A Beginner's Guide to String Theory

String Theory has a long and complicated history, over which it's true interpretation has always been up for debate. While modern study in the field has blossomed to a hundred different topics, the purpose of this project is to explore the *foundations* of the subject. The idea is simple enough. We say that all of the different kinds of particles that we observe, like protons and electrons, are not different objects at all, but are all the *same* object. This object is a little loop of string, which just looks like a particle to us because we're so big! But how can that be? How can one string look like a proton, and another like an electron? The answer comes in two parts, which will roughly form the two parts of this project.

First, we need to understand these strings a bit better. If you imagine a little loop of string wobbling around in space, I'm sure you'll agree that there is a million different ways it could be wobbling. The aim of the first part of this project is to try to get a handle on this. We will show that given a string, we can always fully describe it's movement by simply specifying a set of numbers encoding this motion. To get to this conclusion, we will need to develop several crucial tools of mathematical physics, including differential equations, periodic functions, and eventually, complex numbers.

Then, we need to get from thinking about strings to thinking about particles. This amounts to 'zooming out', until our pieces of strings *look like* particles. What we will show is, if you take two strings that are wobbling differently, then zoom all the way out until they look like particles, then one of them might look like an electron, while the other like a proton! This is the magic of string theory: even though we are now discussing little particles, there is extra information contained in how these tiny strings are moving. But how can the wobbling of this string be interpreted as a particle's mass, or electric charge? Answering this question will be the aim of the second part of the project.

This is the task sheet for Part I. The idea is for you to get together as a group each week and work through the exercises. You should try to complete these tasks *as a team*. This may mean delegating certain things to different people, and then presenting your findings to the rest of the group. Note also that the aim of this task sheet is to give you problems to solve, but not necessarily full details of the tools you'll need to solve them. So, you'll sometimes need to go away and do a bit of research! You may want to look for calculus textbooks in your school library, while Google and Wikipedia are also certainly your friend!

Try to get through as much of this material as possible by the second meeting, but do not worry if you don't make it to the end! The most important thing is that you work together to get a strong understanding of what's going on.

Part I: Solving the Classical String

Our plan is the following. First, we're going to build up a solid foundation in functions and differential equations, which are fundamental building blocks of all of physics. Then, we will write down the particular, and very important, equation governing the dynamics of the string: *the wave equation.* With our newfound knowledge of differential equations, we will then solve the wave equation. We will discuss the implications of the string being *closed* (i.e. a loop rather than having ends), and will discover that the solutions we've found can be written in a nicer way. Finally, we will show that they can be written in an even *nicer* way, if we introduce complex numbers!

I.1 Getting to grips with functions and derivatives

Our starting point will be functions and derivatives. A *set* A is a collection of distinct objects. $\{1, 4, 7\}$ is a set, of three objects. The natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ is a set, with an infinite number of elements. Finally, of interest to us is the *set of real numbers* $\mathbb{R}$. This is simply the set of all numbers! This includes integers and fractions of integers, i.e. rational numbers. It also includes *irrational* numbers, which are defined as numbers that cannot be expressed as a fraction of integers. For example, π is irrational.

Given two sets A, B, a *function* is a map

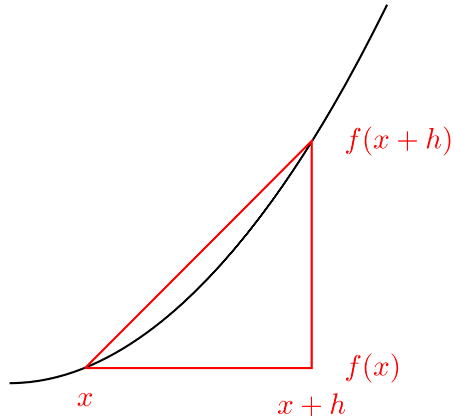
$$f : A \to B \tag{1}$$

i.e. for each element a in the set A, we can act with f to get $f(a)$, which is an element of B. For the remainder of this section, we will be interested in functions $f : \mathbb{R} \to \mathbb{R}$. Given a number $x \in \mathbb{R}$, an example of such a function is $f(x) = 3x^2 - 2x$. Another is $f(x) = 3\cos\left(x^2\right)$. Here, the symbol " $\in$ " just means "in".

Next, we want to define the *derivative* of a function $f(x)$. Given a function $f(x)$, we can define a new function, written $f'(x)$ or $\frac{df}{dx}(x)$, called its *derivative* (with respect to x). The derivative is also sometimes called the *slope* of $f(x)$. Evaluating $f'(x)$ at the point x tells us the gradient of the tangent to $f(x)$ at this point.

You may know some simple derivatives. For instance, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. In this first exercise, you should get an idea of why this is the right answer, for the case of $f(x) = x^2$.

Exercise I.1.1 Consider the function $f(x) = x^2$, and draw it. Now, suppose at some point x we want to work out the slope of $f(x)$. We can get a good approximation by considering two points on the x-axis, x and $x + h$, separated by a small, positive number h. We then draw a triangle, as seen in the diagram below.



i) By considering the gradient of the hypotenuse of this triangle, convince yourself that for h small, we can *approximate*

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

This leads us to the actual formal *definition* of the derivative! In particular, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

Now, the limit symbol is probably new to you. All it means in this context is, we evaluate the fraction

$$\frac{f(x+h) - f(x)}{h}$$

and then we take the value of h to zero. Looking back at our diagram, as we take $h \to 0$, we expect our approximation to become better and better, and indeed, when we take $h \to 0$, the approximation becomes *exact*!

ii) By putting $f(x) = x^2$ into (2), prove that $f'(x) = 2x$

iii) Let's go one step further. Letting $g(x) = x^n$, prove that $g'(x) = nx^{n-1}$

Now, we know that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. Another way to write this is

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{3}$$

We'll also later need to take the derivative of the trigonometric functions $\sin(x)$ and $\cos(x)$. These are

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{4}$$

A final, and very important function is the exponential function, $f(x) = e^x$. This is simply a particular number, $e = 2.71828\ldots$ raised to the power of x. But wait - what does it mean to raise a number to the power x, if x is a general real number? You know what to do if x is an integer, and you might also know what to do if it's rational (e.g. $e^{1/2} = \sqrt{e}$), but what about an *irrational* number? Thankfully, there is a clearer definition for e^x, which nonetheless reproduces the expected results when x is rational.

Exercise I.1.2 We *define* the exponential function $f(x) = \exp(x) = e^x$ (two equivalent notations) by

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \tag{5}$$

i) Go away and make sure you understand all the notation in (5). This might require some Googling or looking in textbooks

ii) Using (5), prove that

$$\frac{d}{dx} e^x = e^x$$

This is indeed the defining property of the exponential function; it is the *unique* solution to $f'(x) = f(x)$, up to rescaling by a constant.

We now need some rules for finding the derivative of some more complicated functions. For instance, we know that $\frac{d}{dx}\sin(x) = \cos(x)$, and $\frac{d}{dx}x^3 = 3x^2$, but what is $\frac{d}{dx}\left(x^3 \sin(x)\right)$, or $\frac{d}{dx}\left(\sin(x^3)\right)$? To tackle these functions, we need the *product rule* and the *chain rule*.

The Product Rule: Let $f(x), g(x)$ be two functions. Then, can form a new function by multiplying f and g, to form $f(x)g(x)$. For example, if $f(x) = \sin(x), g(x) = x^3$, then $f(x)g(x) = x^3\sin(x)$. Then, to find the derivative of this function, we use the *product rule*:

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \left(\frac{df}{dx}\right)(x)\ g(x) + f(x)\left(\frac{dg}{dx}\right)(x) = f'(x)g(x) + f(x)g'(x)$$

The Chain Rule: Given $f(x), g(x)$, there's another new function we can define, called the *composition* of f and g, and written $f(g(x))$. For example, if $f(x) = \sin(x), g(x) = x^3$, then $f(g(x)) = \sin(x^3)$. Then, to find the derivative of this function, we use the *chain rule*:

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x))g'(x)$$

Exercise I.1.3 Let's get some practise of using these rules. Show that

i) $\frac{d}{dx}\left(x^3\sin(x)\right) = 3x^2\sin(x) + x^3\cos(x)$

ii) $\frac{d}{dx}\left(\sin(x^3)\right) = 3x^2\cos(x^3)$

iii) $\frac{d}{dx}\left(\sin(\cos(x))\right) = -\sin(x)\cos(\cos(x))$

iv) $\frac{d}{dx}\left(e^{x\sin(x^2)}\right) = \left(\sin(x^2) + 2x^2\cos(x^2)\right)e^{x\sin(x^2)}$

The final tool we'll need to tackle the differential equations of the string is the idea of a *partial derivative*. As we discussed in our first meeting, sometimes we consider functions not of a single variable, but of multiple. For instance, we could have a function $f(t,x)$ of *both* t and x. In the notation of (1), we would write

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \tag{6}$$

Now, we can take the derivative of f with respect to its two arguments independently. Then, the partial derivative with respect to t is written $\frac{\partial f}{\partial t}$.

Exercise I.1.4 Show that

i) $\frac{\partial}{\partial t}\left(t^2\cos(x)\right) = 2t\cos(x)$

ii) $\frac{\partial}{\partial x}\left(t^2\cos(x)\right) = -t^2\sin(x)$

iii) $\frac{\partial}{\partial x}\left(e^{t^3 x}\right) = t^3 e^{t^3 x}$

iv) $\frac{\partial}{\partial t}\left(\sin(x - vt)\right) = -v\cos(x - vt)$

v) Let $f(t,x) = x\sin(xt)$. Verify that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}\frac{\partial}{\partial t}f = 2x\cos(xt) - x^2 t\sin(xt)$$

vi) Let $f(x,y) = f_1(x) + f_2(y)$ for *any* functions f_1, f_2, and show that $f(x,y)$ satisfies

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

This is in fact the *general* solution to this equation!

Part (v) of this exercise is an example of a general result; that for *any* $f(t,x)$, we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}\frac{\partial}{\partial t}f$$

Make sure you understand this. It's saying, we can take the x derivative *then* the t derivative, or take the t derivative *then* the x derivative, and the result is the same. This is the statement that *partial derivative commute.*

Now, how does the chain rule work for partial derivatives? Suppose we have $f(x,y)$, but both x, y are functions of two further variables s, t. We can write this as $f(x(s,t), y(s,t))$. Then, the chain rule for partial derivatives is

$$\frac{\partial f}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial f}{\partial y} \tag{7}$$

Exercise I.1.5 Let $f(x, y) = x^2 + y$, and $x(s, t) = -s^2$, $y(s, t) = st$

i) Show that $f(x(s, t), y(s, t)) = s^4 + st$, and hence that

$$\frac{\partial f}{\partial s} = 4s^3 + t \tag{8}$$

ii) Show that

$$\frac{\partial f}{\partial x} = 2x = -2s^2, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial x}{\partial s} = -2s, \quad \frac{\partial y}{\partial s} = t$$

and hence verify that (7) reproduces (8)

iii) Do this whole procedure again, but for $\frac{\partial f}{\partial t}$

I.2 Solving some differential equations

In this section, we're going to put what we've learnt to use, by solving some equations! In particular, we are looking to solve some *differential equations*. We'll first look at *ordinary* differential equations (ODEs), which involve functions of a single variable. We'll then upgrade to *partial* differential equations, where we also allow for functions of multiple variables.

While differential equations pop up everywhere in physics, the story is always the same. We write down an equation which some unknown function $f(x)$ must satisfy, then we look for solutions for what this function might be. Let's do an example. Take the equation

$$f''(x) + f'(x) - 2f(x) = 0 \tag{9}$$

Here, the notation $f''(x)$ denotes the second derivative of $f(x)$ with respect to x, i.e.

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

We could call the equation (9) a *second order* ODE, because the highest derivative of f that it contains is the *second* derivative. Then, a very powerful theorem states that an ODE of order n has precisely n solutions. So we expect (9) to have precisely 2 solutions.

Exercise I.2.1

i) Use the *ansatz* (i.e. guess) $f(x) = e^{\lambda x}$ to find the values of the constant λ such that $f(x)$ solves (9). You should indeed find that there are *two* solutions for λ!

ii) Given your two solutions $f_1(x)$ and $f_2(x)$, show that

$$\tilde{f}(x) = a_1 f_1(x) + a_2 f_2(x)$$

is also a solution for any value of the constants a_1, a_2.

iii) Write up your findings from the exercise clearly

What we have shown in this exercise is that given your n solutions to an order n ODE, we can always form what is called a *linear combination* of them, and we still have a solution. This is really just a result of the fact that (9) is *linear*, meaning that it only includes single copies of $f(x)$, and no terms like $f(x)^2$. This ability to form *linear combinations* will be essential to understanding the string!

Finally, we're ready to tackle an important generalisation of what we've learnt already: *partial* differential equations (PDEs). Unsurprisingly, these are just differential equations in which our unknown function f is a function of multiple variables, i.e. $f(x_1, x_2, \dots)$. While there is a very well understood way to solve a huge class of ODEs, the realm of PDEs is vastly more difficult, and very few can be solved completely! Hence, the effort to find new and innovative ways to solve PDEs, which underpin so much of physics, has lead to fascinating new directions in pure mathematics.

Exercise I.2.2 The Heat Equation. There are two particularly important PDEs in physics, and this is one of them. Imagine you have a bar of metal, and you heat up one end. You expect that, over time, this heat disperses through the bar; this process is modelled by the heat equation. To be more precise, let $u(t, x)$ denote the temperature at a point x along the rod, at time t. Then, over time, this temperature is governed by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{10}$$

While we won't study this equation in detail (which would be a whole project in itself!), it gives us the perfect opportunity to verify some solutions to it.

i) Show that $u(t, x) = e^{\lambda^2 t + \lambda x}$ solves (10) for any constant λ

ii) Show that $u(t, x) = e^{-\lambda^2 t} \sin(\lambda x)$ solves (10) for any constant λ

iii) (Harder - only have a look if you have time to spare and are interested!) Show that

$$u(t, x) = \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)$$

solves (10). Don't worry about what happens at $t = 0$!

Before we move on, there is one further trick we can use when solving a differential equation that will prove essential to solving the wave equation; the idea of *changing variables.* Consider the function

$$f(u) = 3u^2 - 2u + 1 \tag{11}$$

Now, rather than using u as the independent variable, we could instead use $v = \frac{1}{2}u - 2$, which is easily inverted to find $u(v) = 2v + 4$. Then, we can define a new function $\tilde{f}(v)$ by

$$\begin{aligned} \tilde{f}(v) &:= f\left(u(v)\right) = 3\left(2v + 4\right)^2 - 2\left(2v + 4\right) + 1 \\ &= 12v^2 + 44v + 41 \end{aligned} \tag{12}$$

Exercise I.2.3

i) Show that the function $f(u)$ as defined in (11) solves the differential equation

$$\frac{df}{du}(u) = 6u - 2 \tag{13}$$

ii) The chain rule then tells us that the new function $\tilde{f}(v) = f(u(v))$ satisfies

$$\frac{d\tilde{f}}{dv}(v) = \frac{df}{du}(u(v))\frac{du}{dv}(v) \tag{14}$$

By calculating the three quantities in (14) individually, verify this equation for our f and $\tilde{f}$

iii) Manipulate (14) to show that

$$\frac{df}{du}(u(v)) = \frac{1}{2}\frac{d\tilde{f}}{dv}(v)$$

Hence, show that in terms of $\tilde{f}$, the differential equation (13) can be written

$$\frac{d\tilde{f}}{dv}(v) = 24v + 44 \tag{15}$$

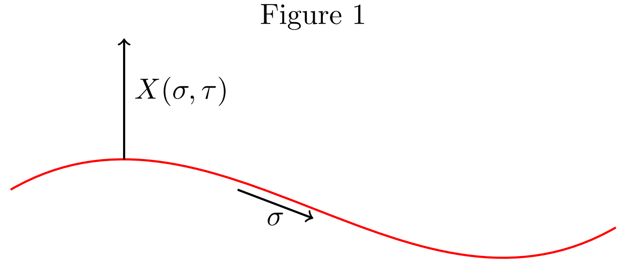
Finally, verify that this differential equation is indeed solved by our $\tilde{f}$, as given in (12)

[*Note: it is standard to drop the* $\sim$ *over* $\tilde{f}$ *and just write* $f(v)$*, with the argument telling us whether we mean the function in terms of* u *or* v*. While this notation is open to misinterpretation, it is (unfortunately) very standard practise!*]

I.3 Strings, and a very special PDE

We have finally got all the tools we need to talk about strings! So, how do we describe a string using functions? Consider a little piece of string going across this piece of paper, shown by the red line below

Figure 1



We use the greek letter σ to tell us where we are along the string. Then, at each σ, we need to specify how far up the piece of paper the string is, which we denote $X(\sigma)$. But, if we want the string to move over time, we also need to allow u to vary over time, which we denote by the greek letter τ. So, we have a function $X(\sigma, \tau)$ of two variables!

Now, one can show that this function $X(\sigma, \tau)$ must satisfy the *wave equation*, which is

$$\frac{\partial^2 X}{\partial \tau^2} - c^2 \frac{\partial^2 X}{\partial \sigma^2} = 0 \ , \tag{16}$$

for some constant $c \neq 0$, which can be written more compactly as

$$X_{\tau\tau} - c^2 X_{\sigma\sigma} = 0 \ .$$

Before finding the general solution to this equation, let's get to grips with a particular solution.

Exercise I.3.1 Show that $X(\sigma, \tau) = \sin(\sigma - c\tau)$ solves the wave equation. On the same graph, plot $X(\sigma, \tau) = \sin(\sigma - c\tau)$ for $c = \frac{1}{2}$, and

- $t = 0$
- $t = \pi$
- $t = 2\pi$

Now do the same for $c = -\frac{1}{2}$, and again for $c = 1$. How should we interpret c?

Exercise I.3.2 Solving the wave equation

We will show that the most general solution to the wave equation is given by

$$X(\sigma, \tau) = f(\sigma - c\tau) + g(\sigma + c\tau) \tag{17}$$

where f, g are *general* functions of one variable.

i) Define $\alpha_+ = \sigma + c\tau$ and $\alpha_- = \sigma - c\tau$. Using the chain rule (7), show that

$$\frac{\partial X}{\partial \sigma} = \frac{\partial X}{\partial \alpha_+} + \frac{\partial X}{\partial \alpha_-}$$

and derive a similar expression for $\frac{\partial X}{\partial \tau}$. Then, by taking a second derivative of these expressions and using the chain rule again, show that

$$\frac{\partial^2 X}{\partial \sigma^2} = \frac{\partial^2 X}{\partial \alpha_+^2} + \frac{\partial^2 X}{\partial \alpha_-^2} + 2\frac{\partial^2 X}{\partial \alpha_+ \alpha_-}$$

and derive a similar expression for $\frac{\partial^2 X}{\partial \tau^2}$. Hence, show that the wave equation (16) can be recast as

$$\frac{\partial^2 X}{\partial \alpha_+ \alpha_-} = 0$$

ii) Use the results of Exercise I.1.4 to write down the general solution (17)

You should try to present your solution in a single document, either hand-written or electronically.

So, we have succeeded in describing a string moving on a sheet of paper, or 'in the plane'. As one final generalisation, we want to describe a string moving in d spatial dimensions, where d is some integer. Now, a line is one dimensional, a plane is two dimensional, and our world is three dimensional. When we described a string in two dimensions, we had one direction along the string, σ, and another direction 'off' the string, X, as shown in Figure 1. So in total, two spatial dimensions. If follows that if we want to describe a string in d spatial dimensions, we should still have the one direction along the string, σ, and then $(d-1)$ more directions X^i off the string, where i is an index that runs $i = 1, 2, \ldots, d-1$. It turns out then then each of these $d-1$ functions $X^i(\sigma, \tau)$ satisfies the wave equation, i.e.

$$\frac{\partial^2 X^i}{\partial \tau^2} - \frac{\partial^2 X^i}{\partial \sigma^2} = 0 \quad \text{for each } i = 1, 2, \ldots, d-1$$

where we have now set the wave speed $c = 1$. It is then convention to write the general solution as

$$X^i(\sigma, \tau) = X^i_L(\sigma_+) + X^i_R(\sigma_-)$$

where $\sigma_\pm = \sigma \pm \tau$, and $X^i_{L/R}$ can be *any* functions! We've used the labels L/R to denote the left- and right-moving parts. Can you work out why these labels make sense? Going forward, we will just set $X^i_L = 0$, and only consider the right-moving part of our solutions, and so we'll write

$$X^i(\sigma, \tau) = X^i(\sigma_-)$$

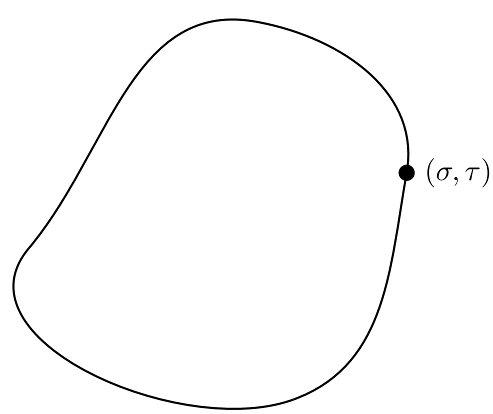
It turns out this will be enough to understand all the important properties of the string!

I.4 Closing the loop

Now, there is one more detail about the string's physics that we have so far swept under the rug. There are two types of strings: *closed* strings, which are loops, and *open* strings, which have endpoints. It turns out the the nuances associated to open strings leads to the notion of *branes*, which single-handedly revolutionised our understanding of string theory. In this project, however, we will be interested in *closed* strings, which in particular give rise to gravity.

So, once again, imagine our functions $X^i(\sigma_-)$ describing the position of our string at position σ and time τ, where recall $\sigma_- = \sigma - \tau$. Further, suppose the string is closed, with length l. The string at a fixed time τ is drawn below in Figure 2.

Figure 2



Now, from the marked point, we could imagine moving along the loop to another point, say $(\sigma + a, \tau)$, and we could ask what the position of our string is at this point. By shifting σ by η, we also shift $\sigma_- = \sigma - \tau$ by a. So, this position is given by $X^i(\sigma_- + a)$. In particular, suppose we choose $\eta = l$, the length of the string, then we come back to the *same point*. Hence, we must have $X^i(\sigma_- + l) = X^i(\sigma_-)$! This is the statement that $X^i(\sigma_-)$ is a *periodic function* of σ_-, with period l.

Exercise I.4.1 We're going to get to grips with periodic functions by looking at some examples. We say a function $f(x)$ is periodic with period L if L is the *minimal* positive real number such that $f(x + L) = f(x)$ for any x.

i) Show that $f(x) = \cos(2\pi x)$ has period $L = 1$, i.e. $f(x + 1) = f(x)$ for any x. You may want to make use of the formula for $\cos(A + B)$

ii) Show that $f(x) = \sin(2\pi x)$ satisfies $f(x + 2) = f(x)$. What is the period of $f(x)$?

iii) Suppose $f(x)$ had period L. Show that $f(x + nL) = f(x)$ for any integer $n \in \mathbb{Z}$

iv) Using parts (i) and (ii), show that the functions

$$f_n(x) = \cos\left(\frac{2\pi n}{l}x\right), \quad g_n(x) = \sin\left(\frac{2\pi n}{l}x\right), \quad n \in \mathbb{Z} \tag{18}$$

each have period l/n. Thus, using part (iii), show that

$$f_n(x + l) = f_n(x), \quad g_n(x + l) = g_n(x) \tag{19}$$

So, we've gotten comfortable with periodic functions by looking at the trigonometric functions $\sin(x)$ and $\cos(x)$. This perhaps felt natural - after all, the fact that these functions are periodic with period 2π is one of the first things we learn about them! The remarkable thing is that these trigonometric functions are the *whole* story! In a certain sense that we will soon define, *all* periodic functions are trigonometric functions! This is the central result of a crucial field called Fourier Analysis.

Let's state the result more clearly. Let $f(x)$ be a function satisfying $f(x + l) = f(x)$. Then, there exists numbers a_i with $i = 0, 1, \ldots$ and b_i with $i = 1, 2, \ldots$ such that

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{l}x\right) + b_n \sin\left(\frac{2\pi n}{l}x\right)\right) \tag{20}$$

You should be able to verify using part (iv) of Exercise I.4.1 that we do indeed have $f(x + l) = f(x)$. Further, we have that the choice of the number $\{a_n, b_n\}$ is *unique*. This means that for every $f(x)$ there is a unique set of number $\{a_n, b_n\}$ corresponding to it, and conversely given a set of numbers $\{a_n, b_n\}$ there is a unique function $f(x)$ corresponding to it. The expression (20) is called the Fourier series for $f(x)$.

Take a second to understand this result. We've gone from having a general function whose only constraint is periodicity, and from it we've found a set of numbers which contain *exactly* the same amount of information!

Exercise I.4.2 Consider the function

$$f(x) = \cos^2\left(\frac{\pi x}{l}\right) \tag{21}$$

i) Show that $f(x+l) = f(x)$

ii) By our theorem, we should therefore be able to find a Fourier series for $f(x)$. Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, show that

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{l}x\right) \tag{22}$$

and hence that the correct choices for the numbers $\{a_n, b_n\}$ is $a_0 = a_1 = \frac{1}{2}$, with all other numbers zero

We can now finally use what we've learnt about Fourier series to write our solution $X^i(\sigma_-)$ in terms of a Fourier series. We have

$$X^i(\sigma_-) = a_0^i + \sum_{n=1}^{\infty}\left(a_n^i \cos\left(\frac{2\pi n}{l}\sigma_-\right) + b_n^i \sin\left(\frac{2\pi n}{l}\sigma_-\right)\right) \tag{23}$$

Note, we now have a second index i on the numbers (or 'Fourier coefficients') a_n^i, b_n^i, since we have to specify these coefficients for each of the directions $i = 1, \ldots, d-1$.