King's College London

University of London

Statistical Physics Approach to Models of Risk

Reimer Kühn

Department of Mathematics, King's College London

with
P Neu (BCG), J Hatchett (Hymans Robertson), K Anand (KCL)

To access published results go to

http://www.mth.kcl.ac.uk/∼kuehn/riskmodeling.html

Fundamental Problem of Risk Analysis

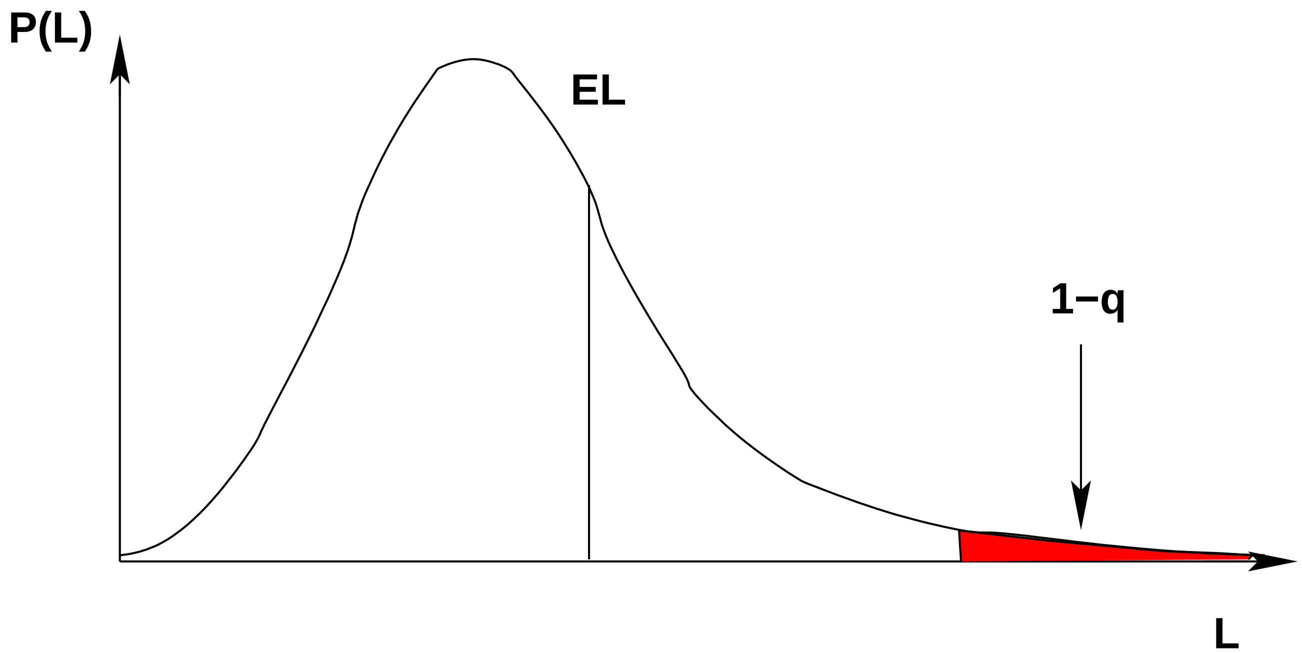
- Estimation of risk
  - Market: potential negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices)
  - Credit: potential change of credit quality, including default (asset values of firms, ratings, stock-prices)
  - Operational: potential losses incurred by process failures (human errors, hardware/software-failures, lack of communication, fraud, external catastrophes)
  - Liquidity: potential losses incurred by rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity

Task

- Estimate PDF of
  - Portfolio value at time T, $PV(T)$, given $PV(0)$
  - Market value of assets of obligors at time T, $A_i(T)$, given $A_i(0)$
  - Losses $L(T)$ due to process failures incurred during risk horizon T
  - Losses $L(T)$ incurred during risk horizon T by need to maintain liquidity in situations of stress
- Quantity of interest: Value at Risk

$$\mathrm{VaR}_{q,T} = (Q_q[L(T)] - EL)\,\mathrm{e}^{-rT} \qquad \mathrm{Prob}(L(T) \leq Q_q[L(T)]) = q$$

- Quantity of interest: Value at Risk



$$\mathrm{VaR}_{q,T} = (Q_q[L(T)] - EL)\, \mathrm{e}^{-rT} \qquad \mathrm{Prob}(L(T) \leq Q_q[L(T)]) = q$$

- Requires good knowledge of tails of loss distributions.

Assessment of PDFs (tails, VaR)

- adjustment of business model, (re)design of processes
  - charged fees, interest rates, rating of clients
  - activities on derivative markets (hedging)
  - insurance policies
- Proper risk control and management is
  - *demanded* by international banking supervision (BASEL)
  - *recognised* by rating agencies, analysts

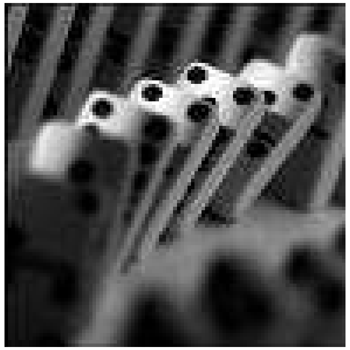
Our Main Interest and Concern: Interactions

- Traditional approaches treat risk elements as independent or *at best* statistically correlated

- Misses functional & dynamic nature of relations: terminal–mainframe/input errors–results/manufacturer–supplier relations . . .

- Effect of interactions between risk elements

- Possibility of avalanches of risk events (process failures, defaults)
- Fat tails in loss distributions
- Volatility clustering in markets (intermittency)



The Case of Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes

- Two state model: processes either up and running ($n_i = 0$) or down ($n_i = 1$)

- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes (quenched disorder); connectivity functionally defined

  $\Longrightarrow$ (lattice gas) model defined on random graph

- losses determined (randomly) each time a process goes down (annealed disorder)

Dynamics

- processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)

- h_{it} total support received by process i at time t

$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$

  with η_{it} random (e.g. Gaussian white noise).

- process i will fail, if the total support for it falls below a critical threshold

$$n_{it+\Delta t} = \Theta\left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it}\right)$$

Meaning of Parameters

- Probability of failure in given 'situation'

$$\text{Prob}\left(n_{it+\Delta t} = 1 \,\middle|\, \boldsymbol{n}(t)\right) = \Phi\left(\sum_j J_{ij}\, n_{jt} - \vartheta_i\right)$$

with

$$\Phi(x) = \frac{1}{2}[1 + \text{erf}(x/\sqrt{2})]$$

- unconditional and conditional probability of failure

$$\begin{aligned} p_i &= \Phi(-\vartheta_i) \\ p_{i|j} &= \Phi(J_{ij} - \vartheta_i) \end{aligned}$$

$$\Rightarrow \quad \vartheta_i = -\Phi^{-1}(p_i)\ , \quad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)$$

Key Features

- For sufficiently strong cooperative interactions, get first order phase transition: coexistence of ‘functioning state and state of ‘catastrophic breakdown (dominoes)

- Critical point at sufficiently high p_i.

- Of special relevance: resilience to (external) stress can be tested.

The Case of Credit Risks — Interacting Companies

- Risk arising from the possibility of obligors going bankrupt or from changes in 'credit quality' ($\Rightarrow$ credit trading)

- Here only influence of defaults

- Two state model: company up and running ($n_i = 0$) or down ($n_i = 1$)

- Probabilities of default and mutual impacts of defaults heterogeneous across the set of companoes (quenched disorder); connectivity functionally defined

  $\Longrightarrow$ (lattice gas) model defined on random graph

- Losses determined (randomly: recovery process) when a company defaults (annealed disorder)

Dynamics

- Companies need “orders” (support, cash inflow) to maintain wealth and avoid default

- h_{it} total wealth of company i at time t,

$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$

  with η_{it} random (e.g. Gaussian white noise).

- company i defaults, if the total wealth falls below zero

$$n_{it+1} = n_{it} + (1 - n_{it}) \Theta \left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

- No recovery within ‘risk horizon’ T: $n_i = 1$ is absorbing state.
  Time unit: 1 month; $T = 12 \Leftrightarrow 1$ year.

Key Features

- Interpretation of parameters ϑ_i and J_{ij} in terms of unconditional and conditional default probabilities

- For sufficiently strong interactions — in particular cooperative interactions — get possibility of collective acceleration of default rates in times of economic stress

- Typical behaviour less sensitive to interactions than rare events $\Rightarrow$ fat tails in loss distribution; important for risk analysis where rare event asymptotics is relevant.

- Due to initial conditions $n_{i0} \equiv 0$ and absorbing state: $\Rightarrow$ no equilibrium dynamics

Analysis for a Stochastic Setting

- Interactions on a random graph

$$J_{ij} = c_{ij} \left(\frac{J_0}{c} + \frac{J}{\sqrt{c}} \, x_{ij} \right)$$

with

$$P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0} \ ,$$

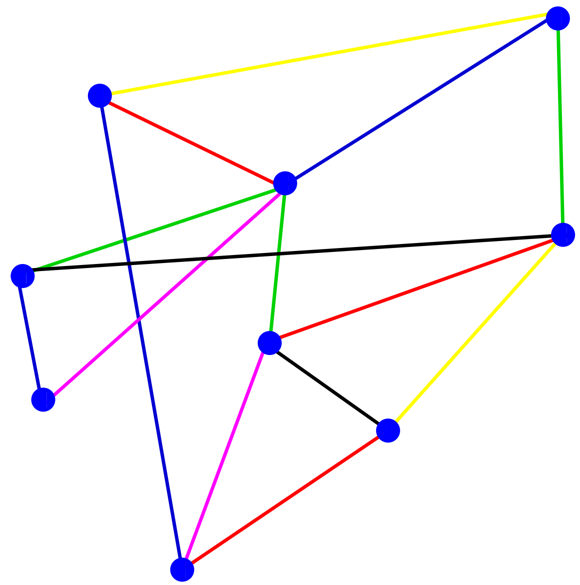
symmetric ($c_{ij} = c_{ji}$), and

$$\overline{x}_{ij} = 0 \ , \quad \overline{x_{ij}^2} = 1 \ , \quad \overline{x_{ij} x_{ji}} = \alpha \ .$$

Study the limit

$$c \gg 1 \ , \qquad N \gg 1 \ .$$

Processes/organizations: small J !



Dynamical Mean-field Analysis — Generating Functions

- Generating function for correlation functions

$$Z[\psi] = \left\langle \mathrm{e}^{-\mathrm{i}\sum_{it}\psi_{it}n_{it}} \right\rangle = \sum_{\boldsymbol{n}_0,\ldots,\boldsymbol{n}_T} P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_T]\ \mathrm{e}^{-\mathrm{i}\sum_{i,t}\psi_{it}n_{it}}$$

- Correlation functions for typical disorder ($\Leftrightarrow$ averaged correlation functions

$$\overline{\langle n_{it}\rangle} = \mathrm{i}\frac{\partial \overline{Z[\psi]}}{\partial \psi_{it}}\bigg|_{\psi\equiv 0} \quad , \qquad \overline{\langle n_{is}n_{jt}\rangle} = \mathrm{i}^2\frac{\partial^2 \overline{Z[\psi]}}{\partial \psi_{is}\partial\psi_{jt}}\bigg|_{\psi\equiv 0}$$

- Credit risk: decompose stochastic force

$$\eta_{it} = \sqrt{\rho}\ \eta_0 + \sqrt{1-\rho}\ \xi_{it}$$

Global component η_0 slowly varying (e.g. fixed over risk horizon of one year); OR: $\rho = 0$

Generating Functions — Solution

- System represented by ensemble of effective single site processes parameterised by ϑ

  - OR ($\rho = 0$)

$$n_{t+1} = \Theta\Big(J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s - \vartheta + \phi_t\Big)$$

  - CR ($\rho \neq 0$, $n_{it} = 1$ absorbing)

$$n_{t+1} = n_t + (1 - n_t)\Theta\Big(J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Big)$$

- Single-site processes exhibit

coloured noise $\{\phi_t\}$, and memory G_{ts}

Generating Functions —Self-Consistency

- Self-consistency equations

$$
\begin{aligned}
\langle\phi_s\phi_t\rangle &= (1-\rho)\delta_{st} + J^2 q_{st} \\
m_t &= \langle\langle n_t\rangle\rangle_\vartheta \\
q_{st} &= \langle\langle n_s n_t\rangle\rangle_\vartheta \\
G_{ts} &= \left\langle \frac{\mathrm{d}\langle n_t\rangle}{\mathrm{d}h_s} \right\rangle_\vartheta
\end{aligned}
$$

- Interpretation of variables

$$m_t = \frac{1}{N}\sum_i \overline{\langle n_{it}\rangle}$$

$$q_{st} = \frac{1}{N}\sum_i \overline{\langle n_{is} n_{it}\rangle} \quad , \quad G_{ts} = \frac{1}{N}\sum_i \frac{\mathrm{d}}{\mathrm{d}h_s}\overline{\langle n_{it}\rangle}$$

Generating Functions —Fully Asymmetric Network ($\alpha = 0$)

- At $\alpha = 0$, no memory effects, and m_t sufficient to describe the dynamics
  - OR ($\rho = 0$)

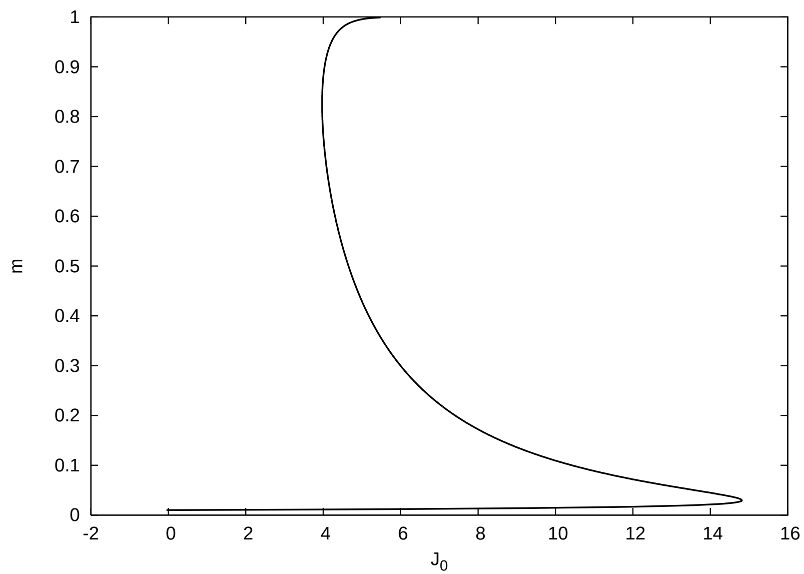
$$m_{t+1} = \left\langle \phi \left(\frac{J_0 m_t - \vartheta}{\sqrt{1 + J^2 m_t}} \right) \right\rangle_\vartheta$$

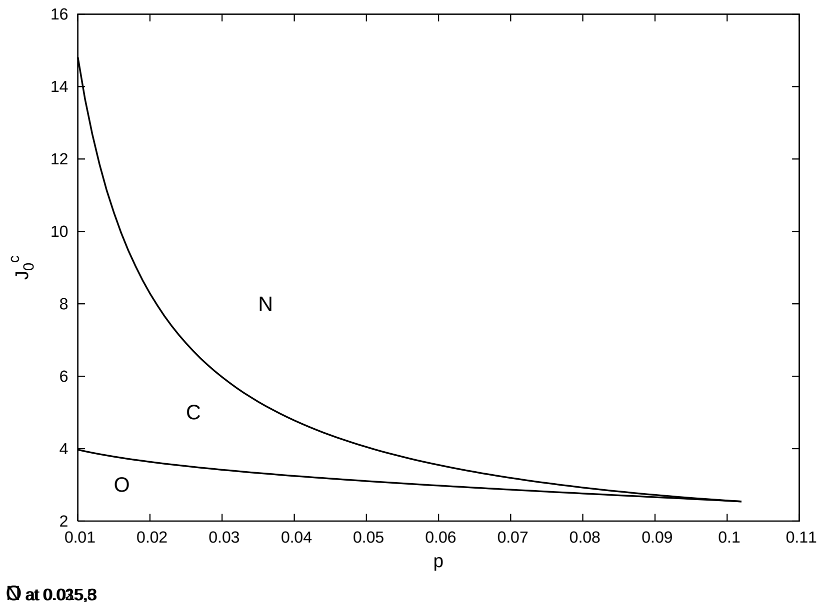
  - CR ($\rho \neq 0$, $n_{it} = 1$ absorbing)

$$\langle n_{t+1} \rangle = \langle n_t \rangle + (1 - \langle n_t \rangle) \Phi \left(\frac{J_0 m_t + \sqrt{\rho}\, \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right)$$

$$m_{t+1} = m_t + \left\langle (1 - \langle n_t \rangle) \Phi \left(\frac{J_0 m_t + \sqrt{\rho}\, \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right) \right\rangle_\vartheta$$

Results OR — Stationary Solution at $\alpha = 0$

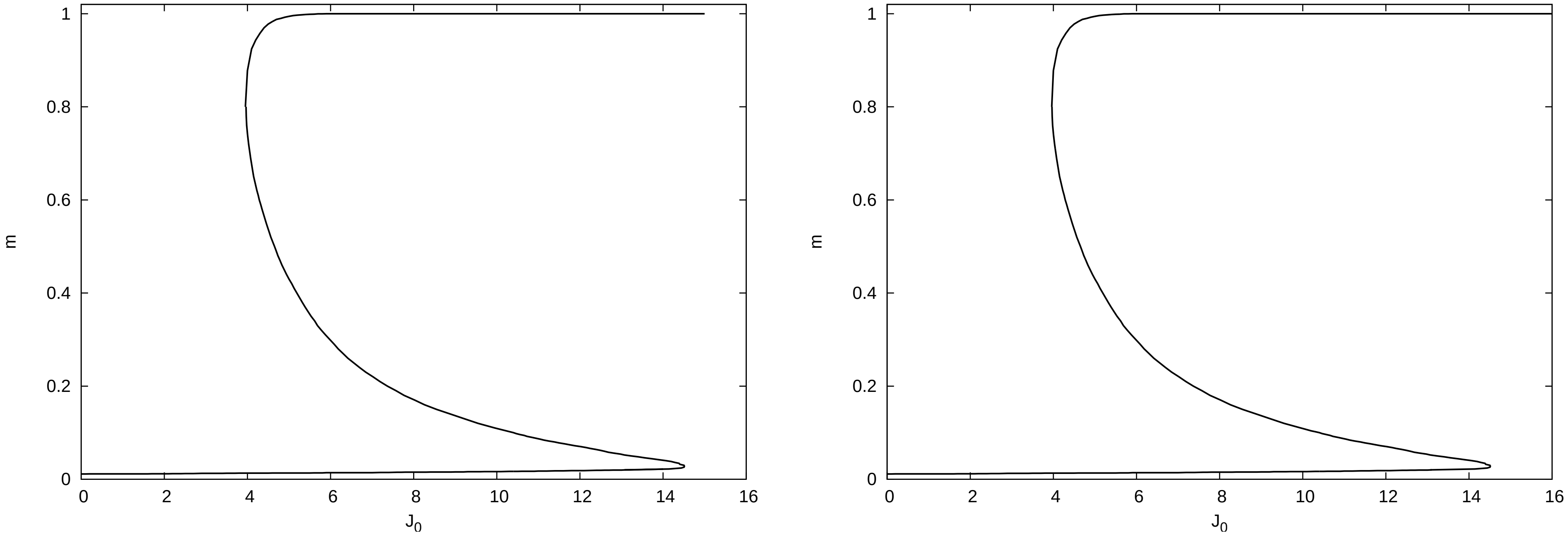




(K Anand and RK, Phys Rev E**75** (2007))

Stationary fraction of down-processes (left) and phase diagram (right) for $p_i = 0.001$, $\alpha = 0$, $J = 0.2$

Results OR — Stationary Solution at $\alpha \neq 0$



(K Anand and RK, Phys Rev E**75** (2007))

Stationary fraction of down-processes for $\alpha = 0.5$ (left) $\alpha = 1.0$ (right) at $J = 0.2$

Results CR — Defaulted Fraction and Loss Distribution

- Recall effective single node process:

$$n_{t+1} = n_t + (1 - n_t)\Theta\Big(J_0 m_t + \alpha J^2 \sum_{s<t} G_{ts} n_s + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Big)$$

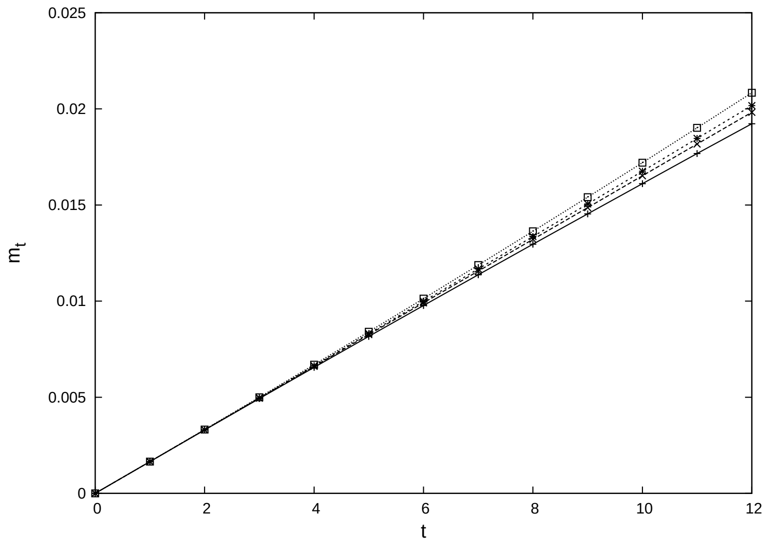
- Memory term vanishes as long as $n_t = 0$, and becomes irrelevant, once $n_t = 1$.

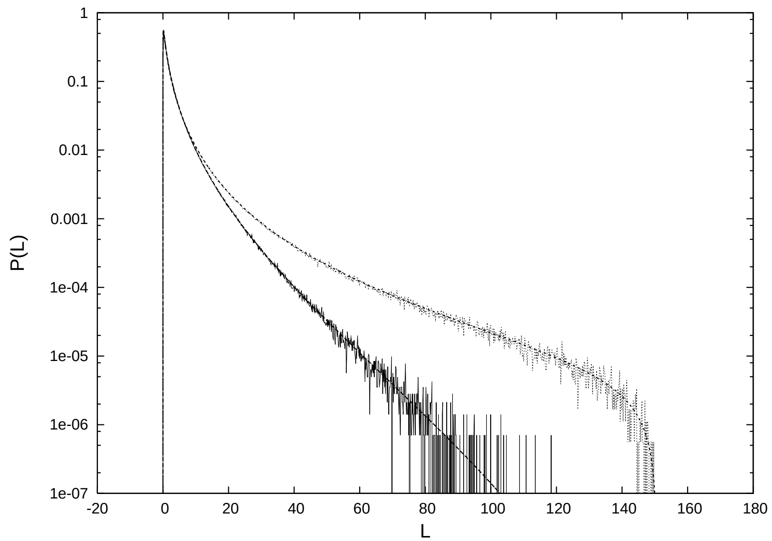
- Can use $\alpha = 0$ theory:

$$m_{t+1} = m_t + \left\langle (1 - \langle n_t \rangle)\Phi\left(\frac{J_0 m_t + \sqrt{\rho}\,\eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}}\right)\right\rangle_\vartheta$$

Results CR — Defaulted Fraction and Loss Distribution

- Loss distribution driven by fluctuating macro-economic condition η_0

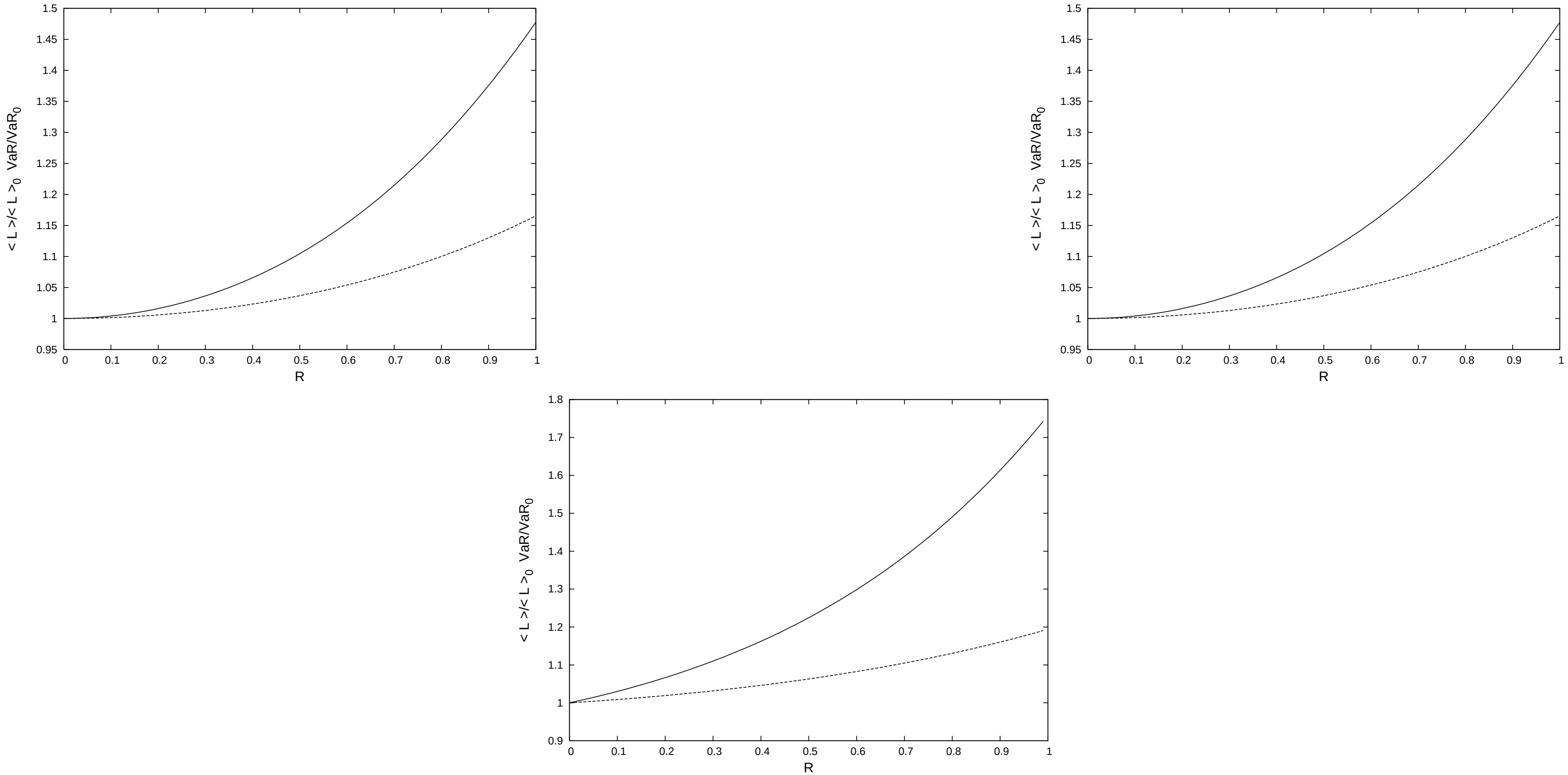




(JPL Hatchett and RK, J Phys A**39** (2006))

Fraction of defaulted firms for neutral macro-economic conditions $\eta_0 = 0$ at $(J_0, J) = (0,0), (1,0), (0,1)$ and $(1,1)$ (left, bottom to top; Loss distribution for a system with $(J_0, J) = (0,0)$ and $(1,1)$ and $\bar{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ (right)

Results CR — Value at Risk



(JPL Hatchett and RK, J Phys A**39** (2006))

Ratios of value at risk (upper curves) and average losses (lower curves) for systems with and without functional interaction evaluated along straight lines in the J_0-J plane, with $R = \sqrt{J_0^2 + J^2}$. Left: $J_0 = 0$; right: $J = 0$; lower: $J_0/J = 1$.

Functional Dependencies in Market Risk

- Standard approach: Geometric Brownian motion (GBM) for Log-returns of risk elements i (stocks, bonds ...)

$$\frac{1}{S_i(t)}\frac{\mathrm{d}S_i(t)}{\mathrm{d}t} = \mu_i + \sigma_i \eta_i(t)$$

- Statistical dependencies via correlated Gaussian noises

$$\langle \eta_i(t)\eta_j(t')\rangle = \rho_{ij}\delta(t-t')$$

- $\Rightarrow$ no functional dependencies, no collective behaviour, no market bubbles, crashes

Functional Dependencies in Market Risk — iGBM

- Functional dependencies between risk elements e.g. via recommendations of analysts, economic dependencies.

- GBM in terms of $h_i(t) = \log(S_i(t)/S_{i0})$

$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t} = \mu_i - \frac{1}{2}\sigma_i^2 + \sigma_i \eta_i(t)$$

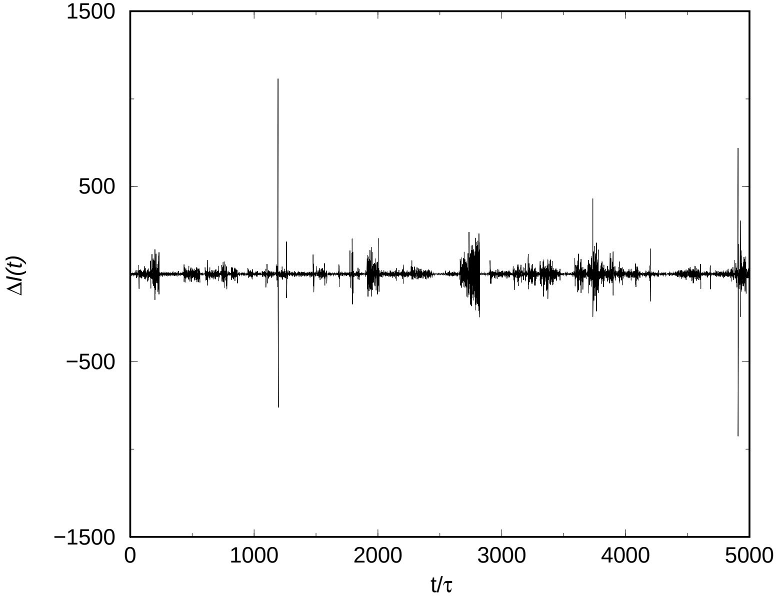
- Minimal interacting generalisation (iGBM): stabilisation and non-linear feedback (e.g. $g(x) = \tanh(x)$)

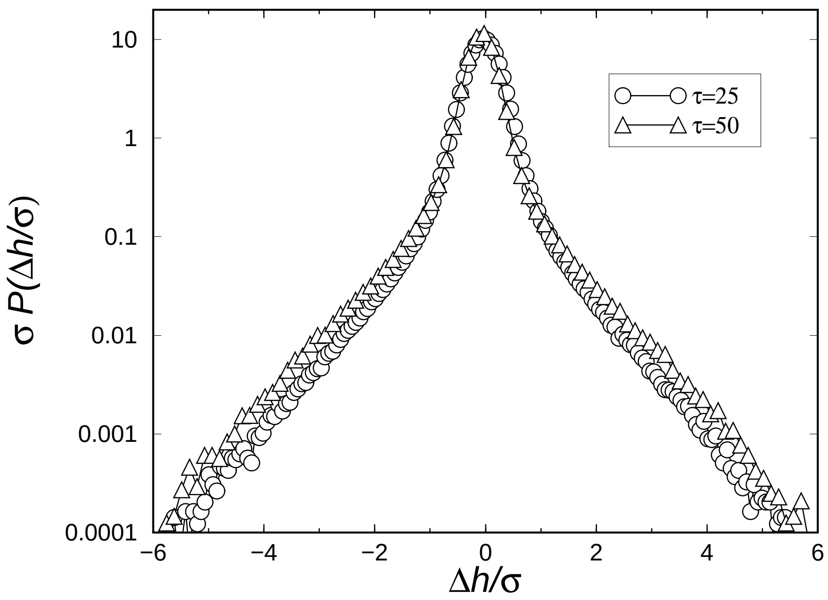
$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t} = -\kappa h_i(t) + \mu_i - \frac{1}{2}\sigma_i^2 + \sum_j J_{ij} g(h_j(t)) + \sigma_i \eta_i(t)$$

- $\Leftrightarrow$ dynamics of of graded-response neurons
  many meta-stable states; transitions between them $\Rightarrow$ intermittent dynamics.

Functional Dependencies in Market Risk – iGBM

- Solve in stochastic setting using GFA as for OR/CR
- Here MC study; trigger transitions by 'unexpected news'





(RK and P Neu, J Phys A**41** (2008))

Change of index $I(t) = N^{-1}\sum_i S_i(t)$ over time increment $\tau = 25$ (left) and normalised distribution of log-returns for $\tau = 25$, and $\tau = 50$ (right).

Summary

- Standard risk models based on statistically correlated risk elements miss dynamically generated functional correlations
- Interacting processes capture functional dependencies in OR.
- Similarly: economic interactions $\Rightarrow$ credit contagion in CR
- Physics analogy: lattice gas model on (random) graph.
- Describes bursts and avalanches of risk events
- Important: coexistence with phases of catastrophic breakdown — no noticeable precursors !
- Analogue Neuron model for MR — intermittent dynamics, fat tails.