Systemic Risk and the Mathematics of Falling Dominoes

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King's College London
University of London

NETADIS
Statistical Physics Approaches to Networks Across Disciplines

Outline

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1. The Laws of Falling Dominoes
2. Risk and Falling Dominoes
3. Fundamental Problem of Risk Analysis
   - Main Types of Risk
   - Main Interest and Concern: Interactions
4. Operational Risks — Interacting Processes
   - Dynamics – Mathematics of Falling Dominoes
   - A Simple Homogeneous Process Network
5. Summary

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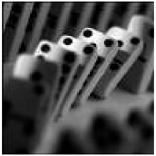

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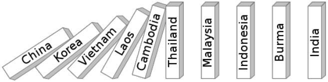
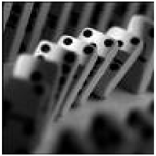


Domino Theory & Spread of Communism



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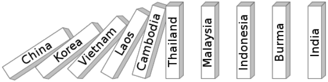

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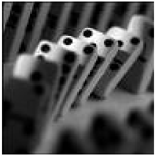

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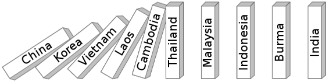

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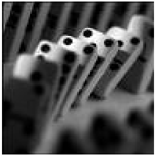

Blackouts in Power Grids

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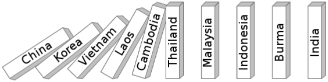

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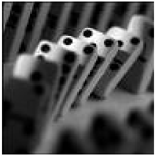

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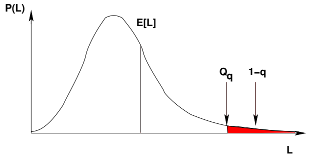
- Popular risk measure:
  Value at Risk

$$\mathrm{VaR}_q = \mathrm{e}^{-rT}\left(Q_q - \mathbb{E}[L]\right)$$

$\Leftrightarrow$ money to set aside now to cover extreme losses at $t = T$.



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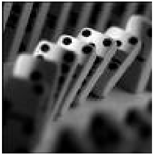

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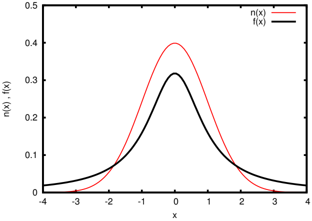

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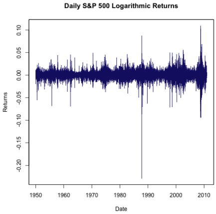
  - Volatility clustering in markets (intermittency)



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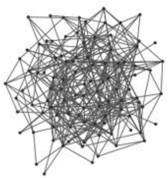

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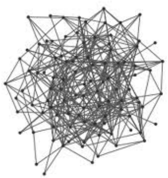

- losses determined (randomly) each time a process goes down

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- Because of the random noise x_{it}, failure is a probabilistic event.

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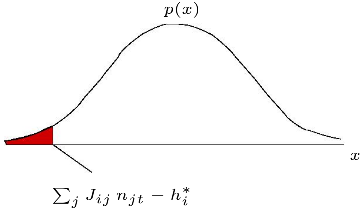

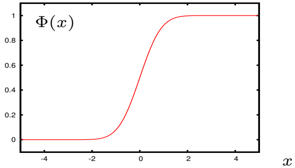

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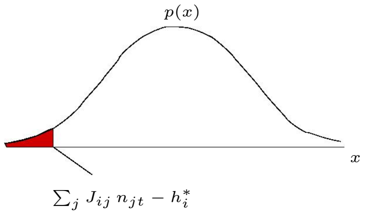

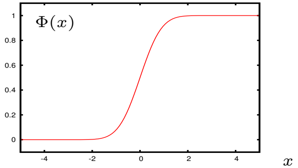

- Unconditional and conditional probability of failure

$$p_i = \Phi(-h_i^*) \quad , \qquad p_{i|k} = \Phi\Big(J_{ik} - h_i^*\Big)$$

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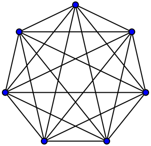

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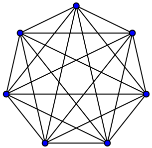

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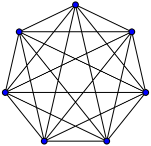

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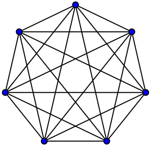

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- Then by Law of Large Numbers (LLN)

$$m_{t+1} = \frac{1}{N} \sum_{i=1}^{N} \Theta\Big(J_0 m_t - h^* - x_{it}\Big) \simeq \Phi\Big(J_0 m_t - h^*\Big)$$

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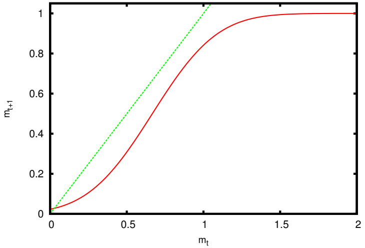

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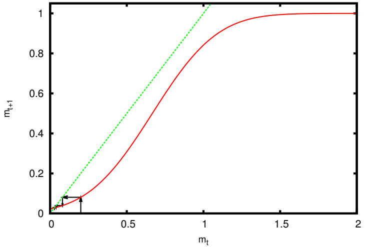

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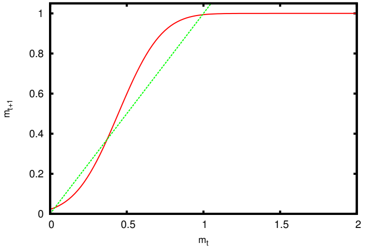

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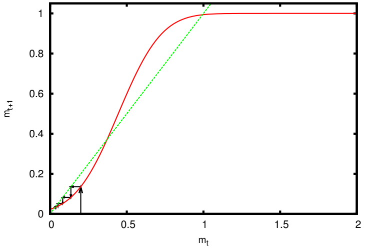
- Analyze the behaviour as a function of the parameters J_0 and h^*



Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and $J_0 = 4.5$

Analysis of the Dynamics

- Iterated function dynamics

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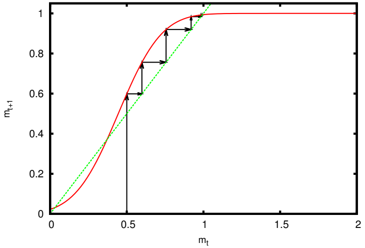

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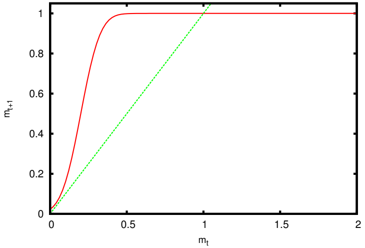

Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and $J_0 = 10$

Analysis of the Dynamics

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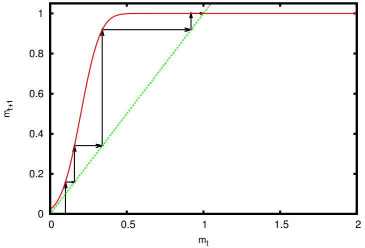

Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and $J_0 = 10$

Analysis of the Dynamics

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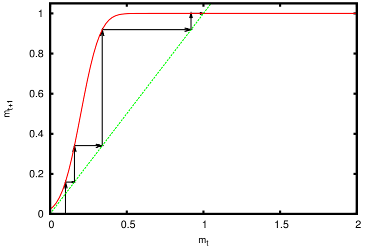
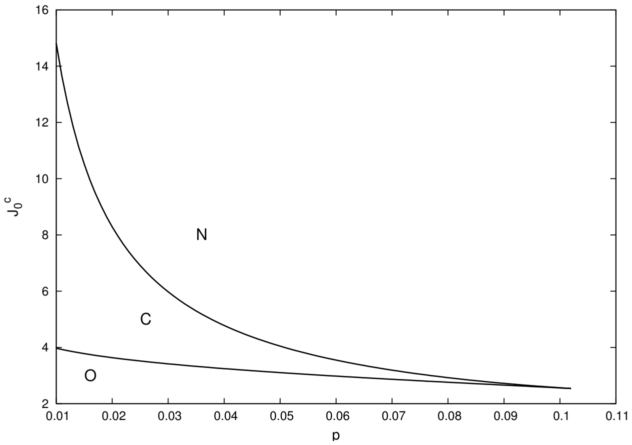


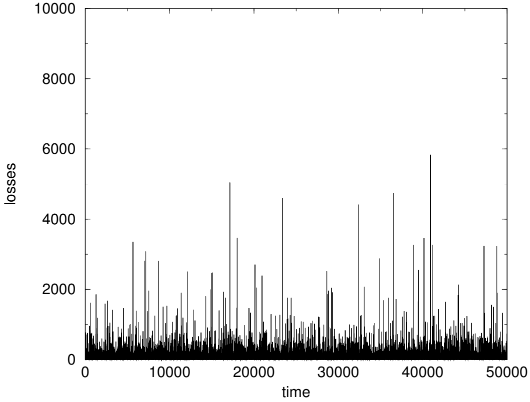
Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and $J_0 = 10$

- By increasing J_0, can change from system with only low-m, via system with coexisting low-m and high-m states, to system with only high-m states.



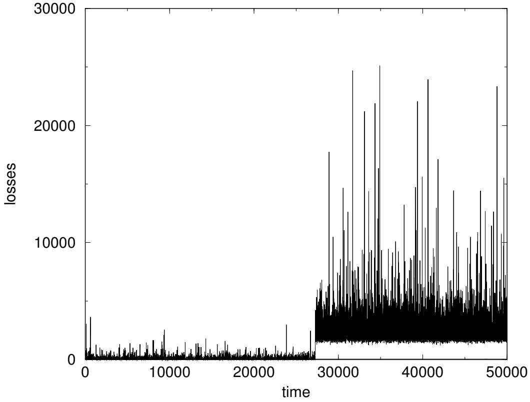
Phase diagram of the OR problem. From K Anand and RK, Phys Rev E**75** (2007)

Spontaneous Breakdown



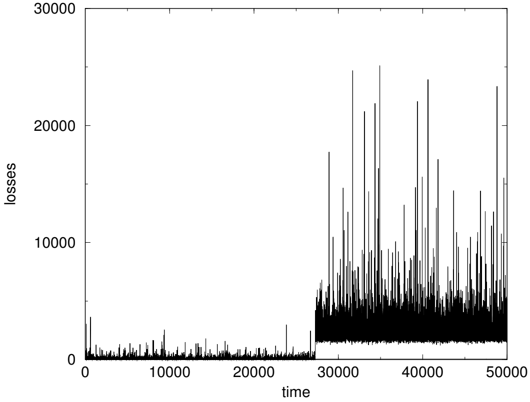
Losses from operational risks in a network of 100 processes: J_0 such that low-m solution is stable

Spontaneous Breakdown



Losses from operational risks in a network of 100 processes: J_0 slightly increased, so low-m solution meta-stable

Spontaneous Breakdown



Losses from operational risks in a network of 100 processes: J_0 slightly increased, so low-m solution meta-stable

- Spontaneous breakdown of meta-stable functioning solution possible in finite systems

Outline

Summary

- Found that process networks can be destabilized by large degrees of interdependency (large J_0) even if all processes are very reliable (large h^*).
- For intermediate levels of dependency (intermediate J_0), functioning and dysfunctional states of the system coexist.
- In systems with finite N, a functioning state can spontaneously switch to the dysfunctional state (without an apparent 'big' perturbation.)
- Results qualitatively unchanged for heterogeneous networks (not all-to-all interactions, heterogeneous levels of reliability, heterogeneous mutual dependency)
- Similar methods for credit risk $\Rightarrow$ ('fat tailed' loss distributions). Crises much more frequent than anticipated if interactions are neglected.
- Not shown: Credit derivatives (CDS) can destabilise a system, if used to expand loan books.
- Can analyze capacity of power-grids (critical loads).

Thank you!

Thank you!

More on this:
`http://www.mth.kcl.ac.uk/~kuehn/riskmodeling`