Systemic Risk and the Mathematics of Falling Dominoes

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Outline

1. The Laws of Falling Dominoes
2. Risk and Falling Dominoes
3. Fundamental Problems of Risk Analysis
   - Main Interest and Concern: Interactions
4. Operational Risks — Interacting Processes
   - Dynamics – Mathematics of Falling Dominoes
   - A Simple Homogeneous Process Network
5. Summary

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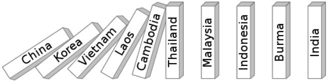
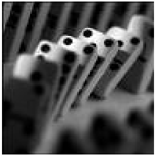
- Avalanches can occur, if dominoes are set too closely.

Risk and Falling Dominoes

Operational Risk



Domino Theory & Spread of Communism



Blackouts in Power Grids

Financial Crisis

Fundamental Problem of Risk Analysis

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- rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity $\leftrightarrow$ liquidity risk

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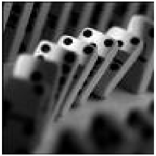

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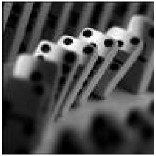

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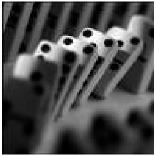
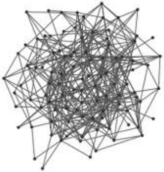
  - Volatility clustering in markets (intermittency)



Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes
- Two state model: processes either up and running ($n_i = 0$) or down ($n_i = 1$)
- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes; connectivity & concept of neighbourhood functionally defined

$\Rightarrow$model defined on random graph



- losses determined (randomly) each time a process goes down

Dynamics – Mathematics of Falling Dominoes

- Processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)
- h_{it} total support received by process i at time t

$$h_{it} = h_i^* - \sum_j J_{ij} n_{jt} + x_{it}$$

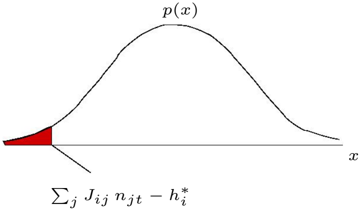
  - h_i^* support in fully functional environment
  - J_{ij} support to process i provided by process j
  - x_{it} random (e.g. Gaussian white noise).
- process i will fail, if the total support for it falls below a critical threshold (if $h_{it} \leq 0$ – domino falls, if kicked too strongly)

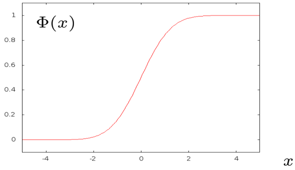
$$n_{it+\Delta t} = \Theta\Big(\sum_j J_{ij} n_{jt} - h_i^* - x_{it} \Big)$$

Probability that a Domino Falls

- Probability of failure/probability of domino falling

$$\text{Prob}\left(n_{it+\Delta t} = 1 \middle| \boldsymbol{n}(t)\right) = \int_{-\infty}^{\sum_j J_{ij}\, n_{jt} - h_i^*} \mathrm{d}x\, p(x) \equiv \Phi\left(\sum_j J_{ij}\, n_{jt} - h_i^*\right)$$





- unconditional and conditional probability of failure

$$\begin{aligned} p_i &= \Phi(-h_i^*) \\ p_{i|j} &= \Phi\left(J_{ij} - h_i^*\right) \end{aligned}$$

A Simple Homogeneous Process Network

- Large systen $1 \leq i \leq N$, $(N \gg 1)$, with all-to-all couplings, and $h_i^* = h^*$ independent of i.

$$J_{ij} = \frac{J_0}{N} \ , \quad \forall \ (i,j) \qquad \Rightarrow \qquad \sum_j J_{ij} n_{jt} = \frac{J_0}{N} \sum_j n_{jt} = J_0 m_t$$

- Dynamics

$$n_{it+\Delta t} = \Theta\Big(\sum_j J_{ij} n_{jt} - h_i^* - x_{it} \Big) = \Theta\Big(J_0 m_t - h^* - x_{it} \Big) \ .$$

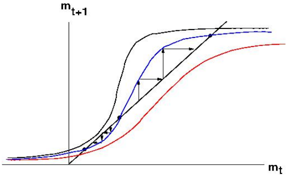
Thus by Law of Large Numbers (LLN)

$$m_{t+\Delta t} = \frac{1}{N} \sum_{i=1}^{N} \Theta\Big(J_0 m_t - h^* - x_{it} \Big) \simeq \Phi\Big(J_0 m_t - h^* \Big)$$

Analysis of the Dynamics

- Iterated function dynamics

$$m_{t+\Delta t} = \Phi\Big(J_0 m_t - h^*\Big)$$



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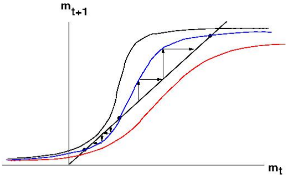

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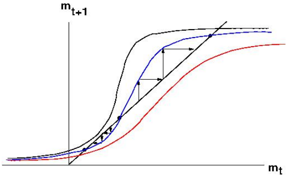

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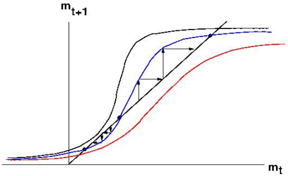

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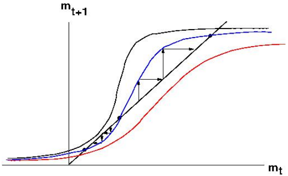

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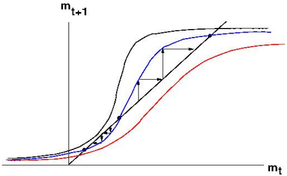

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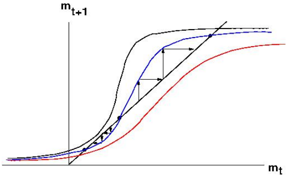

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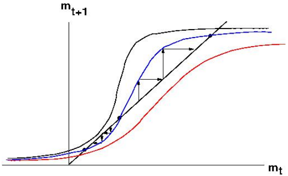

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  - Answer: chain-rule $\frac{\mathrm{d}}{\mathrm{d}x}\Phi(ax) = a\Phi'(ax)$
- For not too small values of h^* can vary between lower and upper curve in above diagram (i.e. between system with only low-m, via system with coexisting low-m and high-m states, to system with only high-m states) by increasing J_0. For small h^* have only high-m state.

Summary

- We found that networks can be destabilized by large degrees of interdependency (large J_0) even if all processes are very reliable (with large h^*).
- For intermediate levels of dependency (intermediate J_0), functioning and dysfunctional states of the system coexist.
- (Not shown): In systems with finite N, a functioning state can spontaneously switch to the dysfunctional state (without an apparent 'big' perturbation.)
- Results qualitatively unchanged for heterogeneous networks (not all-to-all interactions, heterogeneous levels of reliability, heterogeneous mutual dependency)
- Similar methods for credit risk.