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Credit Contagion and Credit Risk

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with

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This Talk

- Part of research programme devoted to understanding

  influence of interactions on risk

  So far looked at

  - Operational Risk (OR): interacting processes
  - Market Risk (MR): interacting Geometric Brownian Motions
  - Credit Risk (CR): economic interaction and credit contagion

- Details at

  http://www.mth.kcl.ac.uk/~kuehn/riskmodeling.html

Fundamental Problem in Risk Analysis

- Estimation of risk
  - Market: potential negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices)
  - Credit: potential change of credit quality, including default (asset values of firms, ratings, stock-prices)
  - Liquidity: potential losses incurred by need to maintain liquidity under strain on cash-flows
  - Operational: potential losses incurred by process failures (human errors, hardware/software-failures, lack of communication, fraud, external catastrophes)

Task

- Estimate PDF of
  - Portfolio value at time T, $PV(T)$, given $PV(0)$
  - Market value of assets of obligors at time T, $A_i(T)$, given $A_i(0)$
  - Losses $L(T)$ due to process failures incurred during risk horizon T
  - Losses $L(T)$ incurred during risk horizon T by need to maintain liquidity in situations of stress
- Quantity of interest: Value at Risk

$$\mathrm{VaR}_{q,T} = (Q_q[L(T)] - \langle L \rangle)\ \mathrm{e}^{-rT} \qquad \mathrm{Prob}(L(T) \leq Q_q[L(T)]) = q$$

Assessment of PDFs (tails, VaR)

- Adjustment of business model, (re)design of processes
  - charged fees, interest rates, rating of clients
  - activities on derivative markets (hedging)
  - insurance policies
- Proper risk control and management is
  - *demanded* by international banking supervision (BASEL)
  - *recognised* by rating agencies, analysts
- For regulator: Implications of regulatory measures.

The case of Credit Risks (CR)

- Risk arising from the possibility of obligors going bankrupt or from changes in 'credit quality' ($\Rightarrow$ credit trading)

- Here only influence of defaults.

- Standard approach:
  - Estimate default probabilities (rating procedure)
  - Estimate loss distributions (recovery process)
  - Correlations of defaults through economy or business sectors: conditional on economy-wide or sectional influences defaults are treated as independent.

Main Critique

- Misses the effects of functional economic dependencies

- No description of collective events (avalanches of defaults)

- Misses influence of loan portfolio structure on its granularity (e.g. mortgages of local bank to people all working for same employer)

& Possible Remedy

- Interacting Companies Model (P Neu, RK 2004)
  - perspective of lending bank
  - economy-wide perspective of regulators/policy makers

Interacting Companies Model

- Two state model:

  company up and running ($n_i = 0$), or defaulted ($n_i = 1$)

- Probabilities of default and mutual impacts of defaults heterogeneous across the set of processes (quenched disorder); connectivity functionally defined

  $\Longrightarrow$ lattice gas model defined on random graph

- Losses determined (randomly: recovery process) when a company defaults (annealed disorder)

Dynamics

- Companies need "orders" (support, cash inflow) to maintain wealth and avoid default

- h_{it} total wealth of company i at time t,

$$h_i(t) = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$

  with η_{it} random (e.g. Gaussian white noise).

- Company i defaults, if the total wealth falls below zero

$$n_{it+1} = n_{it} + (1 - n_{it})\Theta\left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it}\right)$$

- No recovery within 'risk horizon' T: $n_i = 1$ is absorbing state. Time unit: 1 month; $T = 12 \Leftrightarrow 1$ year.

Meaning of Parameters

- Probability of failure in given 'situation'

$$\text{Prob}\left(n_{it+1} = 1 \middle| \boldsymbol{n}_t\right) = \Phi\left(\sum_j J_{ij}\, n_{jt} - \vartheta_i\right)$$

with

$$\Phi(x) = \frac{1}{2}[1 + \text{erf}(x/\sqrt{2})]$$

- unconditional and conditional probability of failure

$$\begin{aligned} p_i &= \Phi(-\vartheta_i) \\ p_{i|j} &= \Phi(J_{ij} - \vartheta_i) \end{aligned}$$

$$\Rightarrow \quad \vartheta_i = -\Phi^{-1}(p_i)\ , \quad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)$$

Key Features

- For sufficiently strong interactions — in particular cooperative interactions — get possibility of collective acceleration of default rates

- Typical behaviour less sensitive to interactions than rare events $\Longrightarrow$ fat tails in loss distribution; important for risk analysis where rare event asymptotics is relevant.

- Due to initial conditions $n_{i0} \equiv 0$ and absorbing state:
  $\Rightarrow$ no equilibrium dynamics

Analysis for a Stochastic Setting

- Consider, e.g., interactions on a random graph

$$J_{ij} = c_{ij}\left(\frac{J_0}{c} + \frac{J}{\sqrt{c}}\, x_{ij}\right)$$

with

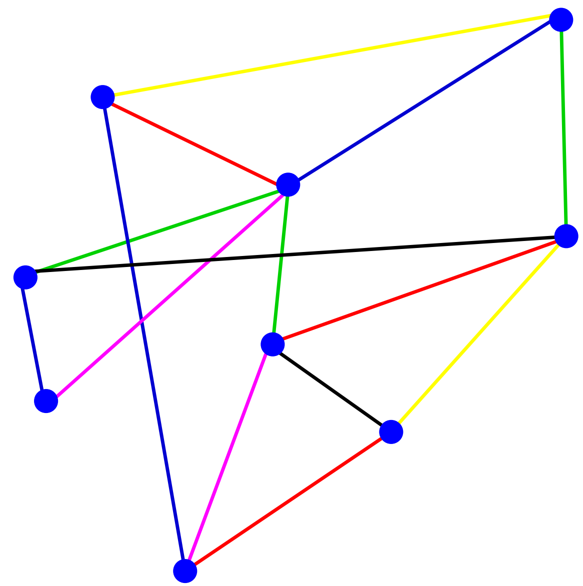
$$P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0} \ ,$$

symmetric ($c_{ij} = c_{ji}$), and

$$\overline{x}_{ij} = 0 \ , \quad \overline{x_{ij}^2} = 1 \ , \quad \overline{x_{ij}x_{ji}} = \alpha \ .$$

Study the limit

$$c \gg 1 \ , \qquad N \gg 1 \ .$$



Heuristic Solution

- In dynamical evolution

$$n_{it+1} = n_{it} + (1 - n_{it})\Theta\left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it}\right)$$

interaction via the local field

$$h_{it} = \sum_j J_{ij} n_{jt} = \sum_j c_{ij}\Big(\frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij}\Big) n_{jt}$$

could be evaluated using LLN (1st term) and CLT (2nd term), if the n_{jt} were uncorrelated.

- Normally prevented by correlations (and memory) through dynamical history: $n_{it} \longrightarrow n_{jt+1} \longrightarrow n_{it+2}$ or $n_{it} \longrightarrow n_{jt+1} \longrightarrow n_{kt+2} \longrightarrow n_{jt+3} \longrightarrow n_{it+4}$ etc. etc.

- Luckily:
  - As long as $n_{it} = 0$ (i is up and running):

$$\text{no influence on } j \qquad \Longleftrightarrow \qquad J_{ji} n_{it} = 0$$

  - Once $n_{it} = 1$ (i is down), it remains so, and correlations in h_{it} are irrelevant. $\Longrightarrow$ Can use limit theorems after all!

- By LLN:

$$h^0_{it} = \frac{J_0}{c} \sum_j c_{ij} n_{jt} \simeq \frac{J_0}{c} \sum_j \overline{c_{ij}} \; \overline{\langle n_{jt} \rangle} = \frac{J_0}{N} \sum_j \overline{\langle n_{jt} \rangle}$$

- By CLT:

$$\tilde{h}_{it} = \frac{J}{\sqrt{c}} \sum_j c_{ij} x_{ij} n_{jt}$$

zero mean Gaussian with

$$\overline{\langle \tilde{h}^2_{it} \rangle} \simeq \frac{J^2}{N} \sum_j \overline{\langle n_{jt} \rangle}$$

- Both depend only on fraction of defaulted companies

$$m_t = \frac{1}{N}\sum_j n_{jt} \longrightarrow \frac{1}{N}\sum_j \overline{\langle n_{jt}\rangle}$$

$\Longrightarrow$ Macroscopic dynamics

$$m_{t+1} = \frac{1}{N}\sum_i n_{it+1} = \frac{1}{N}\sum_i [n_{it} + (1-n_{it})\Theta\,(h_{it} - \vartheta_i + \eta_{it})]$$

- Credit risk: decompose stochastic force (minimal model):

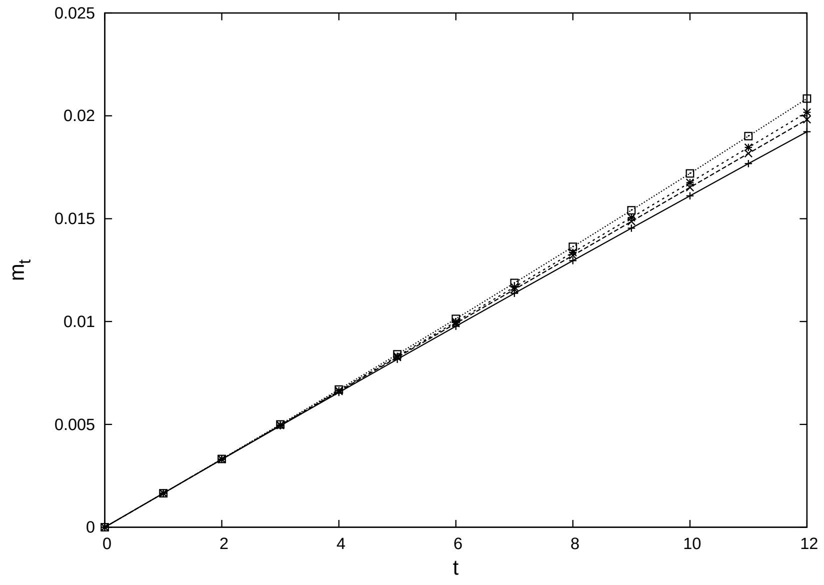
$$\eta_{it} = \sqrt{\rho}\;\eta_0 + \sqrt{1-\rho}\;\xi_{it}$$

Global component η_0 slowly varying.

- Macroscopic dynamics via LLN as average over joint h_{it}, ϑ_i and ξ_{it} distribution (at fixed η_0).

$$m_{t+1} = m_t + \left\langle \left(1 - \langle n_t\rangle_{(\vartheta)}\right)\Phi\left(\frac{J_0 m_t + \sqrt{\rho}\;\eta_0 - \vartheta}{\sqrt{1-\rho+J^2 m_t}}\right)\right\rangle_{\vartheta}$$

Results: Typical Defaulted Fraction and Distribution



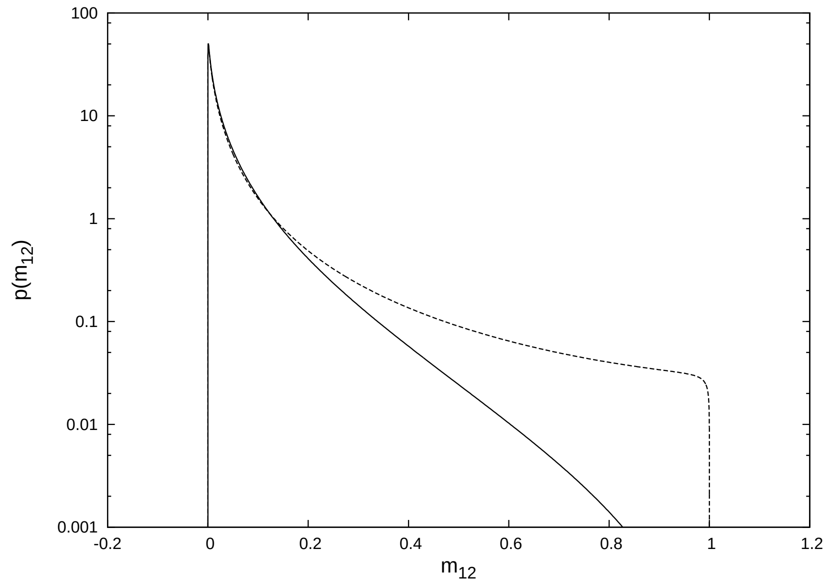


Fig 1. Left: Typical fraction m_t of defaulted firms as a function of time for $(J_0, J) = (0,0), (1,0), (0,1)$, and (1,1) (bottom to top). Right: Distribution of m_t at $t = 12$ for $(J_0, J) = (0,0)$, and (1,1). Bare default rate distribution: $\vartheta_0 = 2.75$, $\sigma_\vartheta^2 = 0.1$. Correlation to macro-economy following Basel II prescription: $\rho = \rho(p_i) \simeq 0.12(1 + \mathrm{e}^{-600 p_i})$.

Results: Phase Diagram

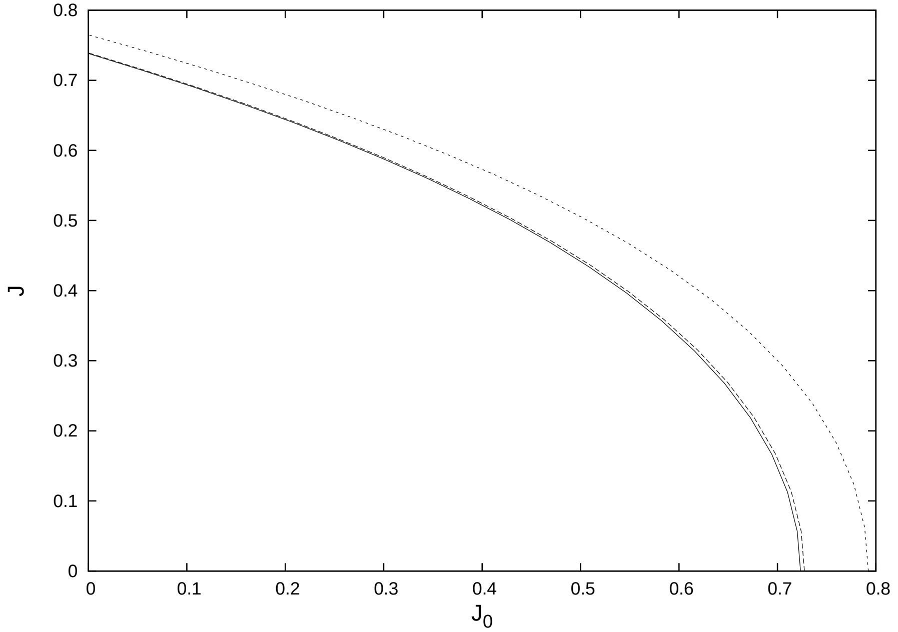


Fig 2. Phase boundaries separating regions without collective acceleration of default rates from regions where acceleration occurs, for $\rho = 0.15$, 0.3 and 0.8 (bottom to top).

Results: Losses per node

$$L(\eta_0) = \frac{1}{N}\sum_i n_{i12}\ \ell_i \longrightarrow \frac{1}{N}\sum_i \overline{\langle n_{i12}\rangle}\ \overline{\ell_i}$$

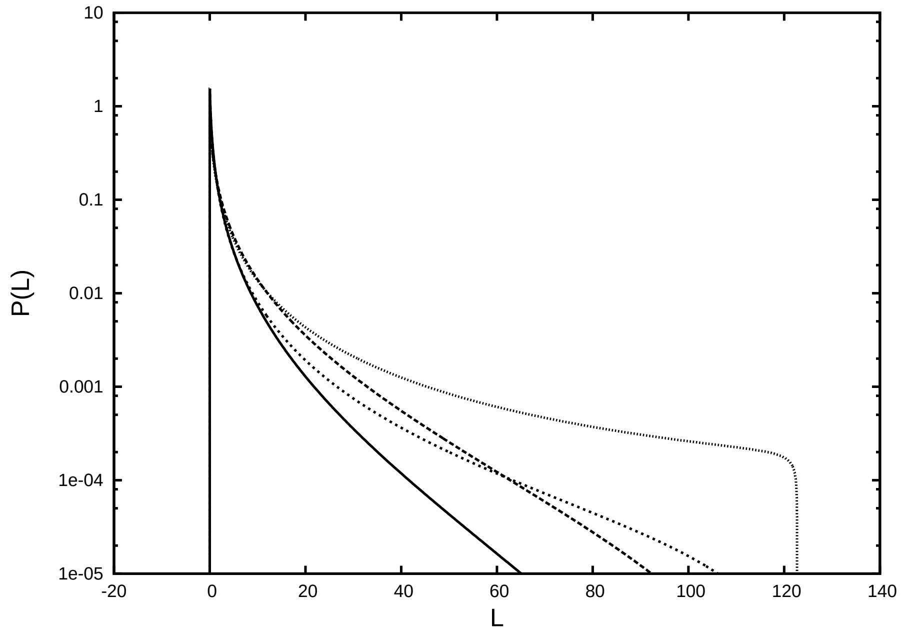


Fig 3. Loss-distribution per node for a system with $\overline{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ at $\varepsilon = 0.005$, at $t = 6$ months and $t = 12$ months, both for the non-interacting, and for the interacting system with $(J_0, J) = (1, 1)$.

Results: Value At Risk

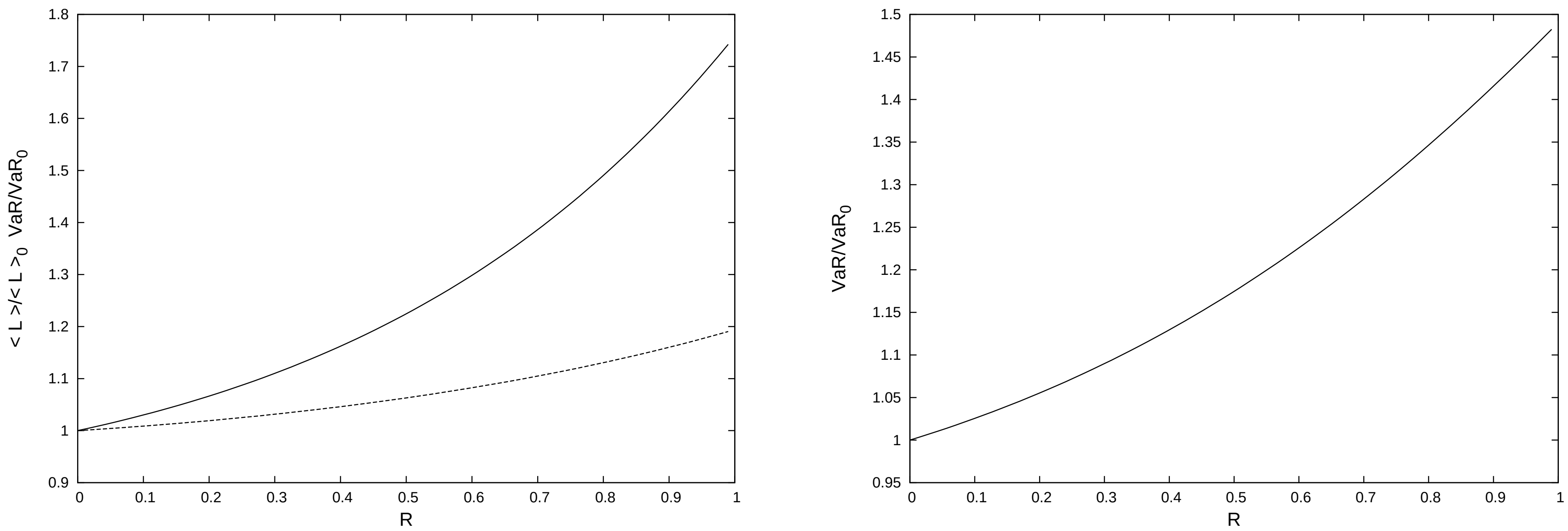


Fig 4. Left: Ratio of value at risk for systems with and without interactions (upper curve) and corresponding ratio of average losses (lower curves) as a function of the strength $R = \sqrt{J_0^2 + J^2}$ of the interactions for $J/J_0 = 1$. Right: Ratio of value at risk for systems with and without interactions as a function of R for $J/J_0 = 1$, with unconditional failure probabilities in interacting system adjusted to keep average fraction of annual defaults constant.

Generalization

- Multi-sector economy; sector specific statistics for mutual dependencies:

$$J_{ij} = c_{ij} \left(\frac{J_0(k_i;k_j)}{c} + \frac{J(k_i;k_j)}{\sqrt{c}}\, x_{ij} \right)$$

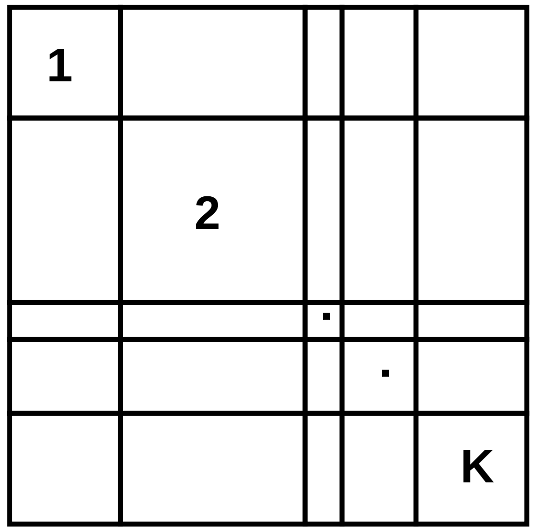
  with $k_\cdot$ sector indicator function.

- sector specific random forces (shocks)

$$\eta_{it} = \sum_{k'} \rho_{k_i,k'} \eta_{k'} + \sqrt{1 - r_{k_i}}\, \xi_{it}$$

  with $r_{k_i} = \sum_{k'} \rho^2_{k_i,k'}$

- Allows to assess effect of sector specific shocks on different parts of the economy.



Issues

- Model calibration
  - (unconditional) failure rates via rating as in traditional models
  - factor loadings $\rho_{k,k'}$, as in traditional factor models
  - New: Interactions. However, only low order statistics needed for global behaviour $\Longrightarrow$ 2 parameters per sector
  - New: Disentangle a-priori and interaction induced effects from single historical data set

  $\Longrightarrow$ ML estimation/Bayesian regression (& model selection(!))

- Model complexity [check whether more heterogeneity (scale-free networks) needed]

Summary

- Standard CR models inadequate to capture functional economic dependencies.
- Interacting companies model (Physics analogy: heterogeneous lattice gas model on functionally defined graph.
- In large portfolios/on economy-wide scale only low order statistics of interaction effects required.
- Typical behavior not strongly modified.
- Describes bursts and avalanches of risk events
- Interesting for risk-management: fat tails in loss distributions created by interactions in situations of economic stress

Generating Function Analysis

- Generating function for correlation functions

$$Z[\psi] = \left\langle \mathrm{e}^{-\mathrm{i}\sum_{t=0}^{12}\sum_i \psi_{it} n_{it}} \right\rangle = \sum_{\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}} P[\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}]\mathrm{e}^{-\mathrm{i}\sum_{t=0}^{12}\sum_i \psi_{it} n_{it}}$$

  Correlation functions as derivatives

$$\langle n_{it}\rangle = \mathrm{i}\frac{\partial Z[\psi]}{\partial \psi_{it}}\bigg|_{\psi\equiv 0} \quad , \qquad \langle n_{is} n_{jt}\rangle = \mathrm{i}^2 \frac{\partial^2 Z[\psi]}{\partial \psi_{is}\partial \psi_{jt}}\bigg|_{\psi\equiv 0}$$

- Correlation functions averaged over disorder ($\Leftrightarrow$ correlation functions for typical realizations of the disorder) from average of generating function.

$$\overline{\langle n_{it}\rangle} = \mathrm{i}\frac{\partial \overline{Z[\psi]}}{\partial \psi_{it}}\bigg|_{\psi\equiv 0} \quad , \qquad \overline{\langle n_{is} n_{jt}\rangle} = \mathrm{i}^2 \frac{\partial^2 \overline{Z[\psi]}}{\partial \psi_{is}\partial \psi_{jt}}\bigg|_{\psi\equiv 0}$$

- Decompose stochastic force: economy & idiosyncratic components

$$\eta_{it} = \sqrt{\rho}\ \eta_0 + \sqrt{1-\rho}\ \xi_{it}$$

- Evaluation uses

$$P[\boldsymbol{n}_0, \ldots, \boldsymbol{n}_{12}] = P[\boldsymbol{n}_0] \prod_{t=0}^{11} P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t)$$

$$P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t) = \int \prod_i \frac{\mathrm{d}\xi_{it}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\sum_i \xi_{it}^2} \prod_i \delta_{n_{it+1}, f_{it}}$$

with

$$f_{it} = n_{it} + (1 - n_{it})\Theta\left(\sum_j J_{ij} n_{jt} + \sqrt{1-\rho}\; \xi_{it} + \sqrt{\rho}\; \eta_0 - \vartheta_i\right)$$

- Introduce

$$\begin{aligned} 1 &= \int \mathrm{d}u_{it}\; \delta\left(u_{it} - \sum_j J_{ij} n_{jt} - \sqrt{1-\rho}\; \xi_{it}\right) \\ &= \int \frac{\mathrm{d}u_{it}\mathrm{d}\hat{u}_{it}}{2\pi}\; \mathrm{e}^{-\mathrm{i}\hat{u}_{it}\left(u_{it} - \sum_j J_{ij} n_{jt} - \sqrt{1-\rho}\; \xi_{it}\right)} \end{aligned}$$

to get quantities to be averaged over into exponent.

- Gives

$$P(\boldsymbol{n}_{t+1}|\boldsymbol{n}_t) = \int \prod_i \frac{\mathrm{d}u_{it}\mathrm{d}\widehat{u}_{it}}{2\pi}\, \mathrm{e}^{\sum_i \left[-\frac{1-\rho}{2}\widehat{u}_{it}^2 - \mathrm{i}\widehat{u}_{it}\left(u_{it}-\sum_j J_{ij}n_{jt}\right)\right]} \prod_i \delta_{n_{it+1},f_{it}}$$

with now

$$f_{it} = n_{it} + (1-n_{it})\Theta\left(u_{it} + \sqrt{\rho}\,\eta_0 - \vartheta_i\right)$$

So

$$Z[\psi|\eta_0] \;=\; \sum_{\boldsymbol{n}_0,\ldots,\boldsymbol{n}_{12}} P[\boldsymbol{n}_0] \int \prod_{it} \frac{\mathrm{d}u_{it}\mathrm{d}\widehat{u}_{it}}{2\pi}\; \exp\left\{\sum_{it}\Big[-\frac{1-\rho}{2}\widehat{u}_{it}^2 - \mathrm{i}\widehat{u}_{it}\Big(u_{it} - \sum_j J_{ij}n_{jt}\Big) - \mathrm{i}\psi_{it}n_{it}\Big]\right\} \prod_{it}\delta_{n_{it+1},f_{it}}$$

- Disorder average affects the J_{ij}; factorizes in (i,j).

$$\prod_{(i,j)} \overline{D_{ij}} = \prod_{i<j} \overline{\exp\left\{\mathrm{i}\sum_t \left(\widehat{u}_{it}J_{ij}n_{jt} + \widehat{u}_{jt}J_{ji}n_{it}\right)\right\}}^{\,c,\,x}$$

- Get

$$\prod_{(i,j)} \overline{D_{ij}} \;=\; \prod_{i<j} \left\{ 1 + \frac{c}{N} \overline{\left[\exp\left\{ \Big(\frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij}\Big) \sum_t \mathrm{i}\hat{u}_{it} n_{jt} \right.\right.} \right.$$

$$\left.\overline{\left.\left. + \Big(\frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ji}\Big) \sum_t \mathrm{i}\hat{u}_{jt} n_{it} \right\} - 1 \right]}^{x} \right\}$$

- Expand exponential using $c \gg 1$

$$\prod_{(i,j)} \overline{D_{ij}} \simeq \exp\left\{ N \left[J_0 \sum_t k_t m_t + \frac{J^2}{2} \sum_{s,t} [Q_{st} q_{st} + \alpha G_{st} G_{ts}] \right] \right\}$$

with

$$k_t = \frac{1}{N} \sum_i \mathrm{i}\hat{u}_{it} \quad , \qquad m_t = \frac{1}{N} \sum_i n_{it}$$

$$Q_{st} = \frac{1}{N} \sum_i \mathrm{i}\hat{u}_{is} \mathrm{i}\hat{u}_{it} \ , \qquad q_{st} = \frac{1}{N} \sum_i n_{is} n_{it} \ , \qquad G_{st} = \frac{1}{N} \sum_i \mathrm{i}\hat{u}_{is} n_{it}$$

Use as integration variables; enforce through δ-functions.
$\Rightarrow$ Generating function evaluated by saddle point method.

- Introduce

$$1 = \int \mathrm{d}Nm_t \ \delta\Big(Nm_t - \sum_j n_{jt}\Big) = \int \frac{\mathrm{d}m_t \mathrm{d}\widehat{m}_t}{2\pi/N} \ \mathrm{e}^{\mathrm{i}\widehat{m}_t\left(Nm_t - \sum_j n_{jt}\right)}$$

and analogous for k_t, q_{st}, Q_{st}, and G_{st}.

- Gives integral representation of $\overline{Z[\psi|\eta_0]}$

$$\overline{Z[\psi|\eta_0]} = \int \mathcal{D}\{\ldots\} \exp\{N[\Xi_1 + \Xi_2 + \Xi_3]\}$$

with

$$\begin{aligned}
\Xi_1 &= J_0 \sum_t k_t m_t + \frac{J^2}{2} \sum_{s,t} [Q_{st} q_{st} + \alpha G_{st} G_{ts}] \\
\Xi_2 &= \mathrm{i} \sum_t [\widehat{m}_t m_t + \widehat{k}_t k_t] + \mathrm{i} \sum_{st} [\widehat{q}_{st} q_{st} + \widehat{Q}_{st} Q_{st} + \widehat{G}_{st} G_{st}] \\
\Xi_3 &= \frac{1}{N} \sum_i \log \sum_{\{n_t\}} \int \prod_t \frac{\mathrm{d}\widehat{u}_t \mathrm{d}u_t}{2\pi} \exp\Big(- \mathcal{S} - \mathrm{i} \sum_t \psi_{it} n_t \Big) \ \prod_t \delta_{n_{t+1}, f_{it}}
\end{aligned}$$

and

$$\mathcal{S} = \sum_t \Big[\frac{1-\rho}{2} \widehat{u}_t^2 + \mathrm{i}\widehat{u}_t u_t + \mathrm{i}\widehat{m}_t n_t + \mathrm{i}\widehat{k}_t \mathrm{i}\widehat{u}_t\Big] + \mathrm{i} \sum_{st} \Big[\widehat{q}_{st} n_s n_t + \widehat{Q}_{st} \mathrm{i}\widehat{u}_s \mathrm{i}\widehat{u}_t + \widehat{G}_{st} \mathrm{i}\widehat{u}_s n_t\Big]$$

- saddle point evaluation $\Longrightarrow$ FPEs

$$\mathrm{i}\hat{m}_t = -J_0 k_t \qquad \mathrm{i}\hat{k}_t = -J_0 m_t$$

$$\mathrm{i}\hat{q}_{st} = -\frac{J^2}{2} Q_{st} \qquad \mathrm{i}\hat{Q}_{st} = -\frac{J^2}{2} q_{st} \qquad \mathrm{i}\hat{G}_{st} = -\alpha J^2 G_{ts}$$

$$m_t = \frac{1}{N}\sum_i \langle n_t \rangle_{(i)} \qquad k_t = \frac{1}{N}\sum_i \langle \mathrm{i}\hat{u}_t \rangle_{(i)} \equiv 0$$

$$q_{st} = \frac{1}{N}\sum_i \langle n_s n_t \rangle_{(i)} \qquad Q_{st} = \frac{1}{N}\sum_i \langle \mathrm{i}\hat{u}_s \mathrm{i}\hat{u}_t \rangle_{(i)} \equiv 0$$

$$G_{st} = \frac{1}{N}\sum_i \langle \mathrm{i}\hat{u}_s n_t \rangle_{(i)} \qquad (= 0\ , \quad s \geq t)$$

with $\langle \ldots \rangle_{(i)}$ denoting averages evaluated w.r.t effective single site dynamics at i.

$$\langle \ldots \rangle_{(i)} = \frac{\sum_{\{n_t\}} \int \prod_t \frac{\mathrm{d}\hat{u}_t \mathrm{d}u_t}{2\pi} (\ldots) \exp\big(-\mathcal{S}\big) \prod_t \delta_{n_{t+1}, f_{it}}}{\sum_{\{n_t\}} \int \prod_t \frac{\mathrm{d}\hat{u}_t \mathrm{d}u_t}{2\pi} \exp\big(-\mathcal{S}\big) \prod_t \delta_{n_{t+1}, f_{it}}}$$

- depend on i only through ϑ_i in $f_{it} = n_t + (1 - n_t)\Theta(u_t + \sqrt{\rho}\eta_0 - \vartheta_i)$

- dynamic action

$$\mathcal{S} = \sum_t \left[\frac{1-\rho}{2}\hat{u}_t^2 + \mathrm{i}\hat{u}_t\Big(u_t - J_0 m_t - \alpha J^2 \sum_{s<t} G_{st} n_s\Big)\right] - \frac{J^2}{2}\sum_{st} q_{st}\mathrm{i}\hat{u}_s\mathrm{i}\hat{u}_t$$

- By LLN

$$\frac{1}{N}\sum_i \langle \ldots \rangle_{(i)} \longrightarrow \int \mathrm{d}\vartheta p(\vartheta)\langle \ldots \rangle_{(\vartheta)} \equiv \Big\langle \langle \ldots \rangle_{(\vartheta)} \Big\rangle_\vartheta$$

- System represented by ensemble of effective single site processes characterized by ϑ-distribution, with coloured noise, and memory

$$n_{t+1} = n_t + (1-n_t)\Theta\Big(J_0 m_t + \alpha J^2 \sum_{s<t} G_{st} n_s + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Big)$$

with self-consistency conditions

$$\langle \phi_t \rangle = 0 \quad , \quad \langle \phi_s \phi_t \rangle = (1-\rho)\delta_{st} + J^2 q_{st}$$

$$m_t = \Big\langle \langle n_t \rangle_{(\vartheta)} \Big\rangle_\vartheta \quad , \quad q_{st} = \Big\langle \langle n_s n_t \rangle_{(\vartheta)} \Big\rangle_\vartheta = m_{\min(s,t)} \quad , \quad G_{st} = \left\langle \frac{\mathrm{d}\langle n_t \rangle_{(\vartheta)}}{\mathrm{d}h_s} \right\rangle_\vartheta$$

Simplifications

- In single site dynamics

$$n_{t+1} = n_t + (1 - n_t)\Theta\Big(J_0 m_t + \alpha J^2 \sum_{s<t} G_{st} n_s + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Big)$$

Memory term zero while $n_t = 0$, and irrelevant once $n_t = 1$.

$$\Longrightarrow \quad n_{t+1} = n_t + (1 - n_t)\Theta\Big(J_0 m_t + \sqrt{\rho}\eta_0 - \vartheta + \phi_t\Big)$$

- To find m_{t+1}, need only $\langle \phi_t^2 \rangle = (1 - \rho) + J^2 q_{tt} = (1 - \rho) + J^2 m_t$

- Get

$$m_{t+1} = \Big\langle \langle n_{t+1} \rangle_{(\vartheta)} \Big\rangle_\vartheta = m_t + \left\langle \frac{1 - \langle n_t \rangle_{(\vartheta)}}{2} \left[1 + \text{erf}\left(\frac{J_0 m_t + \sqrt{\rho}\ \eta_0 - \vartheta}{\sqrt{2(1 - \rho + J^2 m_t)}} \right) \right] \right\rangle_\vartheta$$