Effects of Economic Interactions on Credit Risk

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Abstract. We study a credit risk model which captures effects of economic interactions on a firm's default probability. Economic interactions are represented as a functionally defined graph, and the existence of both cooperative, and competitive, business relations is taken into account. We provide an analytic solution of the model in a limit where the number of business relations of each company is large, but the overall fraction of the economy with which a given company interacts may be small. While the effects of economic interactions are relatively weak in typical (most probable) scenarios, they are pronounced in situations of economic stress, and thus lead to a substantial fattening of the tails of loss distributions in large loan portfolios. This manifests itself in a pronounced enhancement of the Value at Risk computed for interacting economies in comparison with their non-interacting counterparts.

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1. Introduction

The proper quantification of credit risk poses a complex mix of problems, as important credit risk parameters such as default rates, recovery rates or exposures, fluctuate substantially in time even on a high portfolio aggregation level [1]. This results in large unexpected losses in loan portfolios, for which banks are required to hold equity capital as a loss buffer. To determine the appropriate level of equity capital for banks' loan portfolios is one main focus of the regulatory consultive process known as Basel II [2]. Accordingly, credit risk modelling has been a focus of intense research in recent years [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], although considering the risk premium when pricing interest rates goes back some time [22].

The assessment of credit ratings by assessment agencies such as Moody's and S&P allow some statistical assessment of the credit quality of individual offerings or particular companies. However, it is clearly essential when considering the risk of a basket of loans that the correlations between the members of the portfolio are taken into account. One systematic approach is to replace the number of firms in a portfolio with an *effective* number of independent firms [24]. By boosting the contribution of each firm to keep the mean loss constant, this introduces a larger variance of losses, in an attempt to capture the risk caused by correlations between firms. JPMorgan's CreditMetrics approach [3] (see also Credit Suisse Financial Products's CreditRisk$^+$ [4] and [19] for a detailed comparison of the two) tries to model the correlations between firms in credit quality using the observable correlations in equity value of the firms. An intuitively appealing approach is to assume that the default intensity depends on some set of macroscopic economic factors (e.g. interest rates, growth rates, oil prices etc.), the so-called reduced form model [7, 8]. Thus the default rates of different firms are coupled via some limited number of factors, but given the factors the default rates are independent. In structural models [22, 23], the dependence on macro-economic factors is understood in terms of correlations in the dynamics of asset returns of different companies, leading to correlations in default rates via correlated dynamics of returns. More involved approaches have modelled interactions between firms in the wider economy by introducing changes to a firm's default intensity upon the default of another firm [6], or via a copula function [10]. The quantification of these correlations via simulations is discussed in [14].

The main purpose of the present contribution is specifically to expand upon recent modelling and analytic descriptions of the influence of counter-party risk [6, 9, 11, 13, 15, 16, 17, 18, 21]. Counter-party risk addresses the fact that a given firm's economic health is strongly influenced by the performance within the network of companies with which it has direct economic interactions, its partners. We understand the notions of firm and economic partner in a very wide sense: a firm could be any economic entity, a manufacturer, a service provider, a trader of goods or services, even an individual. We shall often use the generic term 'node' to designate these entities. Two firms are partners if the state of one has a material (not necessarily symmetric) effect upon the other, e.g. one is the supplier of the other, performs outsourced services, there exists substantial loans or other financial commitments between the two, or they compete in the same market. Specifically, a defaulting firm within this network of counter-parties will affect a company's own default probability — reducing it, if the defaulting firm was a competitor, or increasing it, if the relation was of a cooperative nature. For instance, if a major manufacturer of PCs were to go out of business tomorrow, this would inevitably have a material impact on the economic performance (i.e. wealth) of other companies operating in the same industry sector. It would on the one hand lead to a deterioration in the financial viability of most of the suppliers or service providers of the PC manufacturer in question — including in particular its own work force! — but on the other hand it would improve the situation for competing producers of PCs, in that they could profit by taking over a share of the defaulted company's market. When the default probability is increased, this process is known as *credit contagion* and has been considered in e.g. [16, 18] while the incorporation of counter-party risk into a reduced form model was introduced in [9]. In what follows we consider the dynamics of individual firm defaults and their influence on loss distributions.

More subtle effects such as credit quality migration are, as yet, not taken into account.

The importance of direct functional interactions in the analysis of risk is not restricted to credit risk. In fact the role of interactions is much more obvious in the context of operational risk, where sequential, functionally induced failures of mutually dependent processes constitute one of the main sources for operational risk. Indeed, an attempt to explore the consequences of interactions for quantifying the capital buffer necessary to cover operational risk [25] has provided major ingredients for the approach to credit risk modelling started by Neu and one of us [21].

In the present paper we provide an analytic solution to the dynamical description of counter-party risk within a heterogeneous, functionally defined network of interacting firms (to be referred to as economy in what follows), see e.g. [18], in the spirit of [21]. We generalize the analysis of that study to capture effects of cooperative as well as competitive business relations within the economy, and we solve the model for a wide degree of dilution of the network of economic dependencies in the sense that we assume each company in the net to have business relations only with a (randomly chosen) subset of the full set of companies. We are interested in quantifying the effect of these interactions from the perspective of a lending bank which would be required to set aside a sufficient amount of capital to cover losses incurred by defaults of its obligors. Another perspective might be that of a central bank, which would base monetary policy decisions in part on their impact on expected default rates at an economy wide scale. The typical risk horizon in these contexts would be one year.

In the present investigation we will always consider the case where the number of interaction partners of each company is large, and for simplicity we shall restrict ourselves here to the case where the graph defining economic connectivity is a Poisson degree distributed Erdös-Rényi random graph [26]. More realistic connectivity distributions reflecting the different connectivity patterns of large and small players in an economy, taking into account small-world effects and fat tailed degree distributions [27] can be handled by methods similar to those used in the present investigation [28, 29], but will be studied in a separate paper.

As in [21], the model parameters are unconditional and conditional default probabilities, which may be thought of as being obtained via a suitable rating procedure. One of the virtues of the present analytic investigation is to highlight the fact that the collective behaviour of the system, which ultimately determines the the loss-distribution on an economy-wide scale, is fairly insensitive to detail. That is, it does not depend on getting individual dependencies correct, but only on the overall distribution of of the unconditional and conditional default probabilities. Our main result is to demonstrate that the effects of economic interactions — while relatively weak in typical scenarios — are pronounced in situations of economic stress, and thus lead to a substantial fattening of the tails of loss distributions even in large loan portfolios.

The remainder of the paper is organized as follows. In Sec. 2 we define our model, and specify the stochastic setting for our analytic investigation. The relation between the model parameters and conditional and unconditional default probabilities as used in [21] is briefly reviewed to make the paper self-contained. Sec. 3 describes a heuristic solution for the dynamical evolution of the fraction of defaulted companies over a risk horizon of one year, starting from an overall healthy situation, a scenario appropriate for the analysis of credit risk. A formal solution in terms of a generating function approach (GFA) [30], which provides a full justification for the heuristic solution is relegated to Appendix A. Both solution methods are based upon techniques developed in the statistical mechanical analysis of dilute neural networks [31]. In Sec. 4 we compute a phase diagram distinguishing regions in parameter space in which economic interactions lead to a collective acceleration of the economy-wide default rate in situations of economic stress from regions where such acceleration is impossible. Distributions of annual fractions of defaulted companies as well as loss distributions, both economy-wide and for finite loan-portfolios are computed and compared with simulations. Sec. 5 summarizes our findings and discusses their implications for the analysis of credit risk.

2. Model Definitions

In this section we define a statistical model that attempts to capture the effects of counter-party risk on credit contagion. In contrast to approaches based on microeconomics, and in keeping with the framework discussed in the introduction, we allow the firms' wealth, macro-economic factors and interactions between firms to all be described probabilistically. This is due to our focus on the characteristic change in behaviour caused by examining interactions between firms in the wider economy.

We analyse an economy which consists of N firms. The state of each firm i at a given time t is described by its 'wealth' $W_{i,t}$; the difference between its assets and its liabilities. Accordingly, a company defaults, if its wealth $W_{i,t}$ falls below zero. We are interested in the influence of economic partners on firms' default probability.

For simplicity, we assume that that within the risk horizon of one year node i experiences an interaction-induced material change of its wealth only if one of its business partners, say j, defaults. In order to formalize this in a dynamical description, we introduce binary indicator variable $n_{j,t}$ which indicates whether node j is solvent at time t ($n_{j,t} = 0$) or has defaulted ($n_{j,t} = 1$).

The value of the ith node's wealth at time t, $W_{i,t}$, is thus taken to be of the form

$$W_{i,t} = \vartheta_i - \sum_{j=1}^{n} J_{ij} n_{j,t} - \eta_{i,t} \tag{1}$$

Here J_{ij} denotes the change in i's wealth which would be induced by a default of node j. One would have $J_{ij} > 0$ if j is a cooperative partner of i, whereas $J_{ij} < 0$ if j is a competitor, while $J_{ij} = 0$ if there is no direct influence of j on i. By ϑ_i we designate i's initial wealth at the beginning of the year, and $\eta_{i,t}$ are (zero mean) random fluctuations caused by both, *external* macro-economic factors (an expanding/shrinking economy, an oil price spike, market sentiment etc), and *firm specific* actions or events.

We take the initial state of the economy to be a set of solvent firms, $n_{i,0} = 0$ for all i, (one could view this as a definition) and say that a firm defaults at time t if $W_{i,t} < 0$.

We define our dynamics such that if a firm goes bankrupt it does not recover within a risk horizon of one year, so the bankrupt state is absorbing. Thus the dynamics of the firms state is given by the equation

$$n_{i,t+1} = n_{i,t} + (1 - n_{i,t})\Theta\Bigg(\sum_j J_{ij} n_{j,t} - \vartheta_i + \eta_{i,t}\Bigg) \tag{2}$$

where $\Theta(\ldots)$ is the Heavyside function. The time step in this dynamical rule will be taken to represent one month.

Note that in equations (1), (2) we are not taking any systematic 'endogenous' drift of a company's wealth into account, in the sense that at constant $\{n_{j,t}\}$ the wealth of company i would just randomly fluctuate about $\vartheta_i - \sum_{j=1}^{n} J_{ij} n_{j,t}$ — a value determined solely by its initial wealth ϑ_i and the solvency status of its economic partners. More realistically one could include some endogenous drift to capture the effect that a compnay makes losses or profits, e.g. by making the ϑ_i in (1), (2) time-dependent, $\vartheta_i \to \vartheta_{i,t}$. A reasonable ansatz for their dynamics would be $\vartheta_{i,t+1} = q_i \vartheta_{i,t}$, with $q_i > 1$ for a company making profits, and $q_i < 1$ for a company making losses. This modification can easily be handled by the methods we use below to investigate the system. However, while requiring further assumptions (about the q_i-distribution), we find that it does not qualitatively alter our main findings, and we therefore do not include this feature in the present study.

We choose the $\eta_{i,t}$ to be Gaussian distributed. Without loss of generality they can — by suitably rescaling the ϑ_i and the J_{ij} — be chosen to have unit variance. We follow widespread practice [3, 5] to account for common fluctuating macro-economic factors by choosing the $\eta_{i,t}$ to be correlated for different i. This could be achieved by taking $\eta_{i,t}$ to be of the form

$$\eta_{i,t} = \sigma_i \xi_{i,t} + \sum_{k=1}^{K} \beta_{ik} Y_{k,t} \tag{3}$$

with uncorrelated Gaussian unit variance white noises $\xi_{i,t}$ and $\{Y_{k,t}\}$, the former describing firm specific wealth fluctuations, whereas the latter could account for the relative effects of fluctuations common to industry sectors, regions, or countries, with prefactors σ_i and β_{ik} describing the relative importance of these fluctuations on i. In what follows we restrict ourselves to a *minimal variant* of this set-up by finally choosing

$$\eta_{i,t} = \sqrt{\rho}\ \eta_t + \sqrt{1-\rho}\ \xi_{i,t}\ . \tag{4}$$

We simplify matters further by assuming that the common economic factor η_t is slow and take it to be constant $\eta_t = \eta_0$ within a risk horizon of one year. One-factor models of this type feature in the regulatory framework laid out in the Basel II accord [2].

None of the simplifying assumptions are necessary for our analysis to go through; the generating function formalism given in Appendix A in particular can easily handle more general cases. However, the simplified setting is sufficient to highlight the important effects of interactions on credit risk, and it does lead to a greatly simplified macroscopic description of the system, as we will see in Sec. 3.

As for the J_{ij} which describe the loss or gain of node i due to a default of node j, in the present paper we will investigate them in a probabilistic setting. It is useful to disentangle the presence or absence of an interaction from its strength by writing

$$J_{ij} = c_{ij}\tilde{J}_{ij} \tag{5}$$

where $c_{ij} \in \{0, 1\}$ describes the absence or presence of a connection $j \to i$, while $\tilde{J}_{ij}$ describes its magnitude, both of which are assumed to be fixed. It is reasonable to assume that connectivity is a symmetric relation, $c_{ij} = c_{ji}$, whereas there is no reason to suppose symmetry of the magnitudes of mutual influences. Considering the case of a small supplier with one large company taking the majority of its orders, if the larger company defaults then the small supplier may well go bust too. However, if the small supplier defaults then the large company is less likely to suffer terminal financial distress, so in general $\tilde{J}_{ij} \neq \tilde{J}_{ji}$.

Specifically, we assume a random connectivity pattern described by

$$P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0} \quad , \; i < j \quad , \; c_{ij} = c_{ji} \tag{6}$$

and we will be interested in the limit of a large economy (the thermodynamic limit), in which the average connectivity c of each node is itself large, $N \to \infty$, $c \to \infty$. We will initially be concerned with the extremely diluted regime, where $c/N \to 0$, taking e.g. $c = \mathcal{O}(\log(N))$. These assumptions have important consequences for the structure of the graph defining the connectivities, namely that each node feels the effects of a large number of other nodes (so that limit theorems will allow us to describe the overall effects of interactions) and, for the extremely diluted regime, that there are only a finite number of loops of finite length even in the infinite economy limit. The graph of interactions between companies for finite N is just an Erdös-Rényi random graph [26].

We take the magnitudes $\tilde{J}_{ij}$ of the interactions to be fixed random quantities. To allow the thermodynamic limit to be taken, the mean and fluctuations of the $\tilde{J}_{ij}$ must scale in a suitable way with the connectivity c. Quite generally, we must have

$$\tilde{J}_{ij} = \frac{J_0}{c} + \frac{J}{\sqrt{c}}x_{ij} \tag{7}$$

in which the x_{ij} are zero-mean unit-variance random variables. The scaling of mean and variance of the $\tilde{J}_{ij}$ is given by the parameters J_0 and J respectively. If $J_0 > 0$ there will be a net cooperative tendency within the economy, which seems to be a reasonable assumption. Finally, we will need to assume that all moments of the x_{ij} are finite and we will choose the x_{ij} to be *independent in pairs*

$$\overline{x}_{ij} = 0 \; , \qquad \overline{x_{ij}^2} = 1 \; , \qquad \overline{x_{ij}x_{ji}} = \alpha \; , \qquad \overline{x_{ij}x_{kl}} = 0 \text{ otherwise} \; . \tag{8}$$

The parameter α, $(-1 \leq \alpha \leq 1)$ describes the degree of correlations between J_{ij} and J_{ji}. Strictly symmetric interactions are obtained only for $\alpha = 1$

At this point let us briefly recall that, after rescaling as described, the model parameters ϑ_i and J_{ij} have a clear meaning in terms of unconditional and conditional default probabilities [21]. We denote by $\boldsymbol{n}_t$ the values of all indicator variables in the economy at time t, assuming $n_{i,t} = 0$. Then by integrating over the unit variance Gaussian $\eta_{i,t}$ in (2) one obtains the conditional probability for node i to default within a month given a configuration $\boldsymbol{n}_t$ of non-defaulted and defaulted firms in the economy at time t as,

$$\text{Prob}\left(n_{i,t+1} = 1 \middle| \boldsymbol{n}_t\right) = \Phi\Big(\sum_j J_{ij}\, n_{j,t} - \vartheta_i\Big)$$

with $\Phi(x) = \frac{1}{2}[1 + \text{erf}(x/\sqrt{2})]$ the cumulative normal distribution. Thus the unconditional probability p_i of default of i within a month in an otherwise healthy economy and the conditional probability $p_{i|j}$ for a default of i within a month, given j and only j has defaulted before are given by

$$p_i \;= \text{Prob}\left(n_{i,t+1} = 1 \middle| \{n_{i,t} = 0\}\right) = \Phi(-\vartheta_i)\ , \tag{9}$$

$$p_{i|j} = \text{Prob}\left(n_{i,t+1} = 1 \middle| n_{j,t} = 1, \{n_{k(\neq j),t} = 0\}\right) = \Phi\Big(J_{ij} - \vartheta_i\Big)\ . \tag{10}$$

These relations may be inverted to express the model parameters in terms of conditional and unconditional default probabilities — quantities that would be estimated in a rating procedure — as

$$\vartheta_i = -\Phi^{-1}(p_i) \quad , \qquad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)\ . \tag{11}$$

While characterising the default of companies is of interest, our primary concern is to examine the distribution of losses accrued over our one year time frame, both in the economy at large, and in a portfolio made up of a finite number of firms within the economy. We assume that the losses caused by default are independent of the month of default, and then examine two different cases. The first simpler case is that the losses at firm i, given that firm i defaults, are uncorrelated with any other variables. The second, perhaps more interesting case, is that the losses at firm i are random but are correlated with the initial monetary reserves ϑ_i. The intuitive reasons are that if a firm has more cash, then the default is less anticipated, and thus will be less priced in by the market; or the firm has larger credit lines and so will default on a larger amount; and finally the firm is likely to be larger, and hence cause a larger loss.

3. Heuristic Solution

In the present section we show that our model has a relatively simple solution that can be obtained by qualitative probabilistic reasoning, appealing to statistical limit theorems. This solution turns out to be exact, as we show using a more involved generating function formalism in Appendix A. Both types of argument have been developed in the analysis of the statistical mechanics of disordered systems, and in particular neural network models [31], while for a more general introduction to emergent collective behaviour see e.g. [32]. Recall the microscopic dynamics as defined by (2). The complications are due to the interactions between firms, namely that the state of a given firm i at time t, depends on the state of the neighbours of i for times $t' < t$ which in turn depend on i at times $t'' < t' < t$. In general this feedback prohibits straightforward analysis, and indeed, it led Jarrow and Yu [9] to eliminate this feedback explicitly by considering an economy of two types of firms: primary firms whose default depended only on macro-economic factors and secondary firms whose default depended on macro-economic factors and the default of primary firms. However, due to the specific structure of our model we are able to push the analysis further. By definition, the overall effect of interaction terms on company i at time t is given by the local field $h_{i,t} = \sum_j J_{ij} n_{j,t}$. From the statistics of the interactions J_{ij} given by (6)-(8) we see that each firm i is connected to, on average, c other firms. Since we consider the large c limit, this means that we could evaluate the statistics of $h_{i,t}$ by appeal to the law of large numbers and the central limit theorem if the contributions to $h_{i,t}$ were independent, or at least sufficiently weakly correlated.

At first sight we cannot expect this condition to hold if we have some degree of symmetry in the interactions, i.e. for $\alpha \neq 0$, even in the extremely diluted regime. Note that there are two ways in which the $n_{j,t}$ of the neighbours interacting with i may become correlated through the dynamics: either they influence each other through firm i, or not through firm i but through some loop of interactions in the economy. In the extremely diluted regime, correlations between the neighbours j of i cannot build up in finite time (within the risk horizon) via loops not involving i, since due to the scaling almost all loops are very long. With symmetry in the interactions, correlations between the $n_{j,t}$ could in principle be induced by the dynamics of i. However, as long as $n_{i,t} = 0$, the $n_{j,t}$ clearly cannot influence each other through site i, whereas once $n_{i,t} = 1$, then firm i is in the absorbing state, and correlations it induces on the dynamics of its neighbours, have become irrelevant for its own microscopic dynamics (2). Thus limit theorems can be used after all to solve the macroscopic dynamics of the system, despite a possible symmetry in the interactions.

Returning to the dynamical evolution equation (2), we observe that the coupling of a node to the economy is via the local field

$$h_{i,t} = \sum_j J_{ij} n_{j,t} = \frac{J_0}{c} \sum_j c_{ij} n_{j,t} + \frac{J}{\sqrt{c}} \sum_j c_{ij} x_{ij} n_{j,t} \ , \tag{12}$$

which is a sum of random quantities (with randomness both due to the Gaussian fluctuating forces (the $\{\eta_{i,t}\}$, respectively the $\{\xi_{i,t}\}$), and due to the heterogeneity of the environment). The first contribution is a sum of terms of non-vanishing average. By the law of large numbers this sum converges to the sum of averages in the large c limit,

$$h^0_{i,t} \equiv \frac{J_0}{c} \sum_j c_{ij} n_{j,t} \to \frac{J_0}{c} \sum_j \overline{c_{ij} \langle n_{j,t} \rangle} \simeq \frac{J_0}{c} \sum_j \overline{c_{ij}}\ \overline{\langle n_{j,t} \rangle} = J_0 \frac{1}{N} \sum_j \overline{\langle n_{j,t} \rangle}$$

in which angled brackets $\langle \ldots \rangle$ denote an average over the fluctuating forces, and the overbar $\overline{(\ldots)}$ an average over the J_{ij}, i.e., the c_{ij} and the x_{ij}. An approximation is made by assuming negligible correlations between the c_{ij} and the $\langle n_{j,t} \rangle$ induced by the heterogeneity of the interactions. The second contribution to (12) is a sum of random variables with zero mean, which we have argued are sufficiently weakly correlated for the central limit theorem to apply for describing the statistics of their sum. Thus the sum

$$\delta h_{i,t} \equiv \frac{J}{\sqrt{c}} \sum_j c_{ij} x_{ij} n_{j,t}$$

is a zero-mean Gaussian whose variance follows from

$$\begin{aligned}\overline{\langle(\delta h_{i,t})^2\rangle} &= \frac{J^2}{c} \sum_{jk} \overline{c_{ij} c_{ik} x_{ij} x_{ik} \langle n_{j,t} n_{k,t}\rangle} \simeq \frac{J^2}{c} \sum_{jk} \overline{c_{ij} c_{ik} x_{ij} x_{ik}}\, \overline{\langle n_{j,t} n_{k,t}\rangle} \\ &= J^2 \frac{1}{N} \sum_j \overline{\langle n_{j,t}\rangle}\end{aligned}$$

An approximation based on assuming negligible correlations has been made as for the first contributions. Thus the local field $h_{i,t}$ is a Gaussian with mean $h^0_{i,t}$ and variance $\overline{\langle(\delta h_{i,t})^2\rangle}$ both scaling with the average fraction of defaulted nodes in the economy. By the law of large numbers this average fraction will be typically realized in a large economy, i.e. we have

$$m_t = \frac{1}{N} \sum_j n_{j,t} \to \frac{1}{N} \sum_j \overline{\langle n_{j,t}\rangle} \tag{13}$$

in the large N limit. The dynamics of the fraction of defaulted nodes then follows from (2),

$$m_{t+1} = \frac{1}{N} \sum_i n_{i,t+1} = m_t + \frac{1}{N} \sum_i (1 - n_{i,t}) \Theta\Big(h_{i,t} - \vartheta_i + \sqrt{\rho}\eta_0 + \sqrt{1-\rho}\; \xi_{i,t}\Big) \ , \tag{14}$$

where the one factor noise model (4) has been used.

The sum in (14) is evaluated as a sum of averages over joint $n_{i,t}$, $h_{i,t}$, and $\xi_{i,t}$ distribution by the law of large numbers. We exploit the fact that $n_{i,t}$, $\xi_{i,t}$ and $h_{i,t}$ are uncorrelated. Noting that the sum $h_{i,t} + \sqrt{1-\rho}\; \xi_{i,t}$ is Gaussian with mean $J_0 m_t$ and variance $1 - \rho + J^2 m_t$, and taking into account that $n_{i,t}$-averages, depend on i through ϑ_i, $\langle n_{i,t}\rangle = \langle n_t\rangle_{(\vartheta_i)}$, we find

$$m_{t+1} = m_t + \frac{1}{N} \sum_i \frac{1 - \langle n_t\rangle_{(\vartheta_i)}}{2} \left[1 + \mathrm{erf}\left(\frac{J_0 m_t + \sqrt{\rho}\; \eta_0 - \vartheta_i}{\sqrt{2(1-\rho+J^2 m_t)}}\right)\right]$$

This version can be understood as an average over the ϑ distribution

$$p(\vartheta) = \frac{1}{N} \sum_i \delta(\vartheta - \vartheta_i) \ ,$$

which maps onto a distribution of unconditional default probabilities as discussed above. Denoting that average by $\langle \ldots \rangle_\vartheta$ we finally get the following evolution equation for the macroscopic fraction of defaulted companies in the economy

$$m_{t+1} = m_t + \left\langle \frac{1 - \langle n_t\rangle_{(\vartheta)}}{2} \left[1 + \mathrm{erf}\left(\frac{J_0 m_t + \sqrt{\rho}\; \eta_0 - \vartheta}{\sqrt{2(1-\rho+J^2 m_t)}}\right)\right]\right\rangle_\vartheta \tag{15}$$

We have thus an explicit dynamic equation for the macroscopic fraction of defaulted nodes in the economy. It involves first propagating ϑ-dependent default probabilities via

$$\langle n_{t+1}\rangle_{(\vartheta)} = \langle n_t\rangle_{(\vartheta)} + \frac{1-\langle n_t\rangle_{(\vartheta)}}{2}\left[1+\mathrm{erf}\left(\frac{J_0 m_t + \sqrt{\rho}\,\eta_0 - \vartheta}{\sqrt{2(1-\rho+J^2 m_t)}}\right)\right] , \qquad (16)$$

which depends only on m_t, thereafter performing an integral over the ϑ distribution to obtain the updated fraction m_{t+1} of defaulted nodes given in (15).

The heuristic solution of the macroscopic dynamics (15), (16) presented here is based on independence assumptions which are not easily justified in a rigorous way via the probabilistic reasoning presented above. However, the solution is supported in full detail by an exact analysis based on generating functions presented in Appendix A.

4. Results

In the present section we explore the consequences of our theory. We studied the dynamics and computed loss distributions for an economy in which the parameters ϑ_i determining unconditional monthly default probabilities according to (9) are normally distributed with mean $\vartheta_0 = 3$, and variance $\sigma_\vartheta^2 = 0.01$ so that typical monthly default probabilities are in the 5×10^{-4} range. Except when stated otherwise we shall use $\rho = 0.15$ for the parameter describing the relative importance of economy-wide fluctuations, a value that is considered to be in an economically acceptable range.

In Fig 1a we show the evolution of the *typical* fraction of defaulted firms over a risk horizon of 12 months for various settings of the interaction parameters J_0 and J; the typical fraction is computed by choosing the most-probable value $\eta_0 = 0$ for the economy-wide influence on the dynamics. Fig 1b shows the probability density of the end of year fraction of defaulted firms driven by fluctuations in economic conditions. It is obvious that interactions cause a significant fattening of the tail of the density at large values of this fraction, which is a clear indication of the significance of counter-party risk in particular in situations of economic stress.

In order to assess whether interactions can lead to a collective acceleration of the rate of defaults we look at the discrete second derivatives

$$\Delta_t = m_{t+1} + m_{t-1} - 2m_t$$

which are always negative for the non-interacting system and maximal at $t = 1$, irrespectively of η_0. Interactions can lead to a collective acceleration of the rate of defaults signified by the possibility that the Δ_t may become positive in unfavourable economic conditions. We define the region in parameter space in which collective acceleration of default rates can occur by the condition that

$$\mathrm{Prob}\Big\{\Delta_t > 0\Big\} > 0 \qquad (17)$$

for some t, with $1 < t < 11$. The concavity of the error function for positive arguments entails that effects of collective acceleration are always strongest at $t = 1$. Evaluating

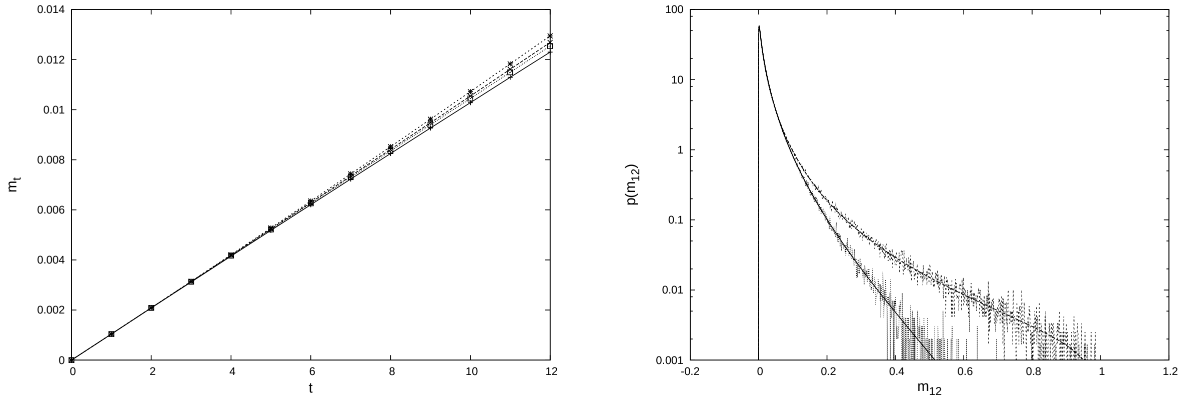

Figure 1. Left: Typical fraction of defaulted companies as a function of time for $(J_0, J) = (0,0), (1,0), (0,1)$, and (1,1) (bottom to top). Right: Distribution of the fraction of defaulted companies at $t = 12$ for $(J_0, J) = (0,0)$ (bottom), and (1,1) (top). Smooth analytic curves are overlaid with results of a simulation in which the distribution is obtained by computing the fraction of defaulted companies for randomly sampled η_0.

this condition for various values of the parameter ρ describing the coupling to the overall economy, we get lines shown in the phase diagram Fig 2a. Note that the influence of ρ is very weak in interesting region of low ρ values. Note also that Δ_1 values are *typically* positive but very small in the region near the phase boundaries shown, as illustrated in In Fig 2b where we exhibit the distribution of discrete second derivatives at $t = 1$, just inside the phase where acceleration of default rates is observed. The tail of negative Δ_1 is found to extend to significantly larger values.

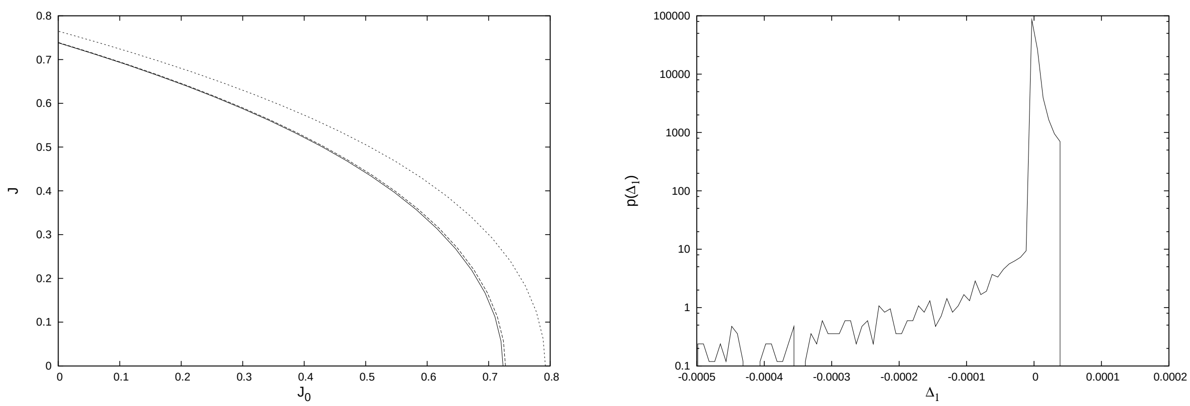

Figure 2. Left: Phase boundaries separating regions without collective acceleration of default rates from regions where acceleration occurs, for $\rho = 0.15$, 0.3 and 0.8 (bottom to top). Right: Distribution of discrete second derivative $\Delta_1 = m_2 + m_0 - 2m_1$ just within the phase with accelerating default rates for $(J_0, J) = (0.5, 0.5)$ (right).

The quantity of central importance from the point of view of credit risk analysis is of course the distribution of losses. Let ℓ_i denote the loss that would be incurred by a default of node i. Then the loss per node for a given state η_0 of the economy is

$$L(\eta_0) = \frac{1}{N}\sum_i n_{i12}\ell_i \tag{18}$$

where ℓ_i is randomly sampled from the loss distribution for node i. We assume that the ℓ_i are independent of the stochastic evolution. In the large system limit, the loss per node at given value of η_0 describing the influence of the overall economy is a non-fluctuating quantity, as it is itself an (empirical) average taken over an (infinitely) large system,

$$L(\eta_0) = \lim_{N\to\infty}\frac{1}{N}\sum_i n_{12}(\vartheta_i)\ell(\vartheta_i) = \int d\vartheta p(\vartheta)\langle n_{12}\rangle_{(\vartheta)}\overline{\ell}(\vartheta) \tag{19}$$

by the law of large numbers, where $\overline{\ell} = \overline{\ell}(\vartheta)$ is the mean of the loss distribution for a node. If loss distributions were identical for each node, with means independent of default probabilities $\overline{\ell}(\vartheta) = \ell_0$, then the distribution of losses driven by the fluctuations of the economic stresses would simply replicate the distribution of the fraction of defaulted firms.

The situation is different if loss-distributions are correlated with default probabilities. As an example we consider the case where average losses are inversely proportional to the unconditional default probabilities $p_d(\vartheta_i) = p_i$ introduced in (9).

$$\overline{\ell}(\vartheta) = \frac{\ell_0}{\varepsilon + p_d(\vartheta)} \tag{20}$$

with a parameter ε introduced as a regularizer to prevent divergence as $p_d \to 0$. That is, the contribution to the total losses incurred by defaulting firms with different unconditional default probabilities is approximately uniform over the default probabilities. In our model we have

$$p_d(\vartheta) = \frac{1}{2}[1 - \mathrm{erf}(\vartheta/\sqrt{2})] \tag{21}$$

Fig. 3 shows loss distributions for such a situation. The analytic curves are computed by noting that the losses per node are monotone increasing functions of η_0 which is itself $\mathcal{N}(0,1)$. Integrated loss distributions are thus simply obtained using error functions

$$\mathrm{Prob}[L(\eta_0) \le L] = \frac{1}{2}[1 + \mathrm{erf}(\eta_0(L)/\sqrt{2})]$$

where $\eta_0(L)$ is the η_0-value giving rise to loss L per node. The probability density function is obtained via a single numerical differentiation.

It would be of some interest to know whether finite sample fluctuations could possibly upset the picture seen so far. To study this issue we look at the losses per node of a finite sample randomly drawn from the nodes of a large economy,

$$L_M(\eta_0) = \frac{1}{M}\sum_{i=1}^{M} n_{12}(\vartheta_i)\ell(\vartheta_i) \tag{22}$$

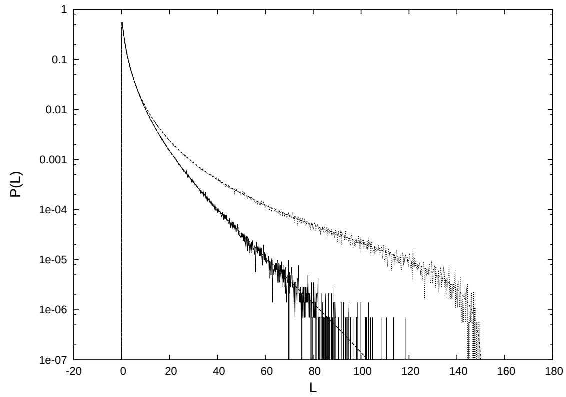

Figure 3. Loss-distribution per node for the infinite system with $\overline{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ at $\varepsilon = 0.005$. As before, smooth analytic curves are overlaid with simulation results. Lower curve: non-interacting system, upper curve: interacting system with $(J_0, J) = (1, 1)$.

Writing this as

$$
\begin{aligned}
L_M(\eta_0) &= \frac{1}{M}\sum_{i=1}^{M}\Big\langle \langle n_{12}\rangle_{(\vartheta)}\overline{\ell}(\vartheta)\Big\rangle_{\vartheta} \\
&\quad + \frac{1}{M}\sum_{i=1}^{M}\Big(\langle n_{12}\rangle_{(\vartheta_i)}\overline{\ell}(\vartheta_i) - \Big\langle \langle n_{12}\rangle_{(\vartheta)}\overline{\ell}(\vartheta)\Big\rangle_{\vartheta}\Big) \\
&\quad + \frac{1}{M}\sum_{i=1}^{M}\Big(n_{12}(\vartheta_i)\ell(\vartheta_i) - \langle n_{12}\rangle_{(\vartheta_i)}\overline{\ell}(\vartheta_i)\Big) \\
&= L_(\eta_0) + L_2(\eta_0, \{\vartheta_i\}) + L_3(\eta_0, \{\vartheta_i\}, \{\phi_{i,t}\})
\end{aligned}
\qquad (23)
$$

we see that it has three components. The first is simply the expectation value describing the loss per node at given η_0 in an infinite system, the third, L_3 has zero mean and is expected to be Gaussian at large M, with variance scaling as M^{-1} describing the noise induced fluctuations about the average for a given collection of $\{\vartheta_i\}$, while the second, L_2 — also a a zero mean Gaussian of variance scaling as M^{-1} at large M — describes the finite sample fluctuations of this average. Since the collection of $\{\vartheta_i\}$ is fixed, these Gaussians are correlated for different η_0. While an analytic evaluation of the loss distribution may still be feasible in principle, it would become very involved in practice.

An approximation to the finite size computation is obtained by assuming that losses per node at given η_0 are Normally distributed about their infinite system η_0-dependent mean with [combining L_2 and L_3] variance (also η_0-dependent)

$$
\sigma_M^2 = M^{-1}\left(\Big\langle \langle n_{12}\rangle_{(\vartheta)}\overline{\ell^2(\vartheta)}\Big\rangle_{\vartheta} - \Big\langle \langle n_{12}(\vartheta)\rangle\overline{\ell}(\vartheta)\Big\rangle_{\vartheta}^2\right) , \qquad (24)
$$

which is an annealed approximation which ignores that the parameters of the individual loss distributions of the nodes in question remain fixed. The results of an evaluation along this line are shown in Fig 4, using the scaling (20) of average losses used above; the approximation suggests that finite size fluctuations give rise to fatter tails in the loss distributions, though the effect is negligible in the interacting system except at the extreme end of the loss distribution, and small but a bit more pronounced in the non-interacting system. A comparison with a simulation shows that the effects of fluctuations are slightly underestimated in our approximation. Note, however, that loan portfolios of typical banks usually contain orders of magnitude more debtors than the $M = 100$ considered in the present example.

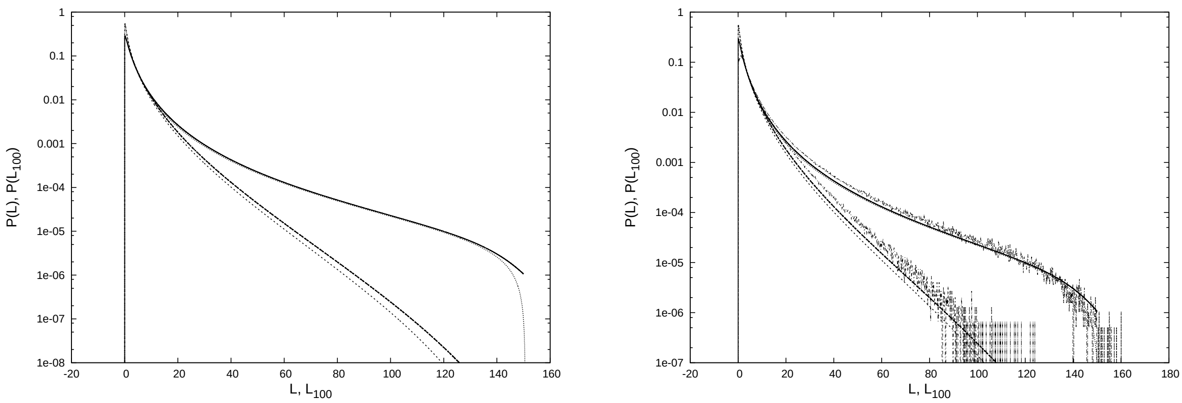

Figure 4. Left: Loss-distribution per node in the large system system limit with $\overline{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ at $\varepsilon = 0.005$, and for a finite sample of $M = 100$ companies randomly taken from the ensemble. For the finite sample, individual loss distributions are taken to be flat in the range $[0, 2\overline{\ell}(\vartheta]$. The two lower curves correspond to (finite and infinite) non-interacting systems, the two upper curves to (finite and infinite) interacting systems with $(J_0, J) = (1, 1)$; for further discussion, see text. Right: The same analytic curves, overlaid with a simulation result for the finite sample.

Let us finally look at the so-called *Value at Risk* in terms of which the capital buffer that banks are required to hold to cover risk is often expressed. It is defined as

$$\mathrm{VaR}_q = (Q_q[L] - \langle L \rangle)\ \mathrm{e}^{-rT} \tag{25}$$

in which $Q_q[L]$ is the q-quantile of the loss distribution at time T, i.e. the loss that is not exceeded with probability q

$$\mathrm{Prob}(L \leq Q_q[L]) = q\ ,$$

while $\langle L \rangle$ is the average loss, and r denotes a risk free interest rate. To highlight the effects introduced by economic interactions we take the ratio of the value at risk computed for economies with and without functional economic interactions $\mathrm{VaR}/\mathrm{VaR}_0$, both computed at confidence level $q = 0.999$ as required by the Basel II regulations [2]. Taking this ratio also eliminates the dependence on the interest rate r.

In Fig 5 we display this ratio alongside with analogous ratios of average losses $\langle L \rangle / \langle L \rangle_0$ for comparison.

As perhaps may be anticipated in view of results displayed in Figs 3 and 4, the Value at Risk is significantly more sensitive to functional interactions in an economy,

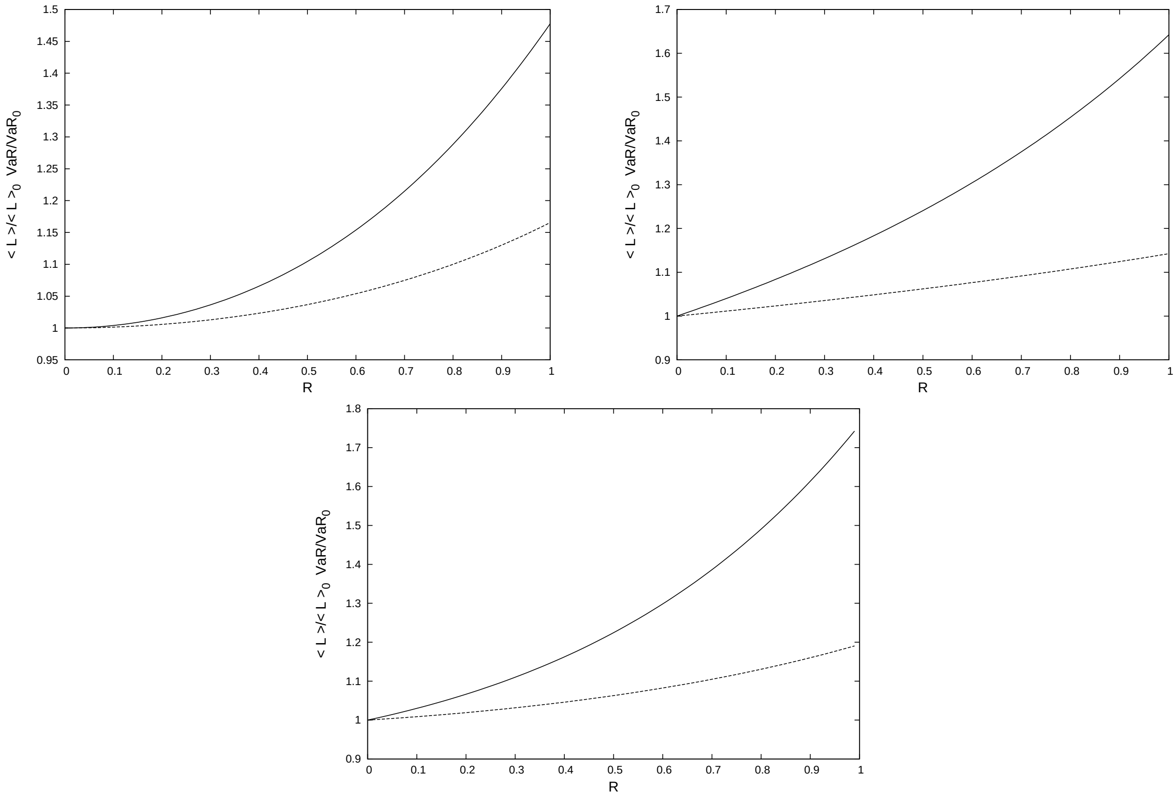

Figure 5. Ratio of Value at Risk for systems with and without functional interaction as a function of the strength of the interaction (upper curves). The analogous ratio for average losses is also shown in each case (lower curves). The curves shown in the three figures are evaluated along straight lines in the $J_0 - J$ plane, and the parameter R measures a distance from the origin $R = \sqrt{J_0^2 + J^2}$. The three figure correspond to the lines $J_0 = 0$ (upper left), $J = 0$ (upper right) and $J_0/J = 1$ (lower).

than the average losses are. This is understandable, as VaR probes the tails of loss distributions, while average losses will be determined mostly by typical results.

For the results displayed in Fig 5, unconditional default probabilities where *not* adjusted with the strength of the interactions so as to keep the average annual default probability constant. In Fig 6, therefore, we take this extra step, displaying an analogous ratio of the Value at Risk of interacting and non-interacting economies, where now unconditional default probabilities in the interacting system are adjusted in such a way that the average (interaction-renormalized) default probability stays constant — at the level chosen for the non-interacting system. To keep matters simple, a homogeneous portfolio, with firm-independent unconditional default probabilities and firm independent average losses was chosen. Clearly the interaction-induced enhancement of the Value at Risk is rather close to the corresponding enhancement computed without adjustment of the unconditional default probabilities.

To summarize, the capital buffer that banks are required to hold according to the Basel II regulations [2] to cover credit risk is *significantly underestimated* when

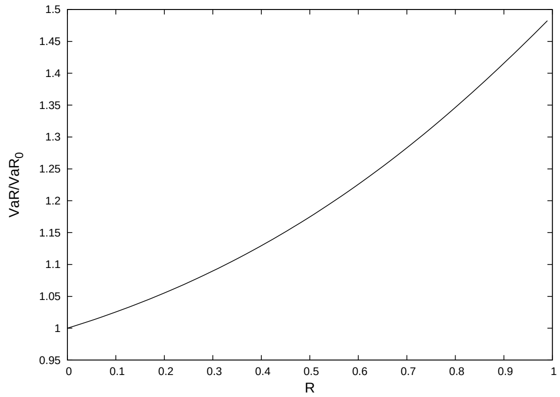

Figure 6. Value at Risk relative to Value at Risk in systems without functional interaction as a function of the strength $R = \sqrt{J_0^2 + J^2}$ of the interaction for $J_0/J = 1$, with unconditional default probabilities in the interacting system adjusted as a function of the strength R of the interaction to keep the annual average default probability constant (in the present case at a value close to 3%.). As in Fig. 5, the VaR ratio is computed at $q = 0.999$.

interaction effects in an economy are not taken into account. It is important to note that this is true already in the regime in which interactions are too weak to cause an overall acceleration of default rates, as can be seen by comparing the phase diagram Fig. 2a with results for the Value at Risk displayed in Figs. 5 and 6.

5. Conclusion

In conclusion, we have studied the effects of economic interactions on credit risks. Though non-equilibrium initial conditions and the fact that the credit-risk problem has an absorbing state would at first sight appear to complicate the analysis, we found, quite to our own surprise, that in particular the presence of the absorbing state simplifies the analysis considerably, as it removes the non-Markovian effects in the macroscopic dynamics that would otherwise be present in systems with some degree of symmetry in the interactions. While the limit of extreme dilution simplified the reasoning within the heuristic solution, we saw in the generating function analysis that the assumption of extreme dilution could be dispensed with. So although the rather heavy machinery of non-equilibrium disordered systems theory is required to rigorously treat the model (due to asymmetry in the inter-firm dependencies and the initial conditions), the resulting effective single-firm process is remarkably simple. This has obvious practical benefits in terms of computational efficiency.

We have seen that the effects of economic interactions are relatively weak in typical economic scenarios, but they are pronounced in situations of economic stress, and thus lead to a substantial fattening of the tails of loss distributions in large loan portfolios. This leads to significant increases in the Value at Risk, i.e. the capital that must be held as a loss buffer, when compared to the non-interacting theory. Importantly, this conclusion remains valid even in the case where there is no overall acceleration in default rates, c.f. Fig. 2a and Figs. 5 and 6.

It is worth paraphrasing these last observations as they address a point of key importance. While credit risk models that do *not* take direct economic interactions into account can provide a very reasonable fit, when calibrated on historical data which reflect normal economic conditions, their predictions would be entirely inadequate when it comes to estimating default rates and losses in situations of significant economic stress.

Note that the model presented here is suitable for detailed and comprehensive stress testing, as explicitly demanded within the regulatory framework of the Basel II accord [2]. The issue of stress testing was addressed in greater detail when the present model was first introduced in [21].

The patterns of economic interactions studied in the present paper are described by an Erdös-Rényi random graph. The large connectivity limit considered in the present investigation further entails that there is no pronounced heterogeneity in the sets of economic partners of any one given node. Connectivity distributions other than Poisson can, however, be handled by suitably adapting the generating function approach explained in Appendix A along the lines developed in [29], and will be investigated in a separate publication [33]. In terms of model fitting there appear to be a vast number of free parameters in terms of the interactions $\{J_{ij}\}$ between firms. However, it is important to realise that to understand the macroscopic behaviour, here only their low order statistics are relevant, reducing the number of parameters that determine collective behaviour in the one factor model to just 6 (!), viz. the three parameters c, J_0 and J characterizing the low order statistics of economic interactions, the parameter ρ describing the relative importance of the macro-economy for the dynamics, and two parameters ϑ_0 and σ_ϑ characterizing the low-order statistics of un-conditional default probabilities.

In the present investigation, we restricted ourselves to analysing the effects of interactions on default-dynamics and, via default rates, on loss distributions. More subtle effects such as credit-quality migration are, as yet, not taken into account, but could be modelled along similar lines using the dynamics of interacting multi-state indicator variables. Further assumptions concerning details of such models would be required, however, and the full complexity of non-Markovian dynamics would resurface in such an analysis.

Appendix A. Generating Function Analysis

In this appendix we describe the generating function approach (GFA) to solve our model, giving full justification to the arguments used in section 3. The reasoning is relatively standard; we include it here to make the paper reasonably self-contained

Appendix A.1. The Generating Function for Correlation Functions

First we introduce the generating function at fixed value of the macro-economic force η_0,

$$Z[\psi|\eta_0] = \left\langle \mathrm{e}^{-\mathrm{i}\sum_{t=0}^{12}\sum_i \psi_{i,t} n_{i,t}} \right\rangle \tag{A.1}$$

where the angled brackets denote averages over the microscopic dynamics (2) of n_i, i.e.

$$Z[\psi|\eta_0] = \sum_{\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}} P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}]\mathrm{e}^{-\mathrm{i}\sum_{t=0}^{12}\sum_i \psi_{i,t} n_{i,t}} \ , \tag{A.2}$$

with $P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}]$ denoting the probability of a sequence of configurations of the entire set of interacting firms over the risk period of 12 months. The generating function can be used to compute expectation values and correlation functions via differentiations with respect to the source fields $\psi_{i,t}$,

$$\langle n_{i,t}\rangle = \mathrm{i}\frac{\partial Z[\psi|\eta_0]}{\partial \psi_{i,t}}\bigg|_{\psi\equiv 0} \quad , \qquad \langle n_{i,s} n_{j,t}\rangle = \mathrm{i}^2\frac{\partial^2 Z[\psi|\eta_0]}{\partial \psi_{i,s}\partial \psi_{j,t}}\bigg|_{\psi\equiv 0}$$

It is expected that correlation functions averaged over the randomness in the couplings J_{ij} are dominated by typical realizations of the disorder, hence to describe typical results an average of the generating function over the disorder,

$$\overline{Z[\psi|\eta_0]} = \int \prod_{i<j} \mathrm{d}P(J_{ij}, J_{ji})\ Z[\psi|\eta_0] \tag{A.3}$$

is computed.

To proceed, the path-probability $P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}]$ at given η_0 is expressed in terms of transition probabilities of the Markovian dynamics,

$$P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}] = P(\boldsymbol{n}_0)\ \prod_{t=0}^{11} P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t)\ ,$$

where

$$P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t) = \prod_i \int \frac{\mathrm{d}\xi_{i,t}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\sum_i \xi_{i,t}^2}\delta_{n_{i,t+1},f_{i,t}} \tag{A.4}$$

with

$$f_{i,t} = n_{i,t} + (1-n_{i,t})\Theta\left(\sum_j J_{ij}n_{j,t} + \sqrt{1-\rho}\ \xi_{i,t} + \sqrt{\rho}\ \eta_0 - \vartheta_i\right) \tag{A.5}$$

The $\xi_{i,t}$ integrations appearing in the transition probabilities and the average over the J_{ij} distribution are facilitated by utilizing δ-distributions to 'extract' the $\xi_{i,t}$ and the J_{ij} from the Heaviside function in $f_{i,t}$, using

$$1 = \int \mathrm{d}u_{i,t}\ \delta\left(u_{i,t} - \sum_j J_{ij}n_{j,t} - \sqrt{1-\rho}\ \xi_{i,t}\right) = \int \frac{\mathrm{d}u_{i,t}\mathrm{d}\hat{u}_{i,t}}{2\pi}\ \mathrm{e}^{-\mathrm{i}\hat{u}_{i,t}\left(u_{i,t} - \sum_j J_{ij}n_{j,t} - \sqrt{1-\rho}\ \xi_{i,t}\right)}\ .$$

This gives

$$P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t) = \int \prod_i \frac{\mathrm{d}u_{i,t}\mathrm{d}\hat{u}_{i,t}}{2\pi}\ \mathrm{e}^{\sum_i\left[-\frac{1-\rho}{2}\hat{u}_{i,t}^2 - \mathrm{i}\hat{u}_{i,t}\left(u_{i,t} - \sum_j J_{ij}n_{j,t}\right)\right]}\prod_i \delta_{n_{i,t+1},f_{i,t}} \tag{A.6}$$

with now

$$f_{i,t} = n_{i,t} + (1 - n_{i,t})\Theta\left(u_{i,t} + \sqrt{\rho}\,\eta_0 - \vartheta_i\right) \tag{A.7}$$

Inserting into the generating function, we get

$$Z[\psi|\eta_0] = \sum_{\boldsymbol{n}_0,\dots,\boldsymbol{n}_{12}} P(\boldsymbol{n}_0) \int \prod_{i,t} \frac{\mathrm{d}u_{i,t}\mathrm{d}\hat{u}_{i,t}}{2\pi}\ \exp\Bigg\{\sum_{i,t}\Bigg[\frac{1-\rho}{2}(\mathrm{i}\hat{u}_{i,t})^2$$
$$-\mathrm{i}\hat{u}_{i,t}\Bigg(u_{i,t} - \sum_j J_{ij} n_{j,t}\Bigg) - \mathrm{i}\psi_{i,t} n_{i,t}\Bigg]\Bigg\}\prod_{i,t}\delta_{n_{i,t+1},f_{i,t}}$$

The disorder average affects the J_{ij}; it factorises in the pairs (i,j) and involves the term

$$\prod_{(i,j)} \overline{D_{ij}} = \prod_{i<j} \overline{\exp\left\{\mathrm{i}\sum_t \left(\hat{u}_{i,t} J_{ij} n_{j,t} + \hat{u}_{j,t} J_{ji} n_{i,t}\right)\right\}}^{c,x}$$

Here the superscripts c and x indicate averages over the c_{ij} and the x_{ij} in the J_{ij} according to the statistics (6)-(8). Performing the c_{ij} average, one gets

$$\prod_{(i,j)} \overline{D_{ij}} = \prod_{i<j}\Bigg\{1 + \frac{c}{N}\overline{\Bigg[\exp\Bigg\{\left(\frac{J_0}{c} + \frac{J}{\sqrt{c}}x_{ij}\right)\sum_t \mathrm{i}\hat{u}_{i,t} n_{j,t}}$$
$$\overline{+\left(\frac{J_0}{c} + \frac{J}{\sqrt{c}}x_{ji}\right)\sum_t \mathrm{i}\hat{u}_{j,t} n_{i,t}\Bigg\} - 1\Bigg]}^{x}\Bigg\}$$

The exponential is expanded using $c \gg 1$. Using (8), keeping dominant terms and re-exponentiating the result one obtains

$$\prod_{(i,j)} \overline{D_{ij}} \simeq \exp\left\{N\left[J_0 \sum_t k_t m_t + \frac{J^2}{2}\sum_{s,t}\Big[Q_{st} q_{st} + \alpha G_{st} G_{ts}\Big]\right]\right\}$$

which depends only on the macro-variables

$$k_t = \frac{1}{N}\sum_i \mathrm{i}\hat{u}_{i,t}\ , \quad m_t = \frac{1}{N}\sum_i n_{i,t} \tag{A.8}$$

$$Q_{st} = \frac{1}{N}\sum_i \mathrm{i}\hat{u}_{i,s}\mathrm{i}\hat{u}_{i,t}\ , \quad q_{st} = \frac{1}{N}\sum_i n_{i,s} n_{i,t}\ , \quad G_{st} = \frac{1}{N}\sum_i \mathrm{i}\hat{u}_{i,s} n_{i,t}\ . \tag{A.9}$$

We thus have

$$\overline{Z[\psi|\eta_0]} = \sum_{\boldsymbol{n}_0,\dots,\boldsymbol{n}_{12}} P(\boldsymbol{n}_0)\int\prod_{i,t}\frac{\mathrm{d}u_{i,t}\mathrm{d}\hat{u}_{i,t}}{2\pi}\ \exp\Bigg\{\sum_{i,t}\left[\frac{1-\rho}{2}(\mathrm{i}u_{i,t})^2 - \mathrm{i}\hat{u}_{i,t}u_{i,t} - \mathrm{i}\psi_{i,t}n_{i,t}\right]$$
$$+N\left[J_0\sum_t k_t m_t + \frac{J^2}{2}\sum_{s,t}\Big[Q_{st}q_{st} + \alpha G_{st}G_{ts}\Big]\right]\Bigg\}\prod_{i,t}\delta_{n_{i,t+1},f_{i,t}}$$

Site factorisation in $\overline{Z[\psi|\eta_0]}$ is achieved as usual by writing it as an integral over the macro-variables, using δ-function identities of the form

$$1 = \int \mathrm{d}(Nm_t)\ \delta\Bigg(Nm_t - \sum_j n_{j,t}\Bigg) = \int \frac{\mathrm{d}m_t \mathrm{d}\hat{m}_t}{2\pi/N}\ \mathrm{e}^{\mathrm{i}\hat{m}_t\left(Nm_t - \sum_j n_{j,t}\right)}$$

and analogous ones for the k_t, q_{st}, Q_{st}, and the G_{st} to compute densities of state. This results in the following compact expression for the average generating function

$$\overline{Z[\psi|\eta_0]} = \int \mathcal{D}\{\ldots\}\exp\{N[\Phi+\Psi+\Xi]\} \tag{A.10}$$

in which $\mathcal{D}\{\ldots\}$ stands for an differentials of all order parameters introduced in (A.8), (A.9) and their conjugate (hatted) parameters introduced via Fourier-representations of δ-functions. The functions Φ, Ψ, and Ξ appearing in (A.10) are given by

$$\Phi = J_0\sum_t k_t m_t + \frac{J^2}{2}\sum_{s,t}\Big[Q_{st}q_{st} + \alpha G_{st}G_{ts}\Big] \tag{A.11}$$

$$\Psi = \mathrm{i}\sum_t[\hat{m}_t m_t + \hat{k}_t k_t] + \mathrm{i}\sum_{st}[\hat{q}_{st}q_{st} + \hat{Q}_{st}Q_{st} + \hat{G}_{st}G_{st}] \tag{A.12}$$

$$\Xi = \frac{1}{N}\sum_i \log\sum_{\{n_t\}}\int\prod_t\frac{\mathrm{d}\hat{u}_t\mathrm{d}u_t}{2\pi}\exp\Big(-\mathcal{S}-\mathrm{i}\sum_t\psi_{i,t}n_t\Big)\prod_t\delta_{n_{t+1},f_{i,t}}\,. \tag{A.13}$$

with $\mathcal{S}$ denoting the 'dynamic action'

$$\mathcal{S} = \sum_t\Big[-\frac{1-\rho}{2}(\mathrm{i}\hat{u}_t)^2 + \mathrm{i}\hat{u}_t u_t + \mathrm{i}\hat{m}_t n_t + \mathrm{i}\hat{k}_t\mathrm{i}\hat{u}_t\Big] + \mathrm{i}\sum_{st}\Big[\hat{q}_{st}n_s n_t + \hat{Q}_{st}\mathrm{i}\hat{u}_s\mathrm{i}\hat{u}_t + \hat{G}_{st}\mathrm{i}\hat{u}_s n_t\Big] \tag{A.14}$$

The third contribution, Ξ, in (A.10) describes an ensemble of independent single site dynamical problems. Thus, to leading order in N we have written our generating function in terms of an integral which may be computed via a saddle point argument.

Appendix A.2. Saddle Point Problem

In the saddle-point, variation of our observables gives

$$\begin{aligned} \mathrm{i}\hat{m}_t &= -J_0 k_t & \mathrm{i}\hat{k}_t &= -J_0 m_t \\ \mathrm{i}\hat{q}_{st} &= -\frac{J^2}{2}Q_{st} & \mathrm{i}\hat{Q}_{st} &= -\frac{J^2}{2}q_{st} \qquad \mathrm{i}\hat{G}_{st} = -\alpha J^2 G_{ts} \end{aligned} \tag{A.15}$$

$$m_t = \frac{1}{N}\sum_i\langle n_t\rangle_{(i)} \qquad k_t = \frac{1}{N}\sum_i\langle \mathrm{i}\hat{u}_t\rangle_{(i)} \tag{A.16}$$

$$q_{st} = \frac{1}{N}\sum_i\langle n_s n_t\rangle_{(i)} \qquad Q_{st} = \frac{1}{N}\sum_i\langle \mathrm{i}\hat{u}_s\mathrm{i}\hat{u}_t\rangle_{(i)} \tag{A.17}$$

$$G_{st} = \frac{1}{N}\sum_i\langle \mathrm{i}\hat{u}_s n_t\rangle_{(i)} \tag{A.18}$$

with $\langle\ldots\rangle_{(i)}$ denoting averages evaluated wrt effective single site dynamics at i.

$$\langle\ldots\rangle_{(i)} = \frac{\sum_{\{n_t\}}\int\prod_t\frac{\mathrm{d}\hat{u}_t\mathrm{d}u_t}{2\pi}(\ldots)\exp\Big(-\mathcal{S}\Big)\prod_t\delta_{n_{t+1},f_{i,t}}}{\sum_{\{n_t\}}\int\prod_t\frac{\mathrm{d}\hat{u}_t\mathrm{d}u_t}{2\pi}\exp\Big(-\mathcal{S}\Big)\prod_t\delta_{n_{t+1},f_{i,t}}} \tag{A.19}$$

In the usual manner [30] averages involving conjugate fields $\mathrm{i}\hat{u}_t$ describe response functions, i.e. perturbations of expectation values wrt external fields, so that averages involving nothing but conjugate variables correspond to perturbations of a constant and will therefore vanish. Moreover, causality implies that G_{st}, which describes the response of the average fraction of defaulted companies to at time t to perturbations at time s must vanish for $s \geq t$. At the saddle point, therefore, we have $k_t \equiv 0$, $\mathrm{i}\hat{m}_t \equiv 0$, $Q_{st} \equiv 0$, $\mathrm{i}\hat{q}_{st} \equiv 0$, and $G_{st} = 0$ for $s \geq t$, thus $\mathrm{i}\hat{G}_{st} = 0$ for $s \leq t$.

With these observations we find that the functions Φ and Ψ appearing in the average generating function (A.10) are zero at the saddle point,

$$\Phi = 0 \ , \qquad \Psi = 0 \ , \tag{A.20}$$

and the dynamic action $\mathcal{S}$ of (A.14) simplifies to

$$\mathcal{S} = -\frac{1}{2}\sum_{st}\Big[(1-\rho)\delta_{st} + J^2 q_{st}\Big]\mathrm{i}\hat{u}_s\mathrm{i}\hat{u}_t + \sum_t \mathrm{i}\hat{u}_t\Bigg(u_t - J_0 m_t - \alpha J^2\sum_{s<t} G_{st} n_s\Bigg) \tag{A.21}$$

With this form of the dynamic action the system-dynamics is described by an ensemble independent effective single-node stochastic process of the form

$$n_{t+1} = f_{\vartheta t} \equiv n_t + (1-n_t)\Theta\Bigg(J_0 m_t + \alpha J^2\sum_{s<t} G_{st} n_s + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Bigg) \tag{A.22}$$

the details of which are self-consistently specified by macroscopic properties of the system via the saddle point equations, in that each single site process (i) depends on the dynamics of the macroscopic fraction of defaulted nodes m_t, (ii) the original Gaussian white noise is replaced by a coloured Gaussian noise ϕ_t with correlations depending on q_{st}

$$\langle \phi_t \rangle = 0 \quad , \quad \langle \phi_s \phi_t \rangle = (1-\rho)\delta_{st} + J^2 q_{st} \ ,$$

and (iii) a memory term appears in the dynamics, if there is some degree of symmetry in the interactions, i.e. if $\alpha \neq 0$.

The only site-dependence in the averages $\langle \ldots \rangle_{(i)}$ appearing the fixed point equations (A.16) - (A.18) comes from the ϑ_i dependence in the update rules $f_{i,t}$. By the law of large numbers, the sums can therefore be evaluated as an average over the ϑ-distribution in the large N limit

$$\frac{1}{N}\sum_i \langle \ldots \rangle_{(i)} \longrightarrow \int \mathrm{d}\vartheta p(\vartheta) \langle \ldots \rangle_{(\vartheta)} \equiv \Big\langle \langle \ldots \rangle_{(\vartheta)} \Big\rangle_\vartheta$$

in which $\langle \ldots \rangle_{(\vartheta)}$ has the same structure as (A.19), except for the fact that the dynamical constraints $f_{i,t}$ of (A.7) are replaced by the $f_{\vartheta t}$ of (A.22). The saddle point equations thus take the form

$$m_t = \Big\langle \langle n_t \rangle_{(\vartheta)} \Big\rangle_\vartheta \quad , \quad q_{st} = \Big\langle \langle n_s n_t \rangle_{(\vartheta)} \Big\rangle_\vartheta \quad , \quad G_{st} = \Big\langle \langle \mathrm{i}\hat{u}_s n_t \rangle_{(\vartheta)} \Big\rangle_\vartheta$$

Appendix A.3. Simplification of the Single Node Equation

The single node equation (A.22) is complicated by the fact that it is non-Markovian, containing a correlation function coupled to the noise term q_{st} and a retarded self-interaction G_{st}. This latter term encodes the physics that a firms performance at time t is influence by its neighbours, themselves dependent on the firm itself at times $s < t$, via loops in our network of corporate interactions — in particular short loops arising through correlated bi-directional interactions. However, as we argued in section 3, if a firm is bankrupt at time s then the performance of partner firms at time t is irrelevant, since the firm will still be bankrupt. In the alternative case when the firm is solvent at time t, it is clear from the definitions in the dynamics that it must have been solvent at time $s < t$ and thus cannot have affected its partner terms at that time. Thus, the retarded self-interaction is zero.

There is a second simplifying feature in (A.22) related to the statistics of the coloured noise within our system. On multiplying (A.22) on both sides by n_s with $s < t$ and and first averaging over the noise $\boldsymbol{\phi}_t = (\phi_1, \phi_2, \ldots, \phi_t)$ one finds $\langle n_s n_t \rangle_{(\vartheta)} = \langle n_{\min(s,t)} \rangle_{(\vartheta)}$ at fixed ϑ, since if $n_s = 1$, then $n_t = 1$ due to the absorbing nature of the defaulted state, whereas if $n_s = 0$, so is the product $n_s n_t$, irrespectively of n_t. As a consequence we have $q_{st} = m_{\min(s,t)}$, and thus

$$\langle \phi_s \phi_t \rangle = (1-\rho)\delta_{st} + J^2 m_s \ , \qquad s \leq t \ , \tag{A.23}$$

Having seen that the memory term in the dynamics vanishes, it transpires that only the equal-time version of the noise correlation $\langle \phi_t \phi_t \rangle = 1 - \rho + J^2 m_t$ is required to propagate the order parameter m_t. One needs

$$m_{t+1} = \Big\langle \langle n_{t+1} \rangle_{(\vartheta)} \Big\rangle_{\vartheta} = m_t + \Big\langle \Big\langle (1-n_t)\Theta(J_0 m_t + \sqrt{\rho}\eta_0 - \vartheta + \phi_t) \Big\rangle_{(\vartheta)} \Big\rangle_{\vartheta}$$

In order to evaluate the average $\langle \ldots \rangle_{(\vartheta)}$ over the correlated noise in the second term, convert the probability density $p(\boldsymbol{\phi}_t) = p(\phi_1, \phi_2, \ldots, \phi_t)$ into $p(n_t, \phi_t)$ — the joint probability density that the node-variable takes value n_t ($n_t = 0$ or $n_t = 1$) and the noise variable at time t is in an infinitesimal interval around ϕ_t; formally one can write this as

$$p(n_t, \phi_t) = \int \mathrm{d}\boldsymbol{\phi}_{t-1} p(\boldsymbol{\phi}_t) \delta_{n_t, n_t(\vartheta, \boldsymbol{\phi}_{t-1})} = \int \mathrm{d}\boldsymbol{\phi}_{t-1} p(\boldsymbol{\phi}_{t-1}, \phi_t) \delta_{n_t, n_t(\vartheta, \boldsymbol{\phi}_{t-1})}$$

where $n_t(\vartheta, \boldsymbol{\phi}_{t-1})$ is the value of n_t for the specific ϑ under consideration, and a given noise history $\boldsymbol{\phi}_{t-1}$. Writing the joint probability in terms of a conditional as

$$p(n_t, \phi_t) = p(n_t | \phi_t) p(\phi_t) \ ,$$

and noting that $p(n_t|\phi_t)$ must be independent of the conditioning by causality, and finally using

$$p(n_t) = \langle n_t \rangle_{(\vartheta)} \delta_{n_t,1} + \Big(1 - \langle n_t \rangle_{(\vartheta)}\Big) \delta_{n_t,0}$$

for a given ϑ one finally obtains

$$m_{t+1} = \Big\langle \langle n_{t+1} \rangle_{(\vartheta)} \Big\rangle_{\vartheta} = m_t + \left\langle \frac{1 - \langle n_t \rangle_{(\vartheta)}}{2} \left[1 + \mathrm{erf}\left(\frac{J_0 m_t + \sqrt{\rho}\ \eta_0 - \vartheta}{\sqrt{2(1 - \rho + J^2 m_t)}} \right) \right] \right\rangle_{\vartheta} \tag{A.24}$$

which agrees with the result of our heuristic reasoning in Sec 3. Note that the condition $c/N \to 0$ is not needed in the present argument.

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[1] S. Keenan, Moody's Investor Services, (2000)

[2] Basel Committee on Banking Supervision 2005 *International Convergence of Capital Measurement and Capital Standards A Revised Framework*, URL: `http://www.bis.org`, Basel

[3] J.P. Morgan Global Research 1997 *CreditMetrics™: The Benchmark for Understanding Credit Risk*, Technical Document, `URL: www.creditmetrics.com`, New York

[4] Credit Suisse First Boston 1997 *Credit Risk+: A Credit Risk Management Framework*, Technical Document, `URL: www.csfb.com/creditrisk`, New York

[5] S. Kealhofer 1998 Net Exposure, Vol. **1** (2); KMV Corporation, *Credit Monitor™ and Portfolio Manager™*, New York, (1996), `URL: www.KMV.com`; O. A. Vasicek 1997 *The loan Loss Distribution*, Technical Report, KMV Corporation

[6] M. Davis and V. Lo 2001 Quantitative Finance **1** 382-387; in Mastering Risk Volume 2: Applications, ed. Carol Alexander, Financial Times Prentice Hall, p. 141-151 (2001); M. Crowder, M. Davis and G. Giampieri 2005 Quantitative Finance **5** 27-34

[7] D. Lando 1998 Rev. Derivatives Res. **2** 99-120

[8] D. Duffie and K. Singletonm 1999 Rev. Financial Stud. **12** 687-720

[9] R. Jarrow and F. Yu 2001 Journal of Finance **56** 1765; F. Yu Mathematical Finance (to appear).

[10] E. Rogge and P. Schönbucher 2003 *Modelling Dynamic Portfolio Credit Risk*, Working Paper, Imperial College, London, ABN Amro Bank, London, and ETH, Zürich ; P. Schönbucher and D. Schubert 2001 *Copula-dependent Default Risk in Intensity Models* Working Paper, Department of Statistics, Bonn University

[11] F. Yu, R. Jarrow and D. Lando 2005 Mathematical Finance **15**(1) 1-26

[12] S. R. Das, D. Duffie and N. Kapadia 2005 *Common Failings: How Corporate Defaults are Correlated*, Working Paper, Santa Clara University, Stanford University, University of Massachusetts

[13] F. Yu 2005 *Default Correlation in Reduced-Form Models*, Journal of Investment Management **3**(4) 33-42

[14] D. Duffie and K. Singleton 1999 *Simulating Correlated Defaults* Working Paper, Graduate School of Business, Stanford University

[15] K. Giesecke 2003 Journal of Fixed Income **13**(3) 74-83; K. Giesecke 2003 Computing in Economics and Finance No 247, Society for Computational Economics

[16] K. Giesecke and S. Weber, *Credit Contagion and Aggregate Losses*, Journal of Economic Dynamics and Control (to appear), K. Giesecke and S. Weber 2004Journal of Banking and Finance **28**(12) 3009-3036

[17] D. Tasche and U. Theiler 2004 in Operations Research 2003, D. Ahr, R. Fahrion, M. Oswald, G. Reinelt (eds.) Springer p. 261-268

[18] D. Egloff, M. Leippold, and P. Vanini 2004 EFA Maastricht Meetings Paper No. 1142

[19] M. B. Gordy 2000 Journal of Banking and Finance, Vol. **24** 119-149

[20] M. B. Gordy 2002 Journal of Financial Intermediation Vol. **12**(3) 199-232

[21] P. Neu and R. Kühn 2004 Physica A **342** 639-655

[22] R. Merton 1974 Journal of Finance Vol. **29** 449-470

[23] Ian Iscoe, Alex Kreinin and Dan Rosen 1999 Algorithmics Research Quarterly Vol. **2** (3) 21-37

[24] Moody's Investment Services 1997 *The Binominal Expansion Technique*, `URL: www.moodys.com`

[25] R. Kühn and P. Neu 2003 Physica A **322** 650-666

[26] B. Bollobàs, *Random Graphs*, (Cambridge Univ. Press, Cambridge, 2001)

[27] A.L. Barabási and R. Albert 1999 Science **286** 509-512; D.J. Watts and S.H. Strogatz 1998 Nature **393** 440-442

[28] J.P.L. Hatchett, B. Wemmenhove, I Pérez Castillo, T. Nikoletopoulos, N.S. Skantzos and A.C.C. Coolen 2004 J. Phys. A: Math. Gen. **37** 6201-6220; I Pérez Castillo, B. Wemmenhove, J.P.L. Hatchett, A.C.C. Coolen, N.S. Skantzos and T. Nikoletopoulos, 2004 J. Phys. A: Math. Gen. **37** 8789-8799; T. Nikoletopoulos, A.C.C. Coolen, I Pérez Castillo, N.S. Skantzos, J.P.L. Hatchett and B. Wemmenhove 2004 J. Phys. A: Math. Gen., **37** 6455-6475

[29] J.P.L. Hatchett 2005 AIP Conf. Proc. **776** 150

[30] C. De Dominicis 1978 Phys. Rev. B **18** 4913-4919

[31] B. Derrida, E. Gardner, A. Zippelius 1987 Europhys. Lett. 4 167-173; R. Kree and A. Zippelius 1991 in E. Domany, J. L. van Hemmen, and K. Schulten eds, Models of Neural Networks. Springer, Berlin; J.P.L. Hatchett and A. C. C. Coolen 2004 J. Phys. A: Math. Gen. **37** 7199-7212; T.L.H. Watkin and D. Sherrington 1991 Europhys.Lett. **14** 791-796; T.L.H. Watkin D. and Sherrington 1991 J. Phys. A:Math. Gen. **24** 5427-5433

[32] L.E. Reichl 1998 *A Modern Course in Statistical Physics* Wiley-VCH

[33] K. Anand, J.P.L. Hatchett, and R. Kühn, work in progress (2005)