_\infty = \mathcal{F}_{+}(\mathcal{E}_\infty) = \mathcal{E}^*$ and, since $\emptyset$ is the smallest d-admissible extension contained in any d-admissible $\mathcal{E} \subseteq \|\mathcal{A}\|$, $\mathcal{E}^*$ is the d-grounded extension. □

So, starting from the empty-set, one establishes the defensibility of elementary arguments subsequently adding those arguments whose committed premises are in the thus established defensible set. Iterating this procedure yields the d-grounded extension $\mathcal{E}_G$, and therefore also the grounded extension $G(DF) = \{(\Delta, \alpha) \mid (\Delta, \emptyset, \alpha) \in \mathcal{E}_G\}$.

[17] Given a finite belief base $\mathcal{B}$, certain formalisations of *Cl-Arg* generate infinitely many classical logic arguments each of which attack the same argument (e.g., in *ASPIC+*, the premise p yields an infinite number of arguments repeatedly chaining the strict inference rule $p \rightarrow p$ so as to claim p, and each of which attack an argument on premise $\neg p$). Since in dialectical *Cl-Arg*, the defined arguments need only satisfy P1-P3 we can assume a finite number of arguments (Fact 1) and hence a finite number of attacks. Furthermore, since the premises of *Cl-Arg* arguments are finite, we have only finitely many epistemic variants of a defeating argument. Thus, the finitary assumption is harmless.

Example 3. *(*Example 1 continued*)*

1. *$\mathbb{G}_1 = (\{g\}, \emptyset, g)$, $\mathbb{G}_2 = (\{g \to \neg a \vee \neg b\}, \emptyset, g \to \neg a \vee \neg b)$ and $\mathbb{D}_1 = (\{(a \wedge b) \to \neg c\}, \emptyset, (a \wedge b) \to \neg c)$, are $\emptyset$-defensible. Hence $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3 = (\{g, g \to \neg a \vee \neg b\}, \emptyset, \neg a \vee \neg b), \mathbb{G}'_4, \mathbb{G}'_5 \in (\mathcal{E}_1 = \mathcal{F}_+(\emptyset))$ and $\{(\emptyset, \Delta, \phi) | (\Delta, \phi) \in \mathcal{A}_\mathcal{B}\} \subseteq (\mathcal{E}_1 = \mathcal{F}_+(\emptyset))$;*
2. *$\mathbb{C} = (\{c\}, \emptyset, c)$ is $\mathcal{E}_1$-defensible, and so is included in $\mathcal{E}_2 = \mathcal{F}_+(\mathcal{E}_1)$;*
3. *$\mathcal{E}_2 = \mathcal{F}_+(\mathcal{E}_2)$ is the d-grounded extension. Hence $\{G_1, G_2, G_3, D_1, C\}$ is the grounded extension.*

Remark 6. *We suggest that the characteristic function $\mathcal{F}_+$ defines a more intuitive notion of epistemic closure than that defined by $\mathcal{F}_p$ in [12]. Indeed, $\mathcal{F}_+$ can be similarly adopted as a characteristic function for DFs instantiated by standard approaches to classical logic argumentation, capturing a notion of epistemic closure that applies to these approaches. This is because it should be obvious to see that:*

> *Let $(\mathcal{A}, \mathcal{C})$ be a DF and $E \subseteq \mathcal{A}$. Then $F(E) = F_+(E) = \{(\Delta, \phi) | \forall \alpha \in \Delta, (\{\alpha\}, \alpha) \in F(E)\}$*

where F is the standardly defined characteristic function for DFs, and F_+ is the analogue of $\mathcal{F}_+$, defined over sets of classical logic arguments. Moreover, adopting the characteristic functions $\mathcal{F}_+$ for dialectical Cl-Arg *and F_+ for standard* Cl-Arg*, observe that the number of arguments whose acceptability one needs to determine w.r.t. any set of (dialectical) arguments E ($\mathcal{E}$) is bounded by the number of formulas in a base $\mathcal{B}$, rather than the number of (dialectical) arguments defined by $\mathcal{B}$.*

We also suggest that preserving admissibility of a d-admissible extension $\mathcal{E}$, when closing under arguments that commit only to $\mathcal{E}$-defensible premises (as shown in Proposition 2-3), captures an intuitive property that we refer to as cautious super-argument closure *(complementing the sub-argument closure property). Again, it should be obvious to see that the same property holds for standard* Cl-Arg.

Finally, note that we make use of $\mathcal{F}_+$ in the result shown in the following section.

4.2. Soundness for Grounded Dialectical Consequence and Sceptical Preferred Subtheories Consequence

Theorem 4 states that if ϕ is a sceptical consequence under the grounded semantics, then ϕ is a sceptical *PS* consequence. The result assumes resource bounded *PS* consequence over a base $\mathcal{B}$ and ordering $\leq$ (Definition 7), the instantiation of a DF $(\mathcal{A}, \mathcal{C})$ by $\mathcal{B}$ (Remark 3), and the argumentation defined sceptical consequence relation (Definition 5).

Theorem 4. *Let $\mathcal{B}$ be a belief base of classical wff, $\leq$ a total ordering over $\mathcal{B}$. Let $\vdash_r \subseteq \vdash_{CL}$ be well-behaved. Then:*

$$\text{if } \mathcal{B} \mathrel{|\!\sim}^{sc}_{\text{grounded}} \phi \text{ then } (\mathcal{B}, \leq) \mathrel{|\!\sim}^{sc}_{rps} \phi \tag{21}$$

Proof. Let $E_1, \ldots, E_m$ be the r-preferred subtheories (*rps*) defined by $(\mathcal{B}, \leq)$ and

$$E = \bigcap_{i=1}^{m} E_i.$$

Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be the stratification of $\mathcal{B}$ induced by $\leq$, and

$$\text{for } i = 1 \ldots n : E^i = (E \cap \mathcal{B}_1) \cup \cdots \cup (E \cap \mathcal{B}_i). \text{ (hence } E = E^n) \tag{22}$$

The iteration of $\mathcal{F}_+$, as described in the proof of Theorem 3, starts with $\mathcal{F}_+$ applied to $\emptyset$, yielding a monotonically increasing sequence that in the limit yields the *lfp* of $\mathcal{F}_+$. Now, $\mathcal{F}_+(\emptyset)$ returns only undefeated arguments ($\{\mathbb{X} \in \|\mathcal{A}_\mathcal{B}\| \mid \neg\exists\mathbb{Y} \in \|\mathcal{A}_\mathcal{B}\|, \mathbb{Y} \Rightarrow_\emptyset \mathbb{X}\}$), which trivially includes the unassailable epistemic variants $\{(\emptyset, \Delta, \phi) | (\Delta, \phi) \in \mathcal{A}\}$. Hence, without loss of generality, we can assume (for convenience) that the iteration of $\mathcal{F}_+$ starts, not with $\emptyset$, but with

$$F^0 = \{(\emptyset, \Delta, \phi) \mid (\Delta, \phi) \in \mathcal{A}_\mathcal{B}\}.$$

Let $F^0, F^1, \ldots, F^\infty$ be the sequence defined by $F^{l+1} = \mathcal{F}_+(F^l)$. We prove by induction that for each elementary argument $(\{\alpha\}, \emptyset, \alpha) \in F^l$, α is in the intersection E of the *rps* $E_1, \ldots, E_k$:

$$\textit{if } (\{\alpha\}, \emptyset, \alpha) \in (F^l = \mathcal{F}_+(F^{l-1})) \text{ then } \alpha \in E \tag{23}$$

Trivially, no $(\{\alpha\}, \emptyset, \alpha)$ such that $\alpha \vdash_r \bot$ can be included in $\mathcal{F}_+(F^{l-1})$, since any such argument would be defeated by the unassailable $(\emptyset, \{\alpha\}, \bot) \in F^0$. We prove Equation 23 by induction on F^l. In what follows, we may write α_p, β_p to respectively denote the single-premise arguments $(\{\alpha\}, \emptyset, \alpha), (\{\beta\}, \emptyset, \beta)$ (Definition 9).

Base case i=1. Suppose $\alpha_p \in F^1 = \mathcal{F}_+(F^0)$ and for some $i = 1 \ldots n$: $\alpha \in \mathcal{B}_i$.

Assume for contradiction that $\alpha \notin E^i$ (recall from Eq. 22 that $E^i = (\bigcup_{k=1}^{i} \mathcal{B}_i) \cap E$), and so by definition of r-preferred subtheories (Def. 7) there is a *rps* E_h s.t.

$$\Pi \subseteq E_h \tag{24}$$

and $\Pi \cup \{\alpha\} \vdash_r \bot$, and

$$\Pi \subseteq \bigcup_{k=1}^{i} \mathcal{B}_k \tag{25}$$

Given the Elitist definition of $\prec$ (see Eg. 1), $\mathbb{Y} = (\Pi, \{\alpha\}, \bot)) \not\prec \alpha_p$. Hence $\mathbb{Y} \Rightarrow_{F^{l-1}} \alpha_p$.

By assumption, $\alpha_p \in F^1 = \mathcal{F}_+(F^0)$. Hence $\exists\mathbb{X} = (\emptyset, \Gamma, \phi) \in F^0$ s.t. $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$ on $\beta_p = (\{\beta\}, \emptyset, \beta)$. Hence $\mathbb{X} \not\prec \beta_p$, and so given $\beta \in \Pi$ and Eq.25:

$$\Gamma \subseteq \bigcup_{k=1}^{i} \mathcal{B}_k \tag{26}$$

Since $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$, $(\sup(\mathbb{X}) = \Gamma) \subseteq \Pi$. Hence if $\texttt{Conc}(\mathbb{X}) = \bot$, $\Pi \vdash_r \bot$. If $\texttt{Conc}(\mathbb{X}) = -\beta$, then since $\beta \in \Pi$, $\Pi \vdash_r \beta, -\beta$ and so $\Pi \vdash_r \bot$. However, $\Pi \subseteq E_h$ (Eq.24), contradicting $E_h \nvdash_r \bot$. Hence $\alpha \in E^i$.

Inductive step for $j < l$: $(\{\alpha\}, \emptyset, \alpha) \in (F^j = \mathcal{F}_+(F^{j-1}))$ *implies* $\alpha \in E$.

General case. Suppose $\alpha_p \in \mathcal{F}_+(F^{l-1})$ and for some $i = 1 \ldots n$: $\alpha \in \mathcal{B}_i$.

Proof is similar to the base case. That is, assume for contradiction that $\alpha \notin E^i$, and so by Def. 7 there is a *rps* E_h s.t.

$$\Pi \subseteq E_h \text{ and } \Pi \cup \{\alpha\} \vdash_r \bot \text{ and } \Pi \subseteq \bigcup_{k=1}^{i} \mathcal{B}_k \tag{27}$$

Hence $\mathbb{Y} = (\Pi, \{\alpha\}, \bot) \not\prec \alpha_p$, and so $\mathbb{Y} \Rightarrow_{F^{l-1}} \alpha_p$.

By assumption, $\alpha_p \in \mathcal{F}_+(F^{l-1})$. Hence $\exists \mathbb{X} = (\Delta, \Gamma, \phi) \in F^{l-1}$ s.t. $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$ on $\beta_p = (\{\beta\}, \emptyset, \beta)$. Hence $\mathbb{X} \not\prec \beta_p$, and so given $\beta \in \Pi$:

$$\Delta \cup \Gamma \subseteq \bigcup_{k=1}^{i} \mathcal{B}_k \tag{28}$$

By inductive hypothesis, definition of $\mathcal{F}_+$, and Eq.28 ($\texttt{Com}(\mathbb{X}) = \Delta) \subseteq E^i$. Since $\mathbb{X} \Rightarrow_\emptyset \mathbb{Y}$, $(\sup(\mathbb{X}) = \Gamma) \subseteq \Pi$. Hence if $\texttt{Conc}(\mathbb{X}) = \bot$, $E^i \cup \Pi \vdash_r \bot$. If $\texttt{Conc}(\mathbb{X}) = -\beta$, then since $\beta \in \Pi$, $E^i \cup \Pi \vdash_r \beta, -\beta$ and so $E^i \cup \Pi \vdash_r \bot$. However, $E^i \subseteq E_h$, $\Pi \subseteq E_h$ (Eq.27), contradicting $E_h \nvdash_r \bot$. Hence $\alpha \in E^i$.

We have shown that Equation 23 holds. The *lfp* of $\mathcal{F}_+$ is the d-grounded extension $\mathcal{E}_G$, which by definition of $\mathcal{F}_+$ includes all (Δ, Γ, ϕ) s.t. $\alpha \in \Delta$, $(\{\alpha\}, \emptyset, \alpha) \in \mathcal{E}_G$. By Definition 5, the grounded extension $G(DF)$ of $DF_\mathcal{B}$ = $(\mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$ is the set $\{(\Delta, \phi) | (\Delta, \emptyset, \phi) \in \mathcal{E}_G\}$. Hence $\bigcup_{(\Delta,\phi)\in \mathrm{G(DF)}} \Delta \subseteq E$. That is to say:

$$\mathcal{B} \mid\!\sim^{sc}_{\textit{grounded}} \phi \textit{ implies } E \vdash_r \phi \tag{29}$$

Recall that $E = \bigcap_{i=1}^{m} E_i$, where $E_{1\ldots m}$ are the *rps*. Hence, by Equation 29:

$$\mathcal{B} \mid\!\sim^{sc}_{\textit{grounded}} \phi \textit{ implies } (\mathcal{B}, \leq) \mid\!\sim^{sc}_{rps} \phi$$

□

4.3. Recovering Sceptical Preferred Subtheories Consequences

In Example 1, $\mathcal{E}$ is the *dialectical* grounded extension shown in Fig. 1i, and so the grounded extension is $\{G_1, G_2, G_3, D_1, C\}$. Intuitively, since the budget g precludes attendance at both conferences a and b (and neither A nor B are in the grounded extension since the ordering over $\mathcal{B}$ does not decide between the two), then attendance at c is justified since only if one attends both a and b is attendance at c precluded. However, in standard approaches to *Cl-Arg* (e.g., the *ASPIC*$^+$ formalisation of *Cl-Arg* [29]) C is *not* in the grounded extension. This is because G_4 or G_5 are required to defend C against D_2. However, since neither A nor B are in the grounded extension, then neither are G_4 or G_5. Moreover, although G_3 is in the grounded extension, G_3 does not defeat D_2 because targeting non-singleton sets of premises (i.e., $\{a, b\}$) is prohibited; attacks can only target single premises. This is because allowing (what [20] call) 'undercut attacks' — X attacks Y if X's conclusion is classically equivalent to the negation of any subset of Y's premises — not only violates consistency postulates (see [20]), but also violates the practical desideratum that whether one argument attacks another should be 'directly inspectable' (i.e., identifiable in linear time) [16].

However, in *Dialectical Cl-Arg* one can effectively simulate undercuts through use of suppositions, while preserving consistency and the requirement that attacks be directly inspectable. Suppose:

> $\mathbb{X} = (\Delta, \emptyset, \phi)$ and $\mathbb{Y} = (\Pi, \Sigma, \gamma)$ where $\phi \dashv\vdash_{\mathsf{CL}} \neg \bigwedge \Theta$ for some $\Theta = \{\alpha_1, \ldots, \alpha_m\} \subseteq \Pi$

then for $i = 1, \ldots, m$ (and assuming sufficient resources): $\exists \mathbb{X}_i = (\Delta, \Theta \setminus \{\alpha_i\}, \neg\alpha_i)$ that targets $\alpha_i \in \Theta$, *where none of the premises* $\Theta \setminus \{\alpha_i\}$ *need be committed.*

Hence, in Example 1, the defeat $\mathbb{D}_2 \Rightarrow_{\mathcal{E}} \mathbb{C}$ is defended by the defeats $\mathbb{G}'_4 \Rightarrow_{\{\mathbb{D}_2\}} \mathbb{D}_2$ and $\mathbb{G}'_5 \Rightarrow_{\{\mathbb{D}_2\}} \mathbb{D}_2$, each of which simulate defeats on the set of committed premises $\{a, b\} \subseteq \mathtt{Com}(\mathbb{D}_2)$. Both $\mathbb{G}'_4$ and $\mathbb{G}'_5$ only commit to g and $g \to \neg a \vee \neg b$, while $\mathbb{G}'_4$, respectively $\mathbb{G}'_5$, need not commit to attendance at a, respectively b. Intuitively, $\mathbb{G}'_4$ ($\mathbb{G}'_5$) argues that given the budget precludes attendance at a *and* b, and supposing $\mathbb{D}_2$'s commitment to attending b (resp. a), one cannot attend a (resp. b).

Remark 7. *Notice that we have illustrated* Dialectical Cl-Arg*'s recovery of the sceptical* PS *consequence* c *in Example 1, assuming resources suffice to construct arguments* G_4 *and* G_5 *(equating with* $\mathcal{B} \vdash_r \neg a$ *and* $\mathcal{B} \vdash_r \neg b$ *in Example 2). However, one need not assume construction of these arguments. We have argument* $F_1 = (\{a, b, g \to \neg a \vee \neg b, g\}, \bot)$ *in Example 1. This equates with* $\{a, b, g \to \neg a \vee \neg b, g\}$ *being r-inconsistent in Example 2 (hence excluding membership of* a and b *in any preferred subtheory, and thus inclusion of* c *in every preferred subtheory). We therefore also have the undefeated* $\mathbb{F}_1 = (\{g \to \neg a \vee \neg b, g\}, \{a, b\}, \bot)$ *in the dialectical grounded extension* $\mathcal{E}$ *shown in Figure 1*[18]. $\mathbb{F}_1$ *also defends* $\mathbb{C}$ *against the defeat by* $\mathbb{D}_2 = (\{a, b, (a \wedge b) \to \neg c\}, \emptyset, \neg c)$*, given that* $\mathbb{F}_1 \Rightarrow_{\emptyset} \mathbb{D}_2$.

We have shown by example, that the grounded consequences (namely, c) obtained by *Dialectical Cl-Arg*'s formalisation of Preferred Subtheories – denoted below by $\mathrel{|\!\!\sim}^{DCl}_{\textit{grounded}}$ – recovers *PS* sceptical consequences that are not obtained by the grounded consequence relation – denoted below by $\mathrel{|\!\!\sim}^{Cl}_{\textit{grounded}}$ – of *Cl-Arg*'s formalisation of Preferred Subtheories. However, it remains to show that the former subsumes the latter, which is to say that:

$$\mathrel{|\!\!\sim}^{Cl}_{\textit{grounded}} \subset \mathrel{|\!\!\sim}^{DCl}_{\textit{grounded}}$$

We show this result in Theorem 5 below. Note that (recall **C1**, **C2** and **C3** as defined in Section 1) since $\mathrel{|\!\!\sim}^{Cl}_{\textit{grounded}}$ assumes omniscience (**C1**), a meaningful comparison requires that we also assume all arguments defined by a base $\mathcal{B}$ when defining $\mathrel{|\!\!\sim}^{DCl}_{\textit{grounded}}$ (in which case P1, P2, and P3 in Definition 6 are trivially satisfied). However, $\mathrel{|\!\!\sim}^{Cl}_{\textit{grounded}}$ also assumes a framework in which arguments with inconsistent premises and premises

[18] This follows from [12, Lemma 9] which states that for any dialectical-complete extension, if $\mathtt{Com}(\mathbb{X}) \subseteq \mathtt{Com}(\mathcal{E})$, then $\mathbb{X} \in \mathcal{E}$. Note that in [12], the notation $\mathtt{Prem}(X)$ and $\mathtt{Prem}(E)$ are used in place of $\mathtt{Com}(\mathbb{X})$ and $\mathtt{Com}(\mathcal{E})$, respectively.

that are not subset minimal are removed (i.e., **C2** and **C3** respectively). Hence, to show the above claim, we need to first more precisely stipulate the respective definitions of $\mid\!\sim^{Cl}_{grounded}$ and $\mid\!\sim^{DCl}_{grounded}$ assumed in the proof of Theorem 5 below.

Definition 11. *Let $DF_{\mathcal{B}} = (\mathcal{A}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}})$ be defined as in Definition 1, where $\mathcal{A}_{\mathcal{B}}$ consists of* all *$\mathcal{B}$-based arguments (i.e., $\mathcal{A}_{\mathcal{B}} = \{(\Delta, \alpha) | \Delta \subseteq \mathcal{B}, \Delta \vdash_{CL} \alpha\}$), and where $\mathcal{B}$ is a set of propositional or first order classical formulae, $\leq$ is a total ordering over $\mathcal{B}$, and $\prec$ is the $\leq$-based Elitist preference ordering over $\mathcal{A}_{\mathcal{B}}$, as defined in Example 1. Let:*

- *$\mathcal{B} \mid\!\sim^{DCl}_{\text{grounded}} \phi$ iff $\mathcal{B} \mid\!\sim^{sc}_{\text{grounded}} \phi$, where $\mathcal{B} \mid\!\sim^{sc}_{\text{grounded}} \phi$ is the dialectical grounded consequence relation as defined in Definitions 3, 4 and 5.*
- *$DF'_{\mathcal{B}} = (\mathcal{A}'_{\mathcal{B}}, \mathcal{C}'_{\mathcal{B}})$ where*
  - *$\mathcal{A}'_{\mathcal{B}}$ are the arguments in $\mathcal{A}_{\mathcal{B}}$ that have subset minimal consistent premises, and $\mathcal{C}'_{\mathcal{B}} = \{(X, Y) | X, Y \in \mathcal{A}'_{\mathcal{B}}, (X, Y) \in \mathcal{C}_{\mathcal{B}}\}$ is the attack relation restricted to arguments in $\mathcal{A}'_{\mathcal{B}}$;*
- *Let $\mathcal{D}'_{\mathcal{B}}$ denote the standard* defeat *relation defined over $\mathcal{A}'_{\mathcal{B}}$:*

$$\mathcal{D}'_{\mathcal{B}} = \{(X, Y) \in \mathcal{C}'_{\mathcal{B}} | \texttt{Conc}(X) = \phi, -\phi \in \texttt{Prem}(Y), X \not\prec (\{-\phi\}, -\phi)\}$$

- *Let E be the grounded extension of $DF'_{\mathcal{B}}$ (as defined in the usual way in [15]). We write $\mathcal{B} \mid\!\sim^{Cl}_{\text{grounded}} \phi$ iff $\exists X \in E$, $\texttt{Conc}(X) = \phi$.*

We leave it to the reader to see that (as in Example 1 in which $\mathcal{A}_{\mathcal{B}}$ are a strict subset of all arguments defined by $\mathcal{B}$) for the totally ordered base $\mathcal{B}$ in Example 1, c is not a $\mid\!\sim^{Cl}_{grounded}$ consequence but is a $\mid\!\sim^{DCl}_{grounded}$ consequence.

The following lemma, which assumes a $DF_{\mathcal{B}}$ and $DF'_{\mathcal{B}}$ as defined above, is used to prove that $\mid\!\sim^{Cl}_{grounded} \subset \mid\!\sim^{DCl}_{grounded}$. The lemma states under which conditions for each defeating argument there exists a defeat preserving subset minimal argument.

Lemma 4. *Let $DF_{\mathcal{B}}$ and $DF'_{\mathcal{B}}$ be defined as in Definition 11. If $\mathbb{Y}, \mathbb{X} \in \|\mathcal{A}_{\mathcal{B}}\|$, $\mathbb{Y} = (\Pi, \Gamma, \phi)$, $\mathbb{X} = (\Delta, \Theta, \chi)$ and:*

- *$X = (\Delta \cup \Theta, \chi) \in \mathcal{A}'_{\mathcal{B}}$*
- *$\mathbb{Y} \Rightarrow_{\mathcal{E}} \mathbb{X}$ (where $\mathcal{E} \subseteq \|\mathcal{A}_{\mathcal{B}}\|$)*
- *$\Pi \cup \Gamma \nvdash_{CL} \bot$*
- *$\exists \Pi' \subset \Pi \cup \Gamma$ s.t. $\Pi' \vdash_{CL} \phi$ and $\neg\exists \Pi'' \subset \Pi'$ s.t. $\Pi'' \vdash_{CL} \phi$*

then there is a subset minimal $Y' = (\Pi', \phi) \in \mathcal{A}'_{\mathcal{B}}$ such that Y' defeats X (i.e., $(Y', X) \in \mathcal{D}'_{\mathcal{B}}$).

Proof. We have $\mathbb{Y} \Rightarrow_{\mathcal{E}} \mathbb{X}$ on some $\beta \in \Delta$, and so by definition of the Elitist preference ordering $\prec$ (see Example 1), $\forall \alpha \in \Pi \cup \Gamma$, $\alpha \not< \beta$. By definition of $\mathcal{A}'_{\mathcal{B}}$, $Y' = (\Pi', \phi)$ is a subset minimal consistent argument in $\mathcal{A}'_{\mathcal{B}}$. Since $\Pi' \subset \Pi \cup \Gamma$, $\forall \alpha \in \Pi'$, $\alpha \not< \beta$, and so $Y \not\prec (\{\beta\}, \beta)$. Hence Y' defeats X on β. □

Theorem 5. *Let $\mathcal{B}$ be a totally ordered belief base of classical wff, and let $\mid\sim^{DCl}_{grounded}$ and $\mid\sim^{Cl}_{grounded}$ be defined as in Definition 11. Then, $\mid\sim^{Cl}_{grounded} \subset \mid\sim^{DCl}_{grounded}$.*

Proof. We have shown by example, that $\mid\sim^{DCl}_{grounded} \not\subseteq \mid\sim^{Cl}_{grounded}$. Let $DF_{\mathcal{B}}$ be as defined in Remark 11. It suffices to show that $\mid\sim^{Cl}_{\text{grounded}} \subseteq \mid\sim^{DCl}_{\text{grounded}}$. That is, the *lfp* of the standard characteristic function F for $DF'_{\mathcal{B}}$ over (classical logic) arguments in $\mathcal{A}'_{\mathcal{B}}$ is a subset of the *lfp* of the dialectical characteristic function $\mathcal{F}_+$ (recall Definition 10) over $\|\mathcal{A}_{\mathcal{B}}\|$ of $DF_{\mathcal{B}}$. Before proceeding, notice that unassailable dialectical arguments of the form $(\emptyset, \Gamma, \psi)$ are trivially included in any admissible extension, and so in the following, we can, without loss of generality, assume that iteration of $\mathcal{F}_+$ starts not with $\emptyset$, but with the set of unassailable arguments.

Let

$$\mathcal{E}^{Cl} = \bigcup_{i=0}^{\infty} \mathcal{E}^{Cl}_i \text{ with } \mathcal{E}^{Cl}_0 = \emptyset \text{ and } \mathcal{E}^{Cl}_{i+1} = F(\mathcal{E}^{Cl}_i)$$

and

$$\mathcal{E}^{DCl} = \bigcup_{i=0}^{\infty} \mathcal{E}^{DCl}_i \text{ with } \mathcal{E}^{DCl}_0 = \{(\emptyset, \Gamma, \psi) | (\Gamma, \psi) \in \mathcal{A}_{\mathcal{B}}\} \text{ and } \mathcal{E}^{DCl}_{i+1} = \mathcal{F}_+(\mathcal{E}^{DCl}_i).$$

We show the following claim by induction on i:

$$\{(\Delta, \emptyset, \phi) \mid (\Delta, \phi) \in \mathcal{E}^{Cl}\} \subseteq \mathcal{E}^{DCl} \tag{30}$$

Base case i=1 Let $X = (\Delta, \chi) \in \mathcal{E}^{Cl}_1$, and so X is undefeated. We show that $\mathbb{X} \in \mathcal{E}^{DCl}_1$, where $\mathbb{X} = (\Delta, \emptyset, \chi) \in \|\mathcal{A}_{\mathcal{B}}\|$, by demonstrating the premise defensibility (Definition 9) of each $\phi \in \Delta$. Suppose some arbitrary $\phi \in \Delta$ and $\mathbb{Y} \Rightarrow_{\mathcal{E}^{DCl}_0} \phi_p$, hence $Y \not\prec (\{\phi\}, \phi)$. Since $\texttt{Com}(\mathcal{E}^{DCl}_0) = \emptyset$, then either:

1. $\mathbb{Y} = (\Pi, \emptyset, -\phi)$.
   - Suppose $\Pi \vdash_{CL} \bot$. We have $\mathbb{Y}' = (\emptyset, \Pi, \bot) \in \mathcal{E}^{DCl}_0$, and $\mathbb{Y}' \Rightarrow_{\emptyset} \mathbb{Y}$.
   - Suppose $Y = (\Pi, -\phi)$ is subset minimal and $\Pi \nvdash_{CL} \bot$. Then $Y = (\Pi, -\phi) \in \mathcal{A}'_{\mathcal{B}}$, $Y \not\prec (\{\phi\}, \phi)$ and Y defeats X on ϕ, contradicting X is undefeated
   - Suppose $Y = (\Pi, -\phi)$ is not subset minimal and $\Pi \nvdash_{CL} \bot$. By Lemma 4, $\exists Y' \in \mathcal{A}'_{\mathcal{B}}$ s.t. Y' defeats X, contradicting X is undefeated.

2. $\mathbb{Y} = (\Pi, \{\phi\}, \psi)$ where $\psi = -\phi$ or $\psi = \bot$.
   (a) Suppose $\psi = -\phi$, and so $(\Pi \cup \{\phi\}, -\phi) \not\prec (\{\phi\}, \phi)$. By *DCPref 2*[19]:

   $$(\Pi, -\phi) \not\prec (\{\phi\}, \phi) \tag{31}$$

   Also, $\Pi \cup \{\phi\} \vdash_{CL} -\phi$ implies (by classical reasoning[20]) $\Pi \vdash_{CL} -\phi$.

[19] Recall (Section 2, p.10) that we assume dialectically coherent preference relations that satisfy DCPref 1 and DCPref 2.

[20] If $\Pi \nvdash_{CL} -\phi$, then $\Pi \nvdash_{CL} \bot$, and so $\Pi \cup \{\phi\} \nvdash_{CL} \bot$, contradicting $\Pi \cup \{\phi\} \vdash_{CL} -\phi$.

Suppose $\Pi \vdash_{CL} \bot$. Then $\mathbb{Z} = (\emptyset, \Pi, \bot) \in \mathcal{E}_0^{DCl}$, and $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{Y}$.

Suppose $\Pi \nvdash_{CL} \bot$.

• Suppose no proper subset of Π entails $-\phi$. Then $Y' = (\Pi, -\phi)$ is a subset minimal consistent argument in $\mathcal{A}'_{\mathcal{B}}$. Given Eq.31 Y' defeats X on ϕ, contradicting X is undefeated.

• Else, let $\Pi' \subset \Pi$ be a minimal strict subset of Π s.t. $\Pi' \vdash_{CL} -\phi$. Then $\exists Y'' = (\Pi', -\phi) \in \mathcal{A}'_{\mathcal{B}}$ s.t. by Lemma 4 Y'' defeats X, contradicting X is undefeated.

(b) Suppose $\psi = \bot$, and so $(\Pi \cup \{\phi\}, \bot) \nprec (\{\phi\}, \phi)$. $\Pi \cup \{\phi\} \vdash_{CL} \bot$ implies (by classical reasoning[21]) $\Pi \vdash_{CL} -\phi$. Then proceed as in (a).

We have shown that if $X = (\Delta, \chi) \in \mathcal{E}_1^{Cl}$, then for any $\phi \in \Delta$, if ϕ_p is defeated by some $\mathbb{Y}$, this either leads to a contradiction or ϕ_p is defensible. Hence $\mathbb{X} = (\Delta, \emptyset, \chi) \in \mathcal{E}_1^{DCl}$.

Inductive step for $j < i$**:** $\{(\Delta, \emptyset, \phi) \mid (\Delta, \phi) \in \mathcal{E}_j^{Cl}\} \subseteq \mathcal{E}_j^{DCl}$.

Assume $X = (\Delta, \chi) \in \mathcal{E}_i^{CL}$. We show that $\mathbb{X} = (\Delta, \emptyset, \chi) \in \mathcal{E}_i^{DCl}$ by demonstrating the premise defensibility of each $\phi \in \Delta$. Suppose $\mathbb{Y} = (\Gamma, \Theta, \psi) \in \|\mathcal{A}_{\mathcal{B}}\|$ such that $\mathbb{Y} \Rightarrow_{\mathcal{E}_{i-1}^{DCl}} \phi_p$ for some $\phi \in \Delta$, and so $\Theta \subseteq \mathtt{Com}(\mathcal{E}_{i-1}^{DCl}) \cup \{\phi\}$ and $Y \nprec (\{\phi\}, \phi)$. There are two cases to consider. $\psi = -\phi$ and $\psi = \bot$:

1. $\psi = -\phi$.

(a) Suppose $(\mathtt{Sup}(\mathbb{Y}) = \Theta) \subseteq \mathtt{Com}(\mathcal{E}_{i-1}^{DCl})$.

Suppose $\boldsymbol{\Gamma \cup \Theta} \vdash_{CL} \bot$. We have:

$$\mathbb{V} = (\Theta, \Gamma, \bot) \in \mathcal{E}_{i-1}^{DCl} \text{ and } \mathbb{V}' = (\Gamma, \Theta, \bot) \in \|\mathcal{A}_{\mathcal{B}}\|.$$

It follows from the properties of dialectically coherent preference relations (in particular *DCPref 1*, p.10), that either:

– $\mathbb{V} \Rightarrow_{\emptyset} \mathbb{V}'$ in which case (since $\mathtt{Com}(\mathbb{Y}) = \mathtt{Com}(\mathbb{V}')$) $\mathbb{V} \Rightarrow_{\emptyset} \mathbb{Y}$, or:

– $\mathbb{V}' \Rightarrow_{\emptyset} \mathbb{V}$. That is, $(\Gamma, \Theta, \bot) \Rightarrow_{\emptyset} (\Theta, \Gamma, \bot)$ on some $\sigma \in \Theta$, and so (given $\Theta \subseteq \mathtt{Com}(\mathcal{E}_{i-1}^{DCl})$), $\mathbb{V}' \Rightarrow_{\mathcal{E}_{i-1}^{DCl}} (\{\sigma\}, \emptyset, \sigma)$. Since $\mathcal{E}_{i-1}^{DCl}$ is admissible (recall Proposition 2-3), $\exists \mathbb{Z} \in \mathcal{E}_{i-1}^{DCl}$ such that $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{V}'$ and so (since $\mathtt{Com}(\mathbb{Y}) = \mathtt{Com}(\mathbb{V}')$) $\mathbb{Z} \Rightarrow_{\emptyset} \mathbb{Y}$.

Suppose $\boldsymbol{\Gamma \cup \Theta} \nvdash_{CL} \bot$.

(a-i) Suppose no proper subset of $\Gamma \cup \Theta$ entails $-\phi$. Then we have the subset minimal consistent $Y = (\Gamma \cup \Theta, -\phi) \in \mathcal{A}'_{\mathcal{B}}$ and (given $Y \nprec (\{\phi\}, \phi)$) Y defeats X on ϕ. By assumption of $X \in \mathcal{E}_i^{CL}$, there is a $Z = (\Sigma, \beta) \in \mathcal{E}_{i-1}^{CL}$ such that Z defeats Y on some $\delta \in (\Gamma \cup \Theta)$ (and so $Z \nprec (\{\delta\}, \delta)$). By inductive hypothesis, $\mathbb{Z} = (\Sigma, \emptyset, \beta) \in \mathcal{E}_{i-1}^{DCl}$.

(a-i-1) Suppose Z defeats Y on some $\delta \in \Gamma$. Since $\mathtt{Com}(\mathbb{Y}) = \Gamma$,

[21] If $\Pi \nvdash_{CL} -\phi$, then $\Pi \nvdash_{CL} \bot$, and so $\Pi \cup \{\phi\} \nvdash_{CL} -\phi$. Contradiction.

$\mathbb{Z} \Rightarrow_\emptyset \mathbb{Y}$.

(a-i-2) Suppose Z defeats Y on $\delta \in \Theta$. Since $\Theta \subseteq \texttt{Com}(\mathcal{E}^{DCl}_{i-1})$ and $\mathbb{Z} \in \mathcal{E}^{DCl}_{i-1}$, this implies that $\mathbb{Z}$ defeats some $\mathbb{D} \in \mathcal{E}^{DCl}_{i-1}$ on δ, contradicting $\mathcal{E}^{DCl}_{i-1}$ is admissible and hence conflict free.

(a-ii) Suppose $(\Gamma' \cup \Theta') \subset (\Gamma \cup \Theta)$ is a minimal strict subset that entails $-\phi$. By Lemma 4, $Y' = (\Gamma' \cup \Theta', -\phi) \in \mathcal{A}'_{\mathcal{B}}$, and Y' defeats X on ϕ. We can then proceed as in **(a-i)**.

(b) Suppose $(\sup(\mathbb{Y}) = \Theta) \subseteq \texttt{Com}(\mathcal{E}^{DCl}_{i-1}) \cup \{\phi\}$ where $\phi \in \Theta$. That is to say, we have $\mathbb{Y} = (\Gamma, \Theta, -\phi) \Rightarrow_{\mathcal{E}^{DCl}_{i-1}} \phi_p$, where $\phi \in \Theta$.

By classical reasoning[22] $\Gamma \cup (\Theta \setminus \{\phi\}) \vdash -\phi$. Let $\Theta' = \Theta \setminus \{\phi\}$. We thus have $Y' = (\Gamma \cup \Theta', -\phi) \in \mathcal{A}_{\mathcal{B}}$ and (by *DCPref 2*, p.10), $Y' \not\prec (\{\phi\}, \phi)$. Hence $\mathbb{Y}' = (\Gamma, \Theta', -\phi) \in \|\mathcal{A}_{\mathcal{B}}\|$, and $\mathbb{Y}' \Rightarrow_{\mathcal{E}^{DCl}_{i-1}} \phi_p$.

Suppose $\Gamma \cup \Theta' \vdash \bot$. We can then repeat the reasoning in the case that $\Gamma \cup \Theta \vdash \bot$ in a), to show that there is a $\mathbb{Z} \in \mathcal{E}^{DCl}_{i-1}$ that defeats $\mathbb{Y}'$ on some $\delta \in \Gamma$, and so $\mathbb{Y}$ on some $\delta \in \Gamma$.

Suppose $\Gamma \cup \Theta' \nvdash \bot$. We can then repeat the reasoning in the case that $\Gamma \cup \Theta \nvdash \bot$ in a), substituting Θ' for Θ and Y' for Y. In particular, we thus show that:

– in the case (a-i) that $\Gamma \cup \Theta'$ is a subset minimal set entailing $-\phi$, there is a $\mathbb{Z} \in \mathcal{E}^{DCl}_{i-1}$ that defeats $\mathbb{Y}'$ on some $\delta \in \Gamma$, and so $\mathbb{Y}$ on some $\delta \in \Gamma$ (case (a-i-1)), or $\mathbb{Z} \in \mathcal{E}^{DCl}_{i-1}$ defeats $\mathbb{Y}'$ on some $\delta \in \Theta'$, which leads to contradicting the admissibility of $\mathcal{E}^{DCl}_{i-1}$ (case (a-i-1)).

– in the case that $\Gamma \cup \Theta'$ is not subset minimal, proceed in the same way as case (a-ii).

2. $\psi = \bot$.
Hence $\mathbb{Y} = (\Gamma, \Theta, \bot) \Rightarrow_{\mathcal{E}^{DCl}_{i-1}} \phi_p$ and $\phi \in \Theta$ and $Y = (\Gamma \cup \Theta, \bot) \not\prec (\{\phi\}, \phi)$. By classical reasoning, $\Gamma \cup (\Theta \setminus \{\phi\}) \vdash_{CL} -\phi$. We can then proceed as in case 1(b).

We have shown that if $X = (\Delta, \chi) \in \mathcal{E}^{CL}_i$, then for any $\phi \in \Delta$, if ϕ_p is defeated by some $\mathbb{Y}$, this either leads to a contradiction, or there is some $\mathbb{Z} \in \mathcal{E}^{CL}_{i-1}$ such that $\mathbb{Z} \Rightarrow_\emptyset \mathbb{Y}$ and so ϕ_p is defensible. Hence $X = (\Delta, \emptyset, \chi) \in \mathcal{E}^{CL}_i$. □

Example 1 also illustrates how *Dialectical Cl-Arg* solves the so called 'foreign commitment problem' [11], that arises when in order to challenge an agent Ag_1's argument (e.g., $D_2 = (\{a, b, (a \wedge b) \rightarrow \neg c\}, \neg c)$) an interlocutor Ag_2 is forced to commit (and thus be held to account when challenged on their commitments) to either a or b in the

[22] Suppose $\Gamma \cup (\Theta \setminus \{\phi\}) \nvdash_{CL} -\phi$. Hence $\Gamma \cup \Theta \nvdash_{CL} \bot$. But $\Gamma \cup \Theta \vdash_{CL} \phi$ and $\Gamma \cup \Theta \vdash_{CL} -\phi$, and so $\Gamma \cup \Theta \vdash_{CL} \bot$. Contradiction. Hence $\Gamma \cup (\Theta \setminus \{\phi\}) \vdash_{CL} -\phi$

arguments G_4 and G_5 respectively, rather than, as one would expect in practice, only suppose them for the sake of argument.

As an aside, notice also that attacks on the conclusions of *Cl-Arg* arguments (so called 'unrestricted rebuts') are also prohibited, in large part because the consistency postulates are again violated [9, 20], but also because it is arguably incoherent to challenge the conclusion of a *deductive* inference. However, it would be pragmatically useful to argue that given one's own commitments Δ in an argument $\mathbb{X}$ concluding ϕ, the commitments Π of the attacked argument $\mathbb{Y}$ concluding $-\phi$ cannot, in toto, be collectively and coherently committed. Again, referring to Example 1, $\mathbb{D}_2$ concludes $\neg c$ and so defeats $\mathbb{C} = (\{c\}, \emptyset, c)$. $\mathbb{C}$ cannot directly unrestrictedly rebut $\mathbb{D}_2 = (\{a, b, (a \wedge b) \rightarrow \neg c\}, \emptyset, \neg c)$. However, $\mathbb{F}_2 = (\{c\}, \{a, b, (a \wedge b) \rightarrow \neg c\}, \bot)$ is a member of the grounded extension $\mathcal{E}$ (given that $\mathbb{C} \in \mathcal{E}$). Suppose then that $F_2 \not\prec X$, for some $X \in \{A = (\{a\}, a), B = (\{b\}, b), D_1 = (\{(a \wedge b) \rightarrow \neg c\}, (a \wedge b) \rightarrow \neg c)\}$ (although according to the Elitist preference ordering, it is in fact the case that $F_2 \prec X$ for each $X \in \{A, B, D_1\}$). Then $\mathbb{F}_2 \Rightarrow_{\{\mathbb{D}_2\}} \mathbb{D}_2$ on a, b and $(a \wedge b) \rightarrow \neg c$, effectively simulating a rebut by arguing that the commitments of $\mathbb{D}_2$, together with the premise c of $\mathbb{C}$, entail $\bot$, but without having to target one of the premises a, b or $(a \wedge b) \rightarrow \neg c$.

While Dialectical *Cl-Arg* recovers sceptical PS consequences that are not obtained under the grounded semantics for standard *Cl-Arg*, completeness still fails to hold for Dialectical *Cl-Arg*. That is:

$$\mathcal{B} \mid\!\sim^{sc}_{rps} \phi \nrightarrow \mathcal{B} \mid\!\sim^{sc}_{grounded} \phi$$

Example 4. *Suppose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where:*

$$\mathcal{B}_1 = \{(\neg a \vee \neg b) \wedge d, (\neg a \vee \neg b) \wedge \neg d\}$$
$$\mathcal{B}_2 = \{a, b, (a \wedge b) \rightarrow \neg c, c\}$$

and suppose a well-behaved $\vdash_r \subseteq \vdash_{CL}$ (recall Definition 7) such that:

1. $\mathcal{B} \vdash_r \alpha$, $\alpha \in \mathcal{B}$
2. $\mathcal{B}_1 \vdash_r \bot$
3. $\{(\neg a \vee \neg b) \wedge d, a, b\} \vdash_r \bot$ *and* $\{(\neg a \vee \neg b) \wedge \neg d, a, b\} \vdash_r \bot$
4. $\{a, b, (a \wedge b) \rightarrow \neg c\} \vdash_r \neg c$ *and* $\mathcal{B}_2 \vdash_r \bot$
5. $\{(\neg a \vee \neg b) \wedge d, a\} \vdash_r \neg b$ *and* $\{(\neg a \vee \neg b) \wedge d, b\} \vdash_r \neg a$

There are four r-preferred subtheories:

$$\Sigma_1 = \{(\neg a \vee \neg b) \wedge d, a, (a \wedge b) \rightarrow \neg c, c\}$$
$$\Sigma_2 = \{(\neg a \vee \neg b) \wedge d, b, (a \wedge b) \rightarrow \neg c, c\}$$
$$\Sigma_3 = \{(\neg a \vee \neg b) \wedge \neg d, a, (a \wedge b) \rightarrow \neg c, c\}$$
$$\Sigma_4 = \{(\neg a \vee \neg b) \wedge \neg d, b, (a \wedge b) \rightarrow \neg c, c\}$$

Now notice that $(\mathcal{B}, \leq) \mid\!\sim^{sc}_{rps} c$, but $(\mathcal{B}, \leq) \not\mid\!\sim^{sc}_{\text{grounded}} c$, since $\mathbb{C} = (\{c\}, \emptyset, c)$ is not in the d-grounded extension $\mathcal{E}_G$. This is because (letting $\mathbb{D}_2 = (\{a, b, (a \wedge b) \rightarrow \neg c\}, \emptyset, \neg c)$ – see Figure 1):

$$\mathbb{D}_2 \Rightarrow_{\emptyset} \mathbb{C} \textit{ and so } \mathbb{D}_2 \Rightarrow_{\mathcal{E}_G} \mathbb{C}$$

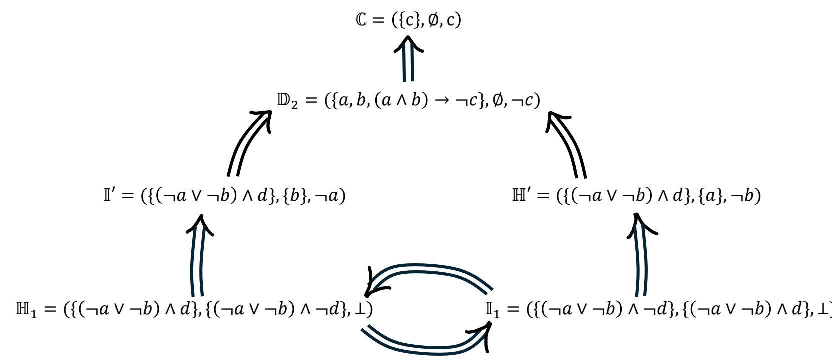

Figure 2: All defeats $\Rightarrow_{\mathcal{E}}$ in the figure occur with respect to $\mathcal{E} = \emptyset$.

and since $\vdash_r$ is a 'well-behaved' approximation of classical consequence (satisfying 1 and 2 above), we have $\mathbb{H} = (\{(\neg a \vee \neg b) \wedge d\}, \emptyset, (\neg a \vee \neg b) \wedge d)$ and $\mathbb{I} = (\{(\neg a \vee \neg b) \wedge \neg d\}, \emptyset, (\neg a \vee \neg b) \wedge \neg d)$, and

$$\mathbb{H}_1 = (\{(\neg a \vee \neg b) \wedge d\}, \{(\neg a \vee \neg b) \wedge \neg d\}, \bot)$$

and

$$\mathbb{I}_1 = (\{(\neg a \vee \neg b) \wedge \neg d\}, \{(\neg a \vee \neg b) \wedge \ d\}, \bot)$$

and so:

$$\mathbb{H}_1 \Rightarrow_{\emptyset} \mathbb{I} \text{ and } \mathbb{I}_1 \Rightarrow_{\emptyset} \mathbb{H} \text{ and } \mathbb{H}_1 \Rightarrow_{\emptyset} \mathbb{I}_1 \text{ and } \mathbb{I}_1 \Rightarrow_{\emptyset} \mathbb{H}_1$$

The symmetric defeat between $\mathbb{H}_1$ and $\mathbb{I}_1$ (shown in Figure 2) means that neither $\mathbb{H}_1$ nor $\mathbb{I}_1$ is in $\mathcal{E}_G$. Hence neither $\mathbb{H}' = (\{(\neg a \vee \neg b) \wedge \neg d\}, \{a\}, \neg b)$ nor $\mathbb{I}' = (\{(\neg a \vee \neg b) \wedge \neg d\}, \{b\}, \neg a)$ are included in $\mathcal{E}_G$, as they are defeated (respectively by $\mathbb{I}_1$ and $\mathbb{H}_1$) and neither of these defeats can be defended by arguments in $\mathcal{E}_G$. Hence no arguments are available in $\mathcal{E}_G$ to defeat $\mathbb{D}_2$ and so defend $\mathbb{C}$ [23].

The above is an instance of the well known syntax sensitivity problem that afflicts syntactic approaches to non-monotonic reasoning. The somewhat unnatural coupling of the 'fate' of $\neg a \vee \neg b$ with d ($\neg d$) through use of the conjunction connective, disrupts the correspondence between sceptical *ps* inference and sceptical inference under the grounded semantics[24].

Notice that this problem is related to the issue of redundantly contaminated arguments (recall Section 2), in the sense that d ($\neg d$) and $\neg a \vee \neg b$ are syntactically disjoint, and so one would not want that d ($\neg d$) 'contaminates' the grounded consequence

[23]We also have that $(\Gamma, \leq) \not\mathrel{|\!\sim}^{sc}_{grounded} c$ in the case that $\vdash_r = \vdash_{CL}$

[24]Observe that *PS* is of course also a syntactic approach to non-monotonic reasoning, and yet obtains c as a sceptical consequence. However, the incremental generation of preferred subtheories, whereby one starts with the maximal consistent subsets $\{(\neg a \vee \neg b) \wedge \neg d\}$ and $\{(\neg a \vee \neg b) \wedge \neg d\}$ of $\mathcal{B}_1$, which in turn means that maximally extending these with subsets of $\mathcal{B}_2$ excludes inclusion of a *and* b, and hence the inference $\neg c$. However, in classical logic argumentation, the arguments defined by $\mathcal{B}$ include $(\{a, b, (a \wedge b) \rightarrow \neg c\}, \neg c)$, which then defeats c.

$\neg a \vee \neg b$. Suppose then that the beliefs in a base $\mathcal{B}$ are translated to conjunctive normal form (*cnf*). That is to say, if $\mathcal{B}$ = $\{\phi_1, \ldots, \phi_n\}$, *CNF*($\mathcal{B}$) = $\{cnf(\phi_1), \ldots, cnf(\phi_n)\}$, where each $cnf(\phi_i)$ is the conjunction $\alpha_i^1 \wedge \cdots \wedge \alpha_i^m$, and where each α_i^j is a disjunction of atomic formulae. Separating out the conjuncts yields

$$\mathcal{B}^* = \{\alpha_1^1, \ldots, \alpha_1^m, \ldots, \alpha_n^1, \ldots, \alpha_n^k\}$$

We conjecture then, that if a belief base $\mathcal{B}$ is transformed to $\mathcal{B}^*$, then completeness can be shown for the dialectical grounded consequence relation:

$$\mathcal{B}^* \mid\!\sim^{sc}_{rps} \phi \rightarrow \mathcal{B}^* \mid\!\sim^{sc}_{grounded} \phi$$

Intuitively, each atomic disjunct β in a disjunction $\alpha_i^j = \beta_1 \vee \cdots \vee \beta_l$ is relevant with respect to the entailment of every other β' in α_i^j. This is of course because for each β_k ($k = 1 \ldots l$), $\beta_1 \vee \cdots \vee \beta_l$ is logically equivalent to $-\beta_1 \wedge \cdots \wedge -\beta_{k-1} \wedge -\beta_{k+1} \cdots \wedge -\beta_l \rightarrow \beta_k$.

In our running example, the transformation would yield

$$\mathcal{B}^* = \{\neg a \vee \neg b, d, \neg d, a, b, \neg a \vee \neg b \vee \neg c, c\}$$

in which case it easy to see that one would recover the grounded consequence c.

5. Related Work and Conclusions

Formalisations of credulous Preferred Subtheories consequence been been established for classical logic instantiations of Dung frameworks, under the stable semantics [2, 29, 32]. In this paper, we have established soundness and completeness under the *preferred* semantics for *Dialectical Cl-Arg*[25]. Hence, agents with bounded inferential capabilities can deploy argument game proof theories and engage in dialogues establishing membership of an argument in dialectical admissible extensions, thereby establishing credulous consequences yielded by resource bounded approximations of Preferred Subtheories.

However, it should be noted that argument game proof theories for Dung AFs cannot straightforwardly be applied to *Dialectical Cl-Arg*. In particular, argument games involve a proponent submitting arguments that define an admissible extension S (containing the argument whose membership in an admissible extension is being decided), so as to defend against arguments moved by an opponent that defeat proponent's submitted arguments (e.g., [28]). Future work is therefore required to adapt such proof theories so that defeats moved by the opponent are parameterised so as to reference (the premises of dialectical arguments in) the set $\mathcal{E}$ that the proponent has thus far committed to.

We observe that many argumentative formalisations of well-known non-monotonic logics are defined in terms of the stable semantics. (See [15] for the earliest of such results, and [4] for a recent overview.) An immediate question is whether results similar

[25] Recall (footnote 4) that we first presented the "stable = preferred" result for both Dialectical and standard *Cl-Arg* in [13].

to that presented in this paper, may be obtained for other such logics (see [34] for one of the first known equivalence results for normal logic programs). Indeed, it would be fruitful to investigate more general conditions (at the level of abstract, rather than instantiated, Dung frameworks) under which the stable and preferred semantics coincide (in this paper we have established equivalence by reference to classical logic instantiations and Elitist argument preferences lifted from a total ordering over beliefs). For example, equivalence may depend on the type of negation employed in the instantiating base logic (characterization results have been investigated for large classes of base logics, not limited to classical and intuitionistic logic, e.g., see [3, 5, 22, 29]) or on the properties of the argument preference ordering 'lifted' from the ordering over the belief base [29]. Investigations in this spirit are likely to point to the presence of problematic odd-loops that disrupt the equivalence between stable and preferred extensions.

This paper has also shown that under the grounded semantics, the dialectical approach is sound with respect to resource bounded sceptical Preferred Subtheories consequence, and more closely approximates the sceptical Preferred Subtheories consequence relations, as compared with standard *Cl-Arg*. We also conjecture that a full correspondence fails only in 'pathological' cases in which syntactically unrelated formulae are conjoined. Future work will more precisely articulate and verify this conjecture. Finally, we note that [32] provide argumentative characterisations of Preferred Subtheories inference using assumption based argumentation [7], and do so for the more general case in which the preferred subtheories are defined by all total orderings that complete a given *partial* ordering over a belief base $\mathcal{B}$ of classical wff. It remains to study characterisations of this more general case using standard and dialectical approaches to *Cl-Arg*.

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