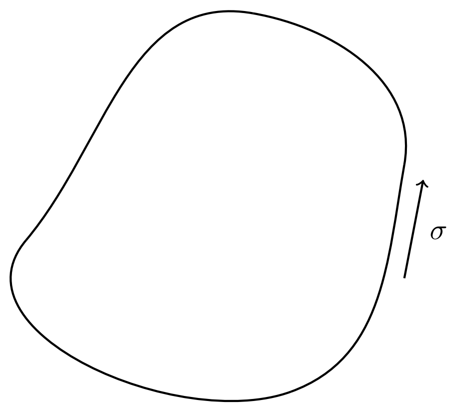
A Beginner's Guide to String Theory
Part II: Exploring the Quantum String

We are now at a halfway point in our investigation into the foundations of string theory. So, before we move on, let's stop for a moment and just get our bearings.

II.0 A brief recap

When we think of physics, we're used to talking about particles. Given some particle, we know everything about it so long as we know where it is, and how it's moving - its position and velocity. So, the first job of any good string theorist is to play the same game not for a particle, but for a *string*. What information do we need to write down to *completely* describe the motion of a string?



So, we first chose a coordinate σ to tell us how far along our string we were. Then, we learnt that if we want to describe a string moving in d spatial dimensions, then we need to write down $d-1$ functions $X^i(\sigma)$ for $i = 1, 2, \ldots d-1$ that well us where the string is in the directions transverse to the string itself. Then, if we want our string to move, we need to introduce a further time coordinate τ, so that our functions are now functions of two variables, $X^i(\sigma, \tau)$.

We then stated a fundamental result: that the functions $X^i(\sigma, \tau)$ must satisfy the *wave equation*,

$$\frac{\partial^2 X}{\partial \tau^2} - \frac{\partial^2 X}{\partial \sigma^2} = 0 \ , \tag{1}$$

where we have here and for the remainder of this project set $c = 1$. So, the whole problem was reduced to solving this one equation. To do so, we introduced new coordinates $\sigma_\pm = \sigma \pm \tau$, called *light-cone coordinates*, to find that the general solution to (1) was given by

$$X^i(\sigma, \tau) = X^i_L(\sigma_+) + X^i_R(\sigma_-) \tag{2}$$

for *any* functions X^i_L and X^i_R. For simplicity, we turned off the left-moving solution, and wrote

$$X^i(\sigma, \tau) = X^i(\sigma_-)$$

The final ingredient we needed to put in was that the string is *closed*, and so if we shift $\sigma \to \sigma + l$ where l is defined as the length of the string, then our coordinates X^i must be unchanged. This lead to the condition

$$X^i(\sigma_- + l) = X^i(\sigma_-) \tag{3}$$

which is to say that $X^i(\sigma_-)$ is periodic, with period l. And so finally we made use of Fourier series to write our general solution as

$$X^i(\sigma_-) = b_0^i + \sum_{n=1}^{\infty} \left(b_n^i \cos\left(\frac{2\pi n}{l}\sigma_-\right) + c_n^i \sin\left(\frac{2\pi n}{l}\sigma_-\right)\right) \tag{4}$$

for some *numbers* $\{b_n^i, c_n^i\}$. So, we've made it! Just as a particle's motion is completely determined by its position and velocity, a string's motion is completely determined by these numbers. But what do they mean? Well, let's turn off b_n^i and c_n^i for $n = 1, 2, \ldots$. Then, we just have $X^i = b_0^i =$ constant, and so the string doesn't move. Then, turning the b_n^i and c_n^i back on creates wobbles, or oscillations, in the string.

> **Exercise II.0.1** By plotting $\cos\left(\frac{2\pi n}{l}\sigma_-\right)$ for various values of n, convince yourself that the higher n goes, the higher frequency the waves on the string are.

So, we've got a string and all its wobbles. You may worry - say we want to describe a string that's not wobbling, but it is moving with constant speed in some direction? It turns out that we eliminated any such motion when we threw away the left-moving bit of our solution. So, we should really think of (4) as describing a string *at rest*.

The focus of the rest of this project is to explore how these numbers $\{b_n^i, c_n^i\}$ are reinterpreted to describe the *mass* of the string when we zoom out and see it as a particle. It is through this mechanism that we hope that string theory can be used to describe the 'particles' our world is made up of!

II.1 Complex numbers are your friend

Before moving onto some quantum physics, its helpful (and fun!) to use some of the basics of complex numbers to write our solution (4) in an even *nicer* form.

So, why do we care about complex numbers? Consider the equation $x^2 - 1 = 0$, which has solutions $x = \pm 1$. Now consider the similar equation $x^2 + 1 = 0$. As we know that any real number x squared gives a positive real number, we know there are no solutions! More accurately, we say that there are no solutions 'over the real numbers $\mathbb{R}$'. So, we just invent a new 'number', say i, which satisfies $i^2 = -1$. Now we have a solution - it's just $x = i$. In fact, we have another solution too, $x = -i$. So once again we have two solutions. Again, more formally we say that the equation has two solutions 'over the complex numbers $\mathbb{C}$'. Note, as you will have seen, given some complex number $z \in \mathbb{C}$, we can always write it as $z = x + iy$ for real numbers $x, y \in \mathbb{R}$.

> **Exercise II.1.1** In Exercise I.1.2, we defined the exponential function by what's called a *power series*. To the first few orders in x, we had
>
> $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \tag{5}$$

It turns out that you can do the same thing with the familar trigonometric functions $\cos(\theta)$ and $\sin(\theta)$. These are

$$\begin{aligned}\cos(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\ \sin(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\end{aligned} \tag{6}$$

To define e^z for a complex number $z \in \mathbb{C}$, we simply stick z into the power series (5). By doing this for the complex number $z = i\theta$ where $\theta \in \mathbb{R}$, show that

$$\begin{aligned}e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)\end{aligned} \tag{7}$$

This then suggests that perhaps we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{8}$$

This is in fact true, and is known as *Euler's formula*.

Exercise II.1.2 For this exercise, you might find it useful to do a little background reading on complex numbers; a good place to start is this NRICH page:
`https://nrich.maths.org/1403`

i) Show that

$$\begin{aligned}\exp\left(\frac{2\pi nx}{l}i\right) &= \cos\left(\frac{2\pi nx}{l}\right) + i\sin\left(\frac{2\pi nx}{l}\right), \quad \text{and} \\ \exp\left(-\frac{2\pi nx}{l}i\right) &= \cos\left(\frac{2\pi nx}{l}\right) - i\sin\left(\frac{2\pi nx}{l}\right)\end{aligned}$$

where remember that $\exp(x)$ is just another way of writing e^x.

ii) Hence, show that

$$\begin{aligned}\cos\left(\frac{2\pi nx}{l}\right) &= \frac{1}{2}\left(\exp\left(\frac{2\pi nx}{l}i\right) + \exp\left(-\frac{2\pi nx}{l}i\right)\right), \quad \text{and} \\ \sin\left(\frac{2\pi nx}{l}\right) &= \frac{1}{2i}\left(\exp\left(\frac{2\pi nx}{l}i\right) - \exp\left(-\frac{2\pi nx}{l}i\right)\right)\end{aligned}$$

iii) Finally, convince yourself that (4) can be written as

$$X^i(\sigma_-) = \sum_{n=-\infty}^{\infty} a_n^i \exp\left(\frac{2\pi n\sigma_-}{l}i\right) \tag{9}$$

for some new set of numbers a_n^i.

iv) (**Harder - only have a go if you're feeling adventurous!)** Show that

$$a_0^i = b_0^i$$
$$a_n^i = \begin{cases} \frac{1}{2}\left(b_n^i - ic_n^i\right) & \text{for } n > 0 \\ \frac{1}{2}\left(b_n^i + ic_n^i\right) & \text{for } n < 0 \end{cases}$$

$$\left[\quad \text{It may be helpful to remember } \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{(-1)} = -i \quad \right]$$

So, the end result of all of this is that we can write our solution as

$$X^i(\sigma_-) = \sum_{n=-\infty}^{\infty} a_n^i \exp\left(\frac{2\pi n \sigma_-}{l} i\right) \tag{10}$$

for a set of complex numbers $\{a_n^i\}$ where $n \in \mathbb{Z}$.

II.2 Quantum states of the string

Let's summarise what we've learnt. We've found that given a little loop of string, we can describe its motion **completely** by a set of numbers a_n^i, where $i = 1, 2, \ldots, D-1$ labels the directions at right angles to the string, while the index n runs over all integers (both positive and negative). To be precise, we saw in the last part of Exercise II.1.2 that a_{-n}^i was the complex conjugate of a_n^i for $n \neq 0$, but we won't worry about that detail here. Also, we discard a_0^i for each i, and consider only those a_n^i for $n \neq 0$.

You'll be happy to hear that these a_n^i are all we need to proceed. From here on, we forget about the X^i and the wave equation they satisfy. We now introduce what it means to **quantise** a theory. In a quantum theory, at any point in time our system lies in a particular *state*, which we write $|\psi\rangle$. For instance, if we use a quantum theory to describe a cup of tea, then after brewing and taking out the teabag it will be in the state $|\psi_{\text{black}}\rangle$. To then change this state, we can act on this state with an *operator*, say A_{milk}, to arrive at a new state $|\psi_{\text{white}}\rangle = A_{\text{milk}} |\psi_{\text{black}}\rangle$.

For our purposes, the state $|\psi\rangle$ describes how our string is moving. Then, our operators are given by the a_n^i, which can act on states. For instance, given state $|\psi\rangle$, we have that $a_n^i |\psi\rangle$ is a different state.

Given two numbers x, y, we are used to being able to reorder the product $xy = yx$. This is called being able to *commute* these numbers. In quantum theory, however, operators in general **do not commute**. To get a handle on this, first define for two operators A, B, the **commutator**

$$[A, B] = AB - BA$$

which for general operators is not zero. Then, for the operators a_n^i, we have the following **commutation relation**:

$$[a_n^i, a_m^j] = \begin{cases} n & \text{if } i = j \text{ and } n + m = 0 \\ 0 & \text{otherwise} \end{cases}$$

In string theory, we define a special state called the **ground state** as the state $|0\rangle$ which satisfies

$$a_n^i |0\rangle = 0 \quad \text{for all } n > 0$$

You'll have noticed I'm sure that this is a completely abstract thing to say. But it turns out it is the *right* thing to say, and given this definition, the ground state $|0\rangle$ can be thought of as describing a **string at rest**. Then, to produce wobbly strings, we act on $|0\rangle$ with a_{-n}^i, with $n > 0$. The a_{-n}^i with $n > 0$ are called **creation operators**, while the a_n^i with $n > 0$ are called **annihilation operators**

Let's do an example to try to make sense of all of this. Consider the quantum state $a_2^3 a_{-1}^1 |0\rangle$. Then, according to the commutation relations, we have $[a_2^3, a_{-1}^1] = 0$. But, $[a_2^3, a_{-1}^1] = a_2^3 a_{-1}^1 - a_{-1}^1 a_2^3$. Hence,

$$a_2^3 a_{-1}^1 |0\rangle = a_{-1}^1 a_2^3 |0\rangle = 0$$

since $a_n^i |0\rangle = 0$ for any $n > 0$. Thus, this quantum state is **zero**.

Exercise II.2.1 Show that

i) $a_3^i a_{-4}^j |0\rangle = 0$ for any i, j

ii) $a_2^1 a_{-2}^1 |0\rangle = 2|0\rangle$

iii) $a_3^2 a_{-3}^1 |0\rangle = 0$

iv) $a_{-3}^1 a_3^1 a_{-3}^1 |0\rangle = 3a_{-3}^1 |0\rangle$

v) $a_{-2}^1 a_{-1}^3 |0\rangle = a_{-1}^3 a_{-2}^1 |0\rangle$ (Note, this is just saying that these two expressions describe the **same** state - we do not care about the order in which creation operators are written)

[*Hint: it is useful to use the manipulation for any two operators A, B given by*]

$$AB = [A, B] + BA$$

How we should think about what we have constructed is that $|0\rangle$ describes our string at rest. Then, we can act with the **creation** operators a_{-n}^i to create **wobbles** (i.e. oscillations) in the string. Remember, the n corresponds roughly to $\cos\left(\frac{2\pi n\sigma}{l}\right)$. You should draw this function for a few small values of n, to see that the higher the value of n, the more wobbly the function is. Another name for these functions are **harmonics**, i.e. $\cos\left(\frac{2\pi n\sigma}{l}\right)$ describes the n^{th} harmonic. This is exactly the same concept you may have seen in physics with standing waves. With this in mind, we would say that a_{-3}^1 creates a 3^{rd} harmonic oscillation in the X^1 direction. Starting with the ground state $|0\rangle$, the set of all states that we can create by acting with creation operators is called the **Fock space** of the theory.

Given the operators a_n^i, there is a very important operator that we can form from them, called the **number operator**, defined by

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-1} a_{-n}^i a_n^i$$

You should convince yourself that $N|0\rangle = 0$. The important property of N is that we have

$$[N, a^i_{-n}] = n a^i_{-n} \qquad \text{for } n > 0$$

This isn't too tricky to prove, but I wouldn't worry about doing so. Going forward, we will **not** need to explicit form of N - we only need this commutation relation.

Exercise II.2.2 Show that

i) $N a^2_{-3}|0\rangle = 3a^2_{-3}|0\rangle$

ii) $N a^1_{-2} a^3_{-1}|0\rangle = 3a^1_{-2}a^3_{-1}|0\rangle$

iii) $N a^i_{-n} a^j_{-m}|0\rangle = (n+m)a^i_{-n}a^j_{-m}|0\rangle$

You should hopefully have started to see a pattern. N counts the total number of oscillations. More precisely, given a quantum state $|\psi\rangle$ in the Fock space, we will have $N|\psi\rangle = n|\psi\rangle$, where remember here N is an operator, but n is just an **integer**. We say the state is at level n. Then, we have

$$N\left(a^{i_1}_{-n_1} a^{i_2}_{-n_2} \dots a^{i_k}_{-n_k}|0\rangle\right) = (n_1 + n_2 + \dots + n_k)\left(a^{i_1}_{-n_1} a^{i_2}_{-n_2} \dots a^{i_k}_{-n_k}|0\rangle\right)$$

This says that the creation operator a^i_{-n} contributes n units to the level of the state!

Exercise II.2.3 Write a little bit (perhaps half a side) on the quantum states of the string. Conclude by showing that there is

i) A single state at level 0

ii) $(D-1)$ states at level 1

iii) $\frac{1}{2}(D-1)(D+2)$ states at level 2

[*Hint: there are two types of states at level 2, namely* $a^i_{-1}a^j_{-1}|0\rangle$ and $a^i_{-2}|0\rangle$. *How many independent (i.e. unique) states are contained in these two expressions? Remember the last part of Exercise II.3.1...*]

II.3 The correct (??!!) dimension of spacetime!

We conclude the project with a fun little calculation - we're going to determine the unique dimension of spacetime in which string theory makes sense. Unfortunately, there are many steps that we will miss. These all have to do with the fact that we just discarded a^i_0, rather than dealing with it properly. It turns out this 'zero mode' encoded the linear momentum, and therefore the mass/energy, of the 'particle' that the string looks like when we zoom out.

Firstly, what do we mean by the dimension of 'spacetime'? Spacetime is simply the domain constructed by taking space and, well, adding time. So if we have D spatial dimensions, as we have done so far, then the dimension of spacetime is $D+1$.

> **Exercise II.3.1** Consider a closed string in a state $|\psi\rangle$. If we zoom out and regard it as a particle, then the mass (squared) of this particle is given by M^2, where
>
> $$\left(N - \frac{D-1}{24}\right)|\psi\rangle = M^2\,|\psi\rangle$$
>
> An important postulate of string theory is that the states at level one should represent **massless particles**. Hence, show that for string theory to be a consistent theory, the dimension of **spacetime** must be 26!

There a few things that have been swept under the carpet...

- Why do we have this shift by $\frac{D-1}{24}$? This seems very strange and arbitrary, and indeed this extra term has a long history. It turns out that when we take a classical theory and try to formulate from it a *quantum* theory, a lot of bad things can happen. These go by the title of **anomalies**. It turns out that in order to make these anomalies go away, we require precisely this shift, or else nothing we do makes any sense!
- What about the level zero states? With this formula, these would supposedly describe particles with *negative* mass squared?! These states go by the name of **tachyons**. They are the basic reason why the 'bosonic' string theory, which we have be describing, is thought to be inconsistent. It turns out that one needs an extra ingredient: an extra kind of symmetry called *supersymmetry.* With this added, we arrive at superstring theory, which is happily tachyon-free!
- But we don't live in 26 dimensions! This would certainly seem to be the case. Firstly, we should really be considering superstring theory, where the correct dimension of spacetime comes out at 10, by essentially the same calculation. But that's still 6 too many - we only live in 4 spacetime dimensions! There has been a huge amount of work done to try to explain these extra dimensions, and how their requirement in string theory could actually be compatible with our world. The basic idea is that of *compactification*, which says that the 6 extra (spatial) dimensions are not infinite as we're used to, but are actually wrapped up in little circles. This should be thought of like how on a cylinder, one direction is long and keeps going (this is one of 'our' dimensions), while the other goes round the cylinder, and is thus *periodic*, or *compact* (this is one of the 'compactified' dimensions). Then, the argument of compactification says that we live on a higher-dimensional analogue of this cylinder, in which we can only see the 'long' dimensions! While elegant in many ways, you runs into yet more troubles when you go down this path...
- **But I heard it was 11 dimensions!** You may have heard that spacetime is actually thought to be 11 dimensional. This is one of the central ideas of what is in many way the successor of string theory: M-Theory! In M-theory, one essentially says that even string theory is not the full picture, and that as a 10d theory it is actually a compactification of one even better 11d theory, which we call M-theory. Much is left to learn about M-theory, but its study has lead to huge leaps in our understanding of the complex interplay between abstract mathematics and physics

This concludes the material for the project. I hope this project has given you a hint of the wonders that lie within string theory. It would be an understatement to say that string theory is a broad field; there are now thousands of physicists (and mathematicians!) working worldwide on what can broadly described as string theory. Yet, there is still much more to learn - so perhaps it'll be you that joins the cause!