Systemic Risk and the Mathematics of Falling Dominoes

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The Laws of Falling Dominoes

- A domino falls, if kicked sufficiently vigorously.

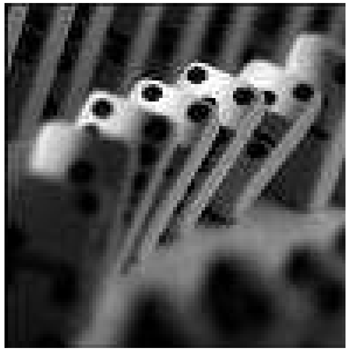
- A domino can be toppled by another domino.

- Avalanches can occur, if dominoes are set too closely.

Risk and Falling Dominoes

Operational Risk

Domino Theory & Spread of Communism



Blackouts in Power Grids

Financial Crisis

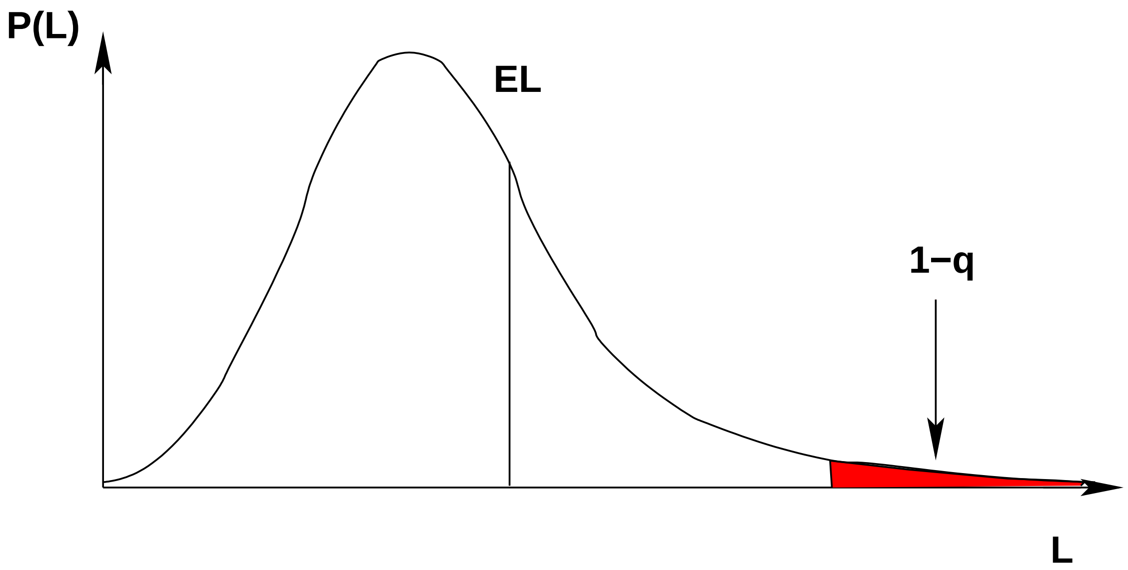
Fundamental Problem of Risk Analysis

- Estimation of risk
  - Market: potential negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices)
  - Credit: potential change of credit quality, including default (asset values of firms, ratings, stock-prices)
  - Operational: potential losses incurred by process failures (human errors, hardware/software-failures, lack of communication, fraud, external catastrophes)
  - Liquidity: potential losses incurred by rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity

Task

- Estimate PDF of
  - Portfolio value at time T, $PV(T)$, given $PV(0)$
  - Market value of assets of obligors at time T, $A_i(T)$, given $A_i(0)$
  - Losses $L(T)$ due to process failures incurred during risk horizon T
  - Losses $L(T)$ incurred during risk horizon T by need to maintain liquidity in situations of stress

- Quantity of interest: Value at Risk



$$\text{VaR}_{q,T} = (Q_q[L(T)] - EL)\, \text{e}^{-rT} \qquad \text{Prob}(L(T) \leq Q_q[L(T)]) = q$$

Money to set aside now, to cover losses in excess of expected loss at time T, which are not exceeded with probability q.

- Requires good knowledge of tails of loss distributions.

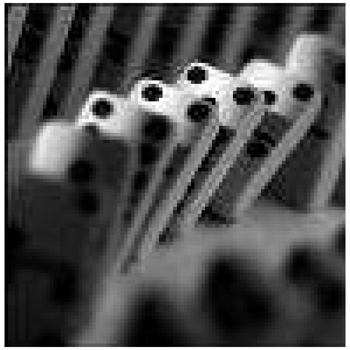
Our Main Interest and Concern: Interactions

- Traditional approaches treat risk elements as independent or *at best* statistically correlated

- Misses functional & dynamic nature of relations: terminal–mainframe/input errors–results/manufacturer–supplier relations . . .

- Effect of interactions between risk elements

  - Possibility of avalanches of risk events (process failures, defaults)
  - Fat tails in loss distributions
  - Volatility clustering in markets (intermittency)



The Case of Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes

- Two state model: processes either up and running ($n_i = 0$) or down ($n_i = 1$)

- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes (quenched disorder); connectivity functionally defined

  $\Rightarrow$ model defined on random graph

- losses determined (randomly) each time a process goes down (annealed disorder)

Dynamics – Mathematics of Falling Dominoes

- processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)

- h_{it} total support received by process i at time t

$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} + \eta_{it}$$

  with η_{it} random (e.g. Gaussian white noise).

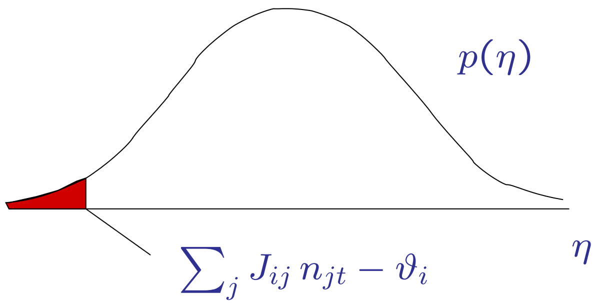
- process i will fail, if the total support for it falls below a critical threshold (domino falls, if kicked too strongly)

$$n_{it+\Delta t} = \Theta\left(\sum_j J_{ij} n_{jt} - \vartheta_i - \eta_{it}\right)$$

Probability that a Domino Falls

- Probability of failure/probability of domino falling

$$\text{Prob}\Big(n_{it+\Delta t} = 1 \Big| \boldsymbol{n}(t)\Big) = \int_{-\infty}^{\sum_j J_{ij}\, n_{jt} - \vartheta_i} \mathrm{d}\eta\, p(\eta) \equiv \Phi\left(\sum_j J_{ij}\, n_{jt} - \vartheta_i\right)$$



- unconditional and conditional probability of failure

$$\begin{aligned} p_i &= \Phi(-\vartheta_i) \\ p_{i|j} &= \Phi(J_{ij} - \vartheta_i) \end{aligned}$$

$$\Rightarrow \quad \vartheta_i = -\Phi^{-1}(p_i)\ , \quad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)$$

All-to-All Interactions — Mean Field Theory

- look at case with all-to-all couplings, $(N \gg 1)$

$$J_{ij} = \frac{J_0}{N} , \quad \forall \ (i,j) \qquad \Rightarrow \qquad \sum_j J_{ij} n_{jt} = \frac{J_0}{N} \sum_j n_{jt} = J_0 m_t$$

- Dynamics

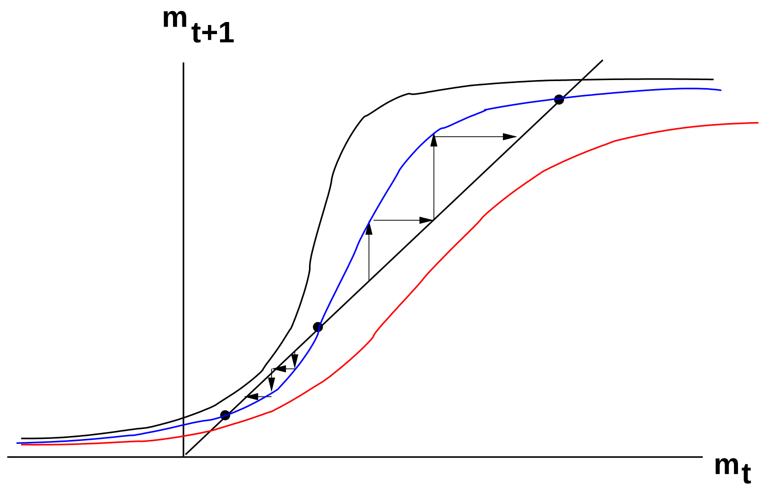
$$n_{it+\Delta t} = \Theta \left(\sum_j J_{ij} n_{jt} - \vartheta_i - \eta_{it} \right) = \Theta \left(J_0 m_t - \vartheta_i - \eta_{it} \right) \ .$$

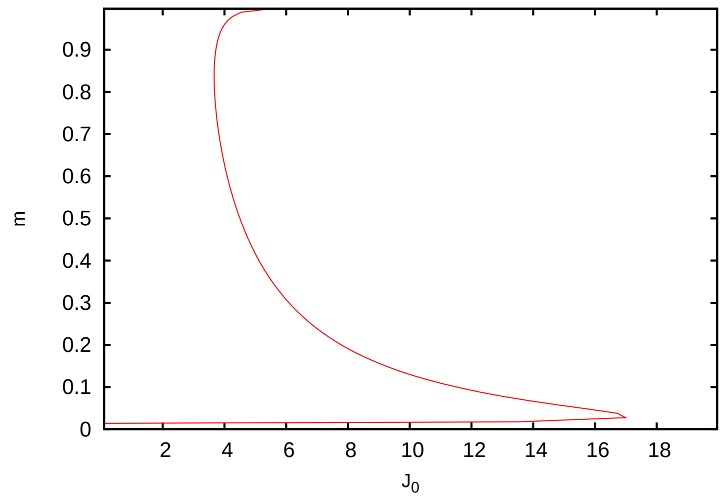
Thus (by LLN)

$$\begin{aligned} m_{t+\Delta t} &= \ = \frac{1}{N} \sum_i \Theta \left(J_0 m_t - \vartheta_i - \eta_{it} \right) \\ &= \ \frac{1}{N} \sum_i \Phi \left(J_0 m_t - \vartheta_i \right) = \Big\langle \Phi(J_0 m_t - \vartheta) \Big\rangle_\vartheta \end{aligned}$$

Mean Field Theory — Graphical Analysis

- Iterative dynamics $m_{t+\Delta t} = \left\langle \Phi(J_0 m_t - \vartheta) \right\rangle_\vartheta$





Left: Graphic representation of iterated dynamics for three different values of J_0 (curves from left to right correspond to decreasing J_0). Right: Location of fixed points as a function of J_0, for nonrandom $\vartheta = 2.25 \leftrightarrow p_i \simeq 0.015$. Backbending part of curve is unstable fixed point.

Analysis for a Stochastic Setting

- Interactions on a random graph

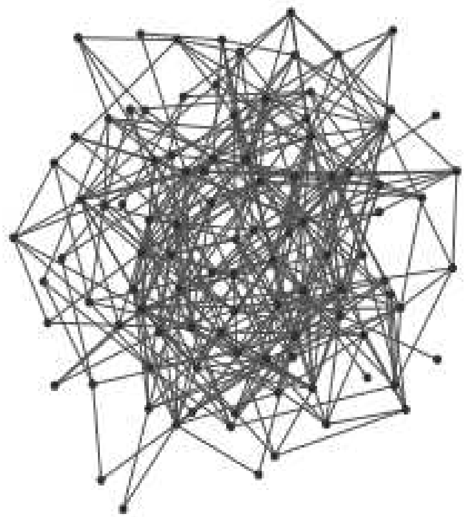
$$J_{ij} = c_{ij}\left(\frac{J_0}{c} + \frac{J}{\sqrt{c}}\, x_{ij}\right)$$

with

$$P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0} \ ,$$

symmetric ($c_{ij} = c_{ji}$), and

$$\overline{x}_{ij} = 0 \ , \quad \overline{x_{ij}^2} = 1 \ , \quad \overline{x_{ij}x_{ji}} = \alpha \ .$$



Study the limit

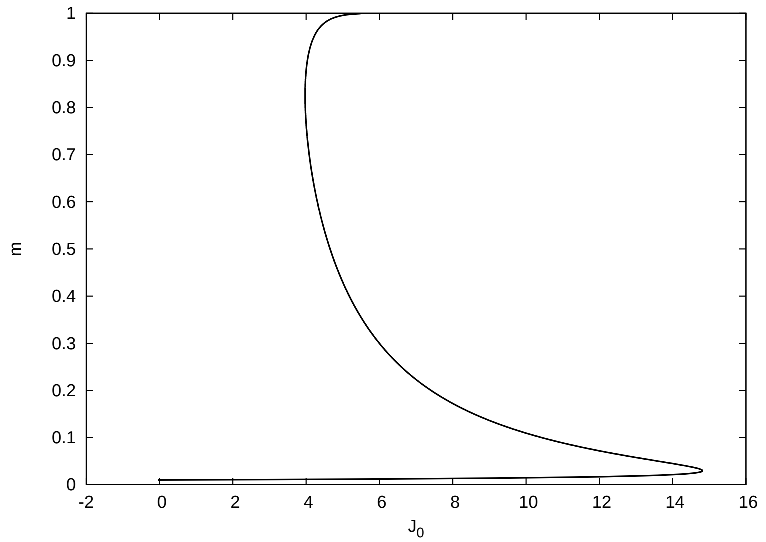
$$c \gg 1 \ , \qquad N \gg 1 \ .$$

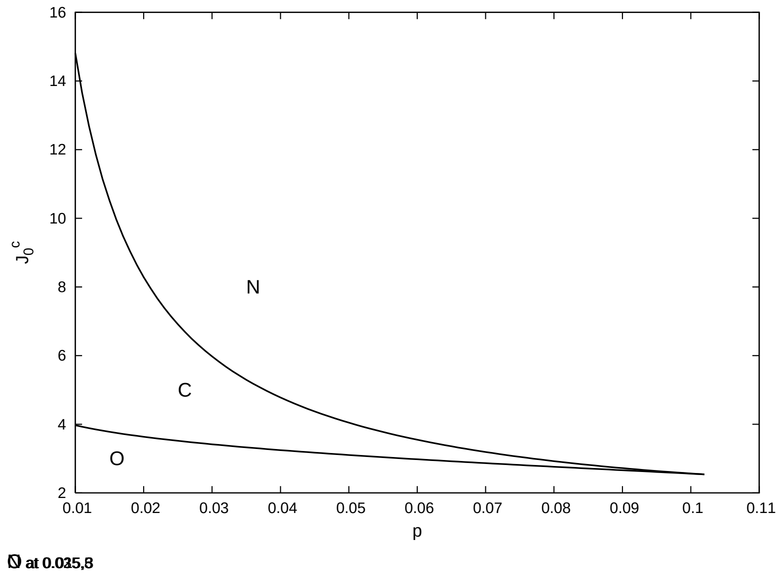
Processes/organizations: small J !

Results for $\alpha = 0$

- Use limit theorems (LLN, CLT) to show $\sum_j J_{ij} n_{jt}$ is Gaussian.
- If noise $\eta_t \sim \mathcal{N}(0, \sigma)$ (also Gaussian), then

$$m_{t+\Delta t} = \left\langle \Phi \left(\frac{J_0 m_t - \vartheta}{\sqrt{\sigma^2 + J^2 m_t}} \right) \right\rangle_\vartheta$$



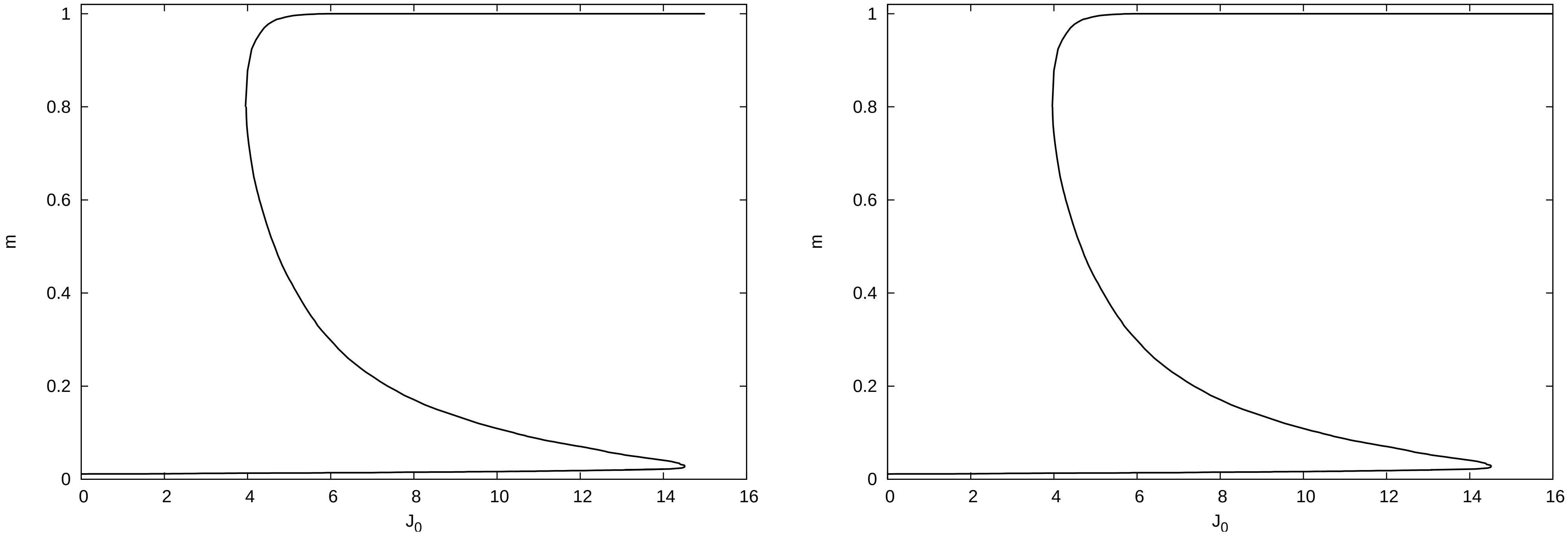


(K Anand and RK, Phys Rev E**75** (2007))

Left: Stationary fraction of down-processes for $p_i = 0.01$, $\alpha = 0$, and $J = 0.2$. Right: phase diagram.

Stationary Solution at $\alpha \neq 0$

- Have to work much harder.

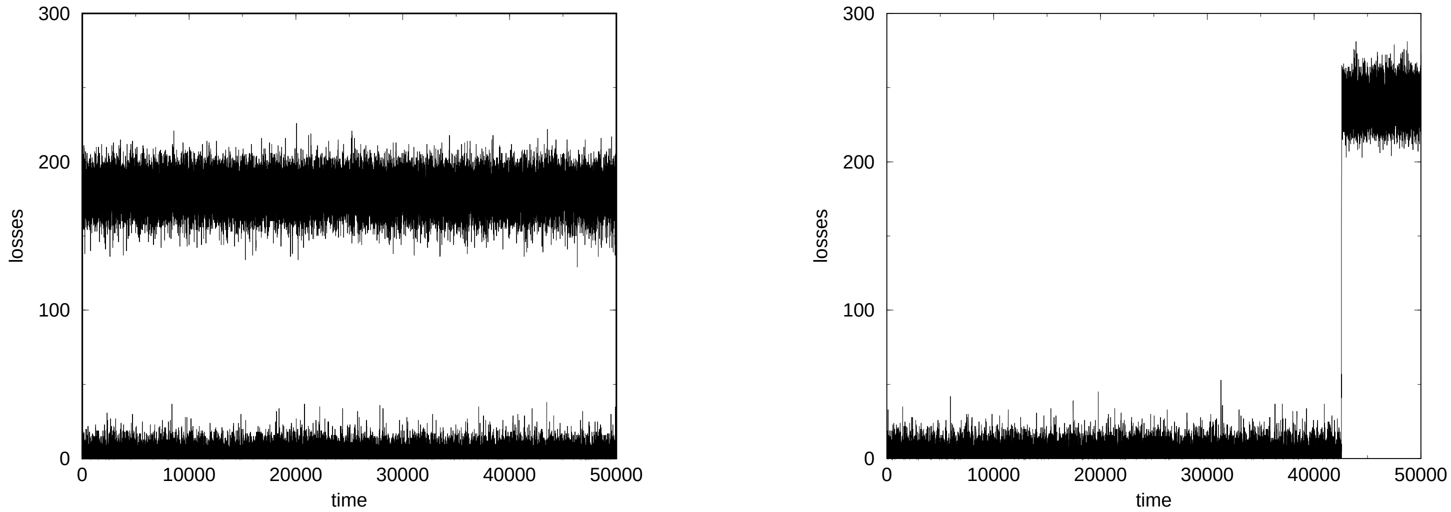


(K Anand and RK, Phys Rev E**75** (2007))

Stationary fraction of down-processes for $\alpha = 0.5$ (left) $\alpha = 1.0$ (right) at $J = 0.2$

Spontaneous Blackouts

- If mutual dependence of processes exceeds a certain level, spontaneous blackouts can occur.



Loss record for a system of $N = 50$ interacting processes. Left panel: Coexisting low-loss and high-loss phases @ $p_i^{\mathsf{max}} = 0.02$ and $(p_{i|j}/p_i)^{\mathsf{max}} = 2.6$. Right panel: Low-loss phase simulated for the same p_i^{max} but $(p_{i|j}/p_i)^{\mathsf{max}} = 3$. The low-loss situation is only meta-stable and spontaneously deays to a high-loss situation via a bubble-nucleation process.

Key Features

- For sufficiently strong cooperative interactions — "dominoes too densely placed" — get a (first order) phase transition: coexistence of 'functioning state' and state of 'catastrophic breakdown'

- There is a critical J_c inside the coexistence region, beyond which the system — while still working — prefers breakdown,

- Critical point at sufficiently high p_i.

- Of special relevance: resilience to (external) stress can be tested.

The Case of Credit Risks — Interacting Companies

- Risk arising from the possibility of obligors going bankrupt or from changes in 'credit quality' ($\Rightarrow$ credit trading)

- Here only influence of defaults

- Two state model: company up and running ($n_i = 0$) or down ($n_i = 1$)

- Probabilities of default and mutual impacts of defaults heterogeneous across the set of firms (quenched disorder); connectivity functionally defined

  $\Rightarrow$ model defined on random graph

- Losses determined (randomly: recovery process) when a company defaults (annealed disorder)

Dynamics — Falling Dominoes Again

- Companies need "orders" (support, cash inflow) to maintain wealth and avoid default

- h_{it} total wealth of company i at time t,
$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$
with η_{it} random (e.g. Gaussian white noise).

- company i defaults, if the total wealth falls below zero
$$n_{it+1} = n_{it} + (1 - n_{it}) \Theta \left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

- No recovery within 'risk horizon' T: $n_i = 1$ is absorbing state. Time unit: 1 month; $T = 12 \Leftrightarrow 1$ year.

Global Analysis

- Analysis in a stochastic setting on random graphs as for OR,

- Loss distribution through fluctuating macro-economic conditions. Noise model (minimal Basel II):

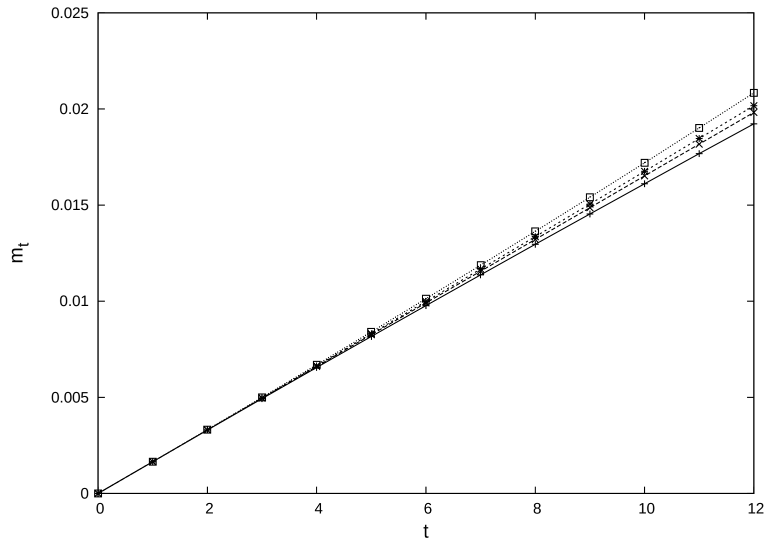
$$\eta_{it} = \sqrt{\rho_i}\,\eta_{0t} + \sqrt{1-\rho_i}\,\xi_{it}$$

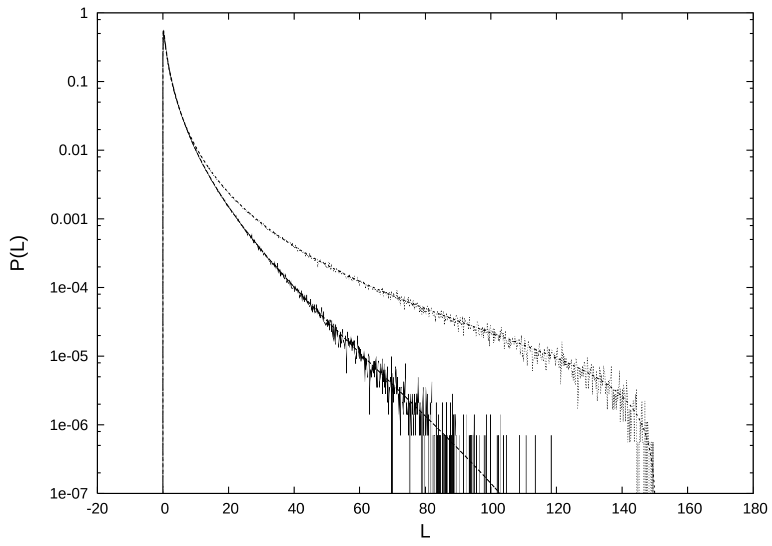
with η_{0t} slow economy-wide component, and ξ_{it} idiosyncratic noise, and $\rho_i = \rho(\vartheta_i)$..

$$\langle n_{t+1}(\vartheta_i)\rangle = \langle n_t(\vartheta_i)\rangle + \Big(1 - \langle n_t(\vartheta_i)\rangle\Big)\Phi\left(\frac{J_0 m_t + \sqrt{\rho_i}\;\eta_0 - \vartheta_i}{\sqrt{1-\rho_i + J^2 m_t}}\right)$$

$$m_{t+1} = m_t + \left\langle \Big(1 - \langle n_t(\vartheta)\rangle\Big)\Phi\left(\frac{J_0 m_t + \sqrt{\rho}\;\eta_0 - \vartheta}{\sqrt{1-\rho + J^2 m_t}}\right)\right\rangle_{\vartheta}$$

Results CR — Defaulted Fraction and Loss Distribution





(JPL Hatchett and RK, J Phys A**39** (2006))

Fraction of defaulted firms for neutral macro-economic conditions $\eta_0 = 0$ at $(J_0, J) = (0,0), (1,0), (0,1)$ and (1,1) (left, bottom to top; Loss distribution for a system with $(J_0, J) = (0,0)$ and $(1,1)$ and $\bar{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ (right)

Key Features

- Interpretation of parameters ϑ_i and J_{ij} in terms of unconditional and conditional default probabilities

- For sufficiently strong interactions — in particular cooperative interactions — get possibility of collective acceleration of default rates in times of economic stress

- Typical behaviour less sensitive to interactions than rare events $\Rightarrow$ fat tails in loss distribution; important for risk analysis where rare event asymptotics is relevant.

- Due to initial conditions $n_{i0} \equiv 0$ and absorbing state: $\Rightarrow$ no equilibrium dynamics

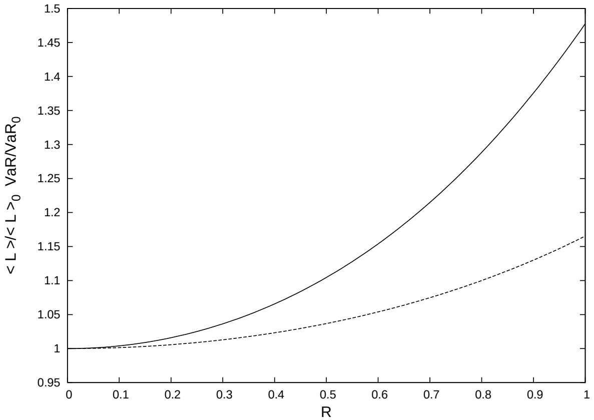
Summary

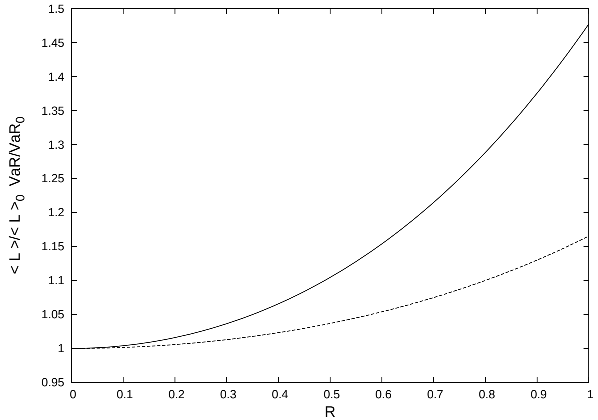
- Standard risk models based on statistically correlated risk elements miss dynamically generated functional correlations
- Interacting processes capture functional dependencies in OR.
- Similarly: economic interactions $\Rightarrow$ credit contagion in CR
- Physics analogy: lattice gas model on (random) graph.
- Describes bursts and avalanches of risk events (**dominoes**)
- Important: coexistence with phases of catastrophic breakdown — no noticeable precursors !
- Generalized Credit risk model to include derivative trading (CDS).
- Neuro-dynamics model for intermittent dynamics & fat-tailed loss distributions in MR

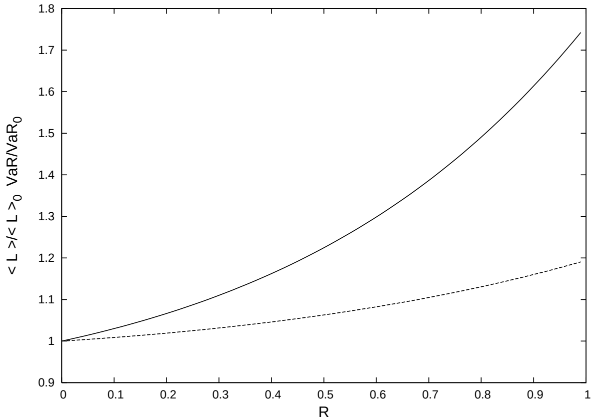
THANK YOU

Supplementary Material

Results CR — Value at Risk



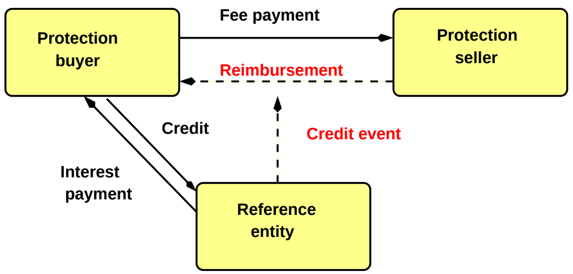


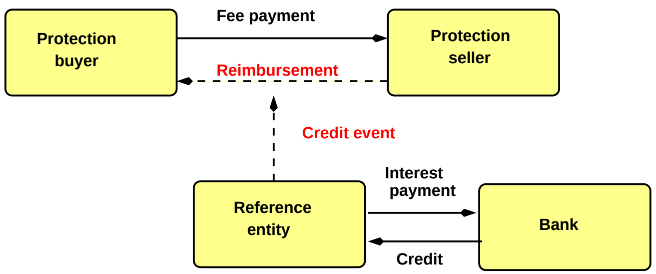


(JPL Hatchett and RK, J Phys A**39** (2006))

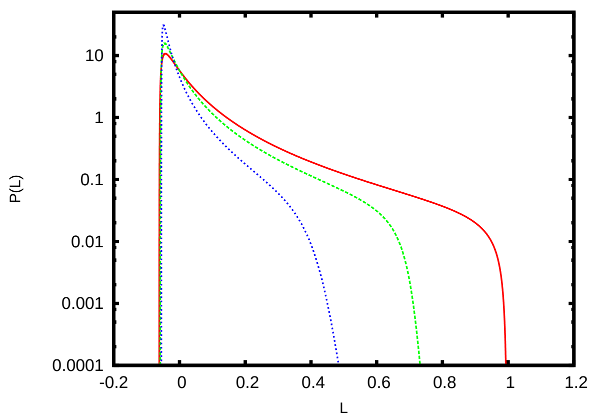
Ratios of value at risk (upper curves) and average losses (lower curves) for systems with and without functional interaction evaluated along straight lines in the J_0-J plane, with $R = \sqrt{J_0^2 + J^2}$. Left: $J_0 = 0$; right: $J = 0$; lower: $J_0/J = 1$.

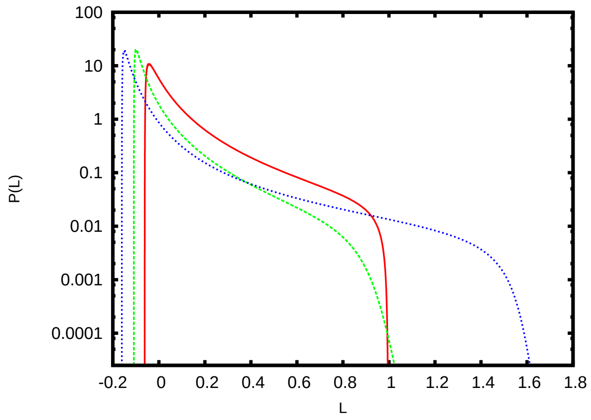
Credit Risk with CDS





Left: Mechanics of a CDS used for hedging. Right: Mechnanics of a speculative CDS.





Baseline-scenario (red full line), and either one third (green long-dashed) or two thirds (blue short-dashed) of the avarage exposure to firms are hedged by CDS, with *insurance sector*, Right: baseline-scenario (red full line) compared with situations where banks have completely hedged their exposures through CDS and used this to either double (green long-dashed) or triple (blue short-dashed) the size of their loan books.

Functional Dependencies in Market Risk

- Standard approach: Geometric Brownian motion (GBM) for Log-returns of risk elements i (stocks, bonds ...)

$$\frac{1}{S_i(t)}\frac{\mathrm{d}S_i(t)}{\mathrm{d}t} = \mu_i + \sigma_i\eta_i(t)$$

- Statistical dependencies via correlated Gaussian noises

$$\langle\eta_i(t)\eta_j(t')\rangle = \rho_{ij}\delta(t-t')$$

- Is basis of JP Morgan's Risk Metrics$^{\mathrm{TM}}$ tool!

- But: no functional dependencies, no collective behaviour, no market bubbles, crashes

Functional Dependencies in Market Risk — iGBM

- Functional dependencies between risk elements e.g. via recommendations of analysts, economic dependencies.

- GBM in terms of $h_i(t) = \log(S_i(t)/S_{i0})$

$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t} = \mu_i - \frac{1}{2}\sigma_i^2 + \sigma_i\eta_i(t)$$

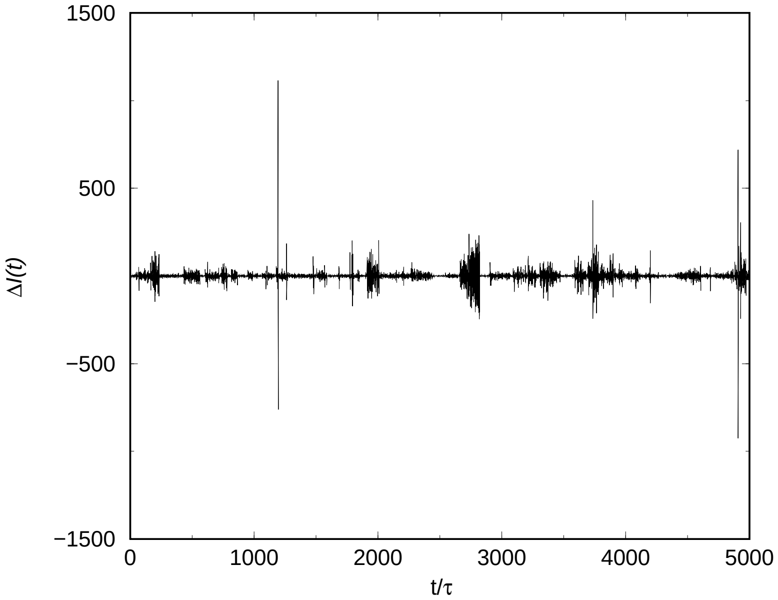
- Minimal interacting generalisation (iGBM): stabilisation and non-linear feedback (e.g. $g(x) = \tanh(x)$)

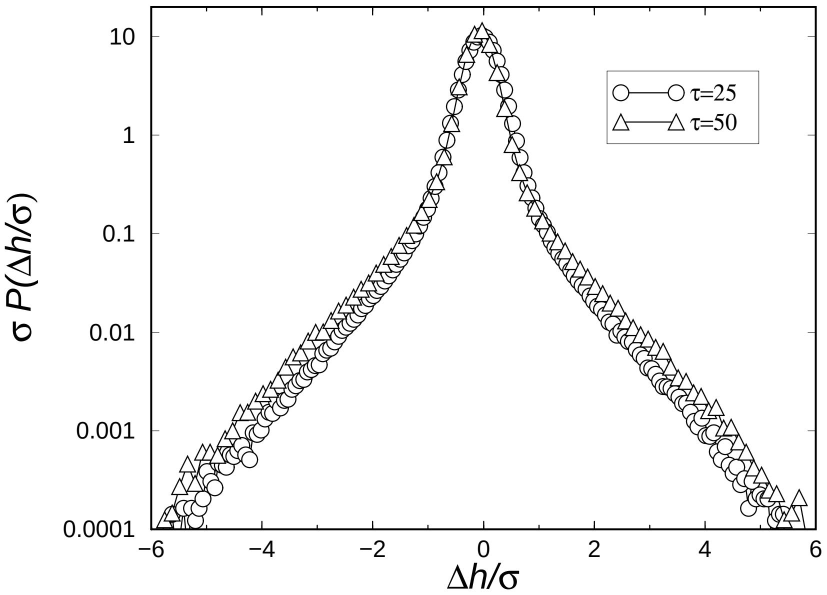
$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t} = -\kappa h_i(t) + \mu_i - \frac{1}{2}\sigma_i^2 + \sum_j J_{ij}g(h_j(t)) + \sigma_i\eta_i(t)$$

- $\Leftrightarrow$ dynamics of of graded-response neurons
  many meta-stable states; transitions between them $\Rightarrow$ intermittent dynamics.

Functional Dependencies in Market Risk – iGBM

- Solve in stochastic setting as for OR/CR
- Here MC study; trigger transitions by ‘unexpected news’





(RK and P Neu, J Phys A**41** (2008))

Change of index $I(t) = N^{-1}\sum_i S_i(t)$ over time increment $\tau = 25$ (left) and normalised distribution of log-returns for $\tau = 25$, and $\tau = 50$ (right).