

University of London

Spectra of Sparse Random Matrices and Localization on Random Graphs

Reimer Kühn

Department of Mathematics, King's College London

Overview

- Look at spectra of sparse symmetric random matrices
 - Follow replica formulation of Edwards and Jones (76), Rodgers and Bray (88)
 - Use techniques recently developed for StatMech of finitely coordinated random systems
 - Use different representation of replica symmetric ansatz
 - Identify DOS of localized and extended states
 - Deconvolution: local DOS of vertices with different coordination
- Explore for various ensembles
- Some details in
 - RK, J Phys A41, 295002, (2008), cond-mat/0803.2886
 - T. Rogers, I. Perez Castillo, RK, and K. Takeda Phys Rev E 78, 031116 (2008), cond-mat/0803.1553

Spectral Density and Resolvent

• Spectral density of random matrix \boldsymbol{M} from resolvent

$$\overline{\rho(\lambda)} = \lim_{N \to \infty} \frac{1}{\pi N} \operatorname{Im} \operatorname{Tr} \overline{[\lambda_{\varepsilon} I - M]^{-1}}, \qquad \lambda_{\varepsilon} = \lambda - i\varepsilon$$

• **EXPRESS** (S F Edwards & R C Jones, JPA, 1976) **as**

$$\overline{\rho(\lambda)} = \lim_{N \to \infty} \frac{1}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \operatorname{Tr} \overline{\ln [\lambda_{\varepsilon} I - M]}$$
$$= \lim_{N \to \infty} -\frac{2}{\pi N} \operatorname{Im} \frac{\partial}{\partial \lambda} \overline{\ln Z_N} ,$$

where Z_N is a Gaussian integral:

$$Z_N = \int \prod_{i=1}^N \frac{\mathrm{d}u_i}{\sqrt{2\pi/i}} \exp\left\{-\frac{i}{2}\sum_{i,j}u_i(\lambda_\varepsilon\delta_{ij} - M_{ij})u_j\right\}$$

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Sparse Random Matrices

• Sparse symmetric matrix M given, e.g. by

$$M_{ij} = c_{ij} K_{ij}$$

with $\{c_{ij}\}$ adjacency matrix of a random graph. E. g.

$$c_{ij} = \begin{cases} 0 ; \text{with prob} & 1 - \frac{c}{N} \\ 1 ; \text{with prob} & \frac{c}{N} \end{cases}$$

- \equiv Posisssonian (Erdös Renyi) random graph. Others: regular, scale-free, small-world ...
- Distribution of K_{ij} arbitrary (Gaussian, bimodal, non-random ...)
- Exploit StatMech techniques for sparsely coordinated amorphous systems. (RK, J van Mourik, M Weigt, A Zippelius J Phys A,2007)

Performing the Average — Replica Method

• Replica identity

$$\overline{\ln Z_N} = \lim_{n \to 0} \frac{1}{n} \ln \overline{Z_N^n}$$

• For integer n, Z_N^n is partition function of n identical copies of the system (n-th power of Gaussian integral)

$$\overline{Z_N^n} = \int \prod_{ia} \frac{\mathrm{d}u_{ia}}{\sqrt{2\pi/i}} \exp\left\{-\frac{i}{2}\lambda_{\varepsilon}\sum_{i,a}u_{ia}^2 + \frac{c}{2N}\sum_{ij}\left(\left\langle\exp\left(iK\sum_a u_{ia}u_{ja}\right)\right\rangle_K - 1\right)\right\}$$

• Decoupling of sites by introducing the replicated density

$$\rho(\boldsymbol{u}) = \frac{1}{N} \sum_{i} \prod_{a} \delta(u_a - u_{ia})$$

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• Enforce definition via (functional) $\delta\text{-distribution}$

$$1 = \int \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp\left\{-i \int d\boldsymbol{u}\hat{\rho}(\boldsymbol{u}) \left(N\rho(\boldsymbol{u}) - \sum_{i} \prod_{a} \delta(u_{a} - u_{ia})\right)\right\}$$

• Gives

$$\overline{Z_N^n} = \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \exp\left\{N\left[\frac{c}{2}\int d\rho(\boldsymbol{u})d\rho(\boldsymbol{v})\left(\left\langle\exp\left(iK\sum_a u_a v_a\right)\right\rangle_K - 1\right)\right.\\\left. - \int d\boldsymbol{u}\,i\hat{\rho}(\boldsymbol{u})\rho(\boldsymbol{u}) + \ln\int\prod_a \frac{du_a}{\sqrt{2\pi/i}}\exp\left(i\,\hat{\rho}(\boldsymbol{u}) - \frac{i}{2}\,\lambda_{\varepsilon}\sum_a u_a^2\right)\right]\right\}$$

• Evaluation of $N^{-1} \ln \overline{Z_N^n}$ by saddle point method

• Stationarity w.r.t. ρ and $\hat{\rho}$

$$\frac{\delta}{\delta\rho(\boldsymbol{u})}: \quad i\hat{\rho}(\boldsymbol{u}) = c\int d\rho(\boldsymbol{v}) \left(\left\langle \exp\left(iK\sum_{a}u_{a}v_{a}\right)\right\rangle_{K} - 1 \right) \quad (*) \\
\frac{\delta}{\delta\hat{\rho}(\boldsymbol{u})}: \quad \rho(\boldsymbol{u}) = \frac{\exp\left(i\hat{\rho}(\boldsymbol{u}) - \frac{i}{2}\lambda_{\varepsilon}\sum_{a}u_{a}^{2}\right)}{\int d\boldsymbol{u}\exp\left(i\hat{\rho}(\boldsymbol{u}) - \frac{i}{2}\lambda_{\varepsilon}\sum_{a}u_{a}^{2}\right)} \quad (**)$$

• Problem: $n \rightarrow 0$ limit. (GJ Rodgers, AJ Bray, PRB 37, 1988) Ansatz: permutation & rotational symmetry in replica space

$$i\,\widehat{\rho}(\boldsymbol{u}) = cg(|\boldsymbol{u}|)$$

• Exploit to perform 'angular integrals in (*),(**)

• For $K \in \{\pm 1\}$ get

$$g(u) = -u \int_0^\infty dv \exp\left[cg(v) - \frac{i}{2}\lambda_{\varepsilon}v^2\right] J_1(uv)$$
, as $n \to 0$
Independent SuSy derivation (YV Fyodorov, AD Mirlin, JPA 24, 1991

- Rodgers-Bray Equation extremely difficult to analyze.
- Here: different representation of permutation & rotational symmetry. Superpositions of Gaussians:

$$\rho(\boldsymbol{u}) = \int d\pi(\omega) \prod_{a} \frac{\exp\left[-\frac{\omega}{2}u_{a}^{2}\right]}{Z(\omega)}$$
$$i\hat{\rho}(\boldsymbol{u}) = c \int d\hat{\pi}(\hat{\omega}) \prod_{a} \frac{\exp\left[-\frac{\hat{\omega}}{2}u_{a}^{2}\right]}{Z(\hat{\omega})}$$

 \Leftrightarrow solve (*),(**) in terms of an integral transformation

• Get saddle point equations for π and $\hat{\pi}$

Population Dynamics

• Self-consistency equations for π and $\hat{\pi}$: pair of non-linear integral equations

$$\widehat{\pi}(\widehat{\omega}) = \int d\pi(\omega) \left\langle \delta(\widehat{\omega} - \widehat{\Omega}(\omega, K)) \right\rangle_{K}$$

$$\pi(\omega) = \sum_{k \ge 1} \frac{k}{c} p_{c}(k) \int \prod_{\ell=1}^{k-1} d\widehat{\pi}(\widehat{\omega}_{\ell}) \, \delta(\omega - \Omega_{k-1})$$

with

$$\widehat{\Omega}(\omega, K) = \frac{K^2}{\omega}$$
, $\Omega_{k-1} = i\lambda_{\varepsilon} + \sum_{\ell=1}^{k-1} \widehat{\omega}_{\ell}$

• Structure suggests solving via stochastic population based algorithm; note: get complex $\omega, \hat{\omega}$, but $\operatorname{Re}(\omega) \geq 0$, $\operatorname{Re}(\hat{\omega}) \geq 0$ selfconsistently in population.

Spectral Density

• Spectral density from solution (using $\{\hat{\omega}\}_k = \sum_{\ell=1}^k \hat{\omega}_\ell$)

$$\overline{\rho(\lambda)} = \frac{1}{\pi} \operatorname{Im} \sum_{k=0}^{\infty} p_c(k) \int \prod_{\ell=1}^k d\widehat{\pi}(\widehat{\omega}_\ell) \frac{i}{i\lambda_{\varepsilon} + \{\widehat{\omega}\}_k}$$
$$= \frac{1}{\pi} \sum_{k=0}^{\infty} p_c(k) \int \prod_{\ell=1}^k d\widehat{\pi}(\widehat{\omega}_\ell) \frac{\operatorname{Re}\{\widehat{\omega}\}_k + \varepsilon}{(\operatorname{Re}\{\widehat{\omega}\}_k + \varepsilon)^2 + (\lambda + \operatorname{Im}\{\widehat{\omega}\}_k)^2}$$

• Define

$$P(a,b) = \sum_{k} p_c(k) \int \prod_{\ell=1}^{k} \mathrm{d}\hat{\pi}(\hat{\omega}_{\ell}) \,\delta\left(a - \mathrm{Re} \,\{\hat{\omega}\}_k\right) \delta\left(b - \mathrm{Im} \,\{\hat{\omega}\}_k\right) \,,$$

and get

$$\overline{\rho(\lambda)} = \int \frac{\mathrm{d}a \, \mathrm{d}b}{\pi} \, P(a,b) \, \frac{a+\varepsilon}{(a+\varepsilon)^2 + (b+\lambda)^2} \, .$$

• Note: singular nature of integrand for a = 0, as $\varepsilon \to 0$.

Results — **Sampling**

• Integral defining $\overline{\rho(\lambda)}$ is evaluated by sampling from population

$$\overline{\rho(\lambda)} \simeq \frac{1}{\mathcal{N}} \left[\frac{1}{\pi} \sum_{\substack{i=1\\a_i=0}}^{\mathcal{N}} \frac{\varepsilon}{\varepsilon^2 + (b_i + \lambda)^2} + \frac{1}{\pi} \sum_{\substack{i=1\\a_i>0}}^{\mathcal{N}} \frac{a_i + \varepsilon}{(a_i + \varepsilon)^2 + (b_i + \lambda)^2} \right]$$

• Note: $\varepsilon \rightarrow 0$ -limit singular in the first contribution:

 $b_i + \lambda \neq 0 \quad \forall i \quad \text{vs.} \quad b_i + \lambda = 0 \text{ for some } i$.

 Identify with pure point (singular) and continuous spectrum respectively. (R Abou-Chacra, PW Anderson, DJ Thouless, JPC 6, 1973)

Need to keep small $\varepsilon > 0$ (and $\mathcal{N}\varepsilon \gg 1$) to 'see' first contribution.

Results — Poisson Random Graphs



Spectral densities for $\langle K_{ij}^2 \rangle = 1/c$, on Poissonian random graphs with c = 4 (left), and c = 2 (right) using $\varepsilon = 10^{-300}$ (full line); in both panels: numerical diagonalization results for graphs of size N = 2000 (dashed).

More on the Posisson c = 2 case



Upper left: zoom into the central region; upper right: results on logarithmic scale; lower: results regularized at $\varepsilon = 10^{-3}$. In all panels: numerical diagonalization results for graphs of size N = 2000 (dashed). Localization for $|\lambda| > 2.295$! 13/22 Localization — IPRs

$$IPR(v) = \frac{\sum_{i=1}^{N} v_i^4}{(\sum_{i=1}^{N} v_i^2)^2}$$

• IPR = $\mathcal{O}(1)$ for localized, $\mathcal{O}(N^{-1})$ for de-localized states



Continuous and full densities of state, and average IPRs for Poissonian random graphs with c = 2 (left) and c = 4 (right). Average IPRs from numerical diagonalization of matrices with N = 250, N = 500, N = 1000 and N = 2000. Scaling of IPRs confirms location of mobility edges seen in DOS. 14/22

IPRs — Scaling with System Size



Scaling of average IPRs with system size for Poisson Random graphs with c = 2 (upper) and c = 2 (lower). The fraction of sites not in the giant cluster is $x_i \simeq 0.205$ at c = 2 and $x_i \simeq 0.02$ at c = 4.

Orther Ensembles

- Poisson random graphs with bimodal couplings
- Regular random graphs with Gaussian or bimodal couplings (recover Wigner semi-circle law in the $c \gg 1$ limit)
- Scale free graphs (power law degree distribution) — For $p(k) = P_0 k^{-\gamma}$ confirm $\overline{\rho(\lambda)} \sim \lambda^{1-2\gamma}$ at large λ .
- In all cases: localization & mobility edges.

Results — Graph-Laplacians

• Spectra of matrices with row-constraints



Spectral density for the Laplacian on a Poissonian random graph with c = 2 as computed via the present algorithm. Left: $\varepsilon = 10^{-3}$ -results; right: results from numerical diagonalisation of $N \times N$ matrices of the same type with N = 2000.

Continuous Spectrum and 'Low-Energy' Lifshithz Tail



Spectral density for the Laplacian on a Poissonian random graph with c = 2. Left: continuous part of the spectrum obtained using $\varepsilon = 10^{-300}$ as a regularizer. Right: zoom into the small $|\lambda|$ region, exhibiting a mobility edge and a localized DOS ($\varepsilon = 10^{-5}$ and 10^{-6}) compatible with Lifshitz tail behaviour.

Results — Unfolding Spectral Densities



Spectral density for the Laplacian on a Poissonian random graph with c = 2 (full upper line), shown together with its unfolding according to contributions of different coordination. Identifiable humps from left tor right: k = 9, k = 8, ..., k = 3. Several notable humps from k = 2, together with the k = 1 contribution mainly responsible for dip at $\lambda = -1$. The k = 0contribution is mainly responsible for the δ -peak at $\lambda = 0$

Results — Random Schrödinger Operators

• Spectral properties of discrete random Schrödinger operator

$$H = -\Delta + V$$
, $V_{ij} = v_i \delta_{ij}$, $v_i \in [-W, W]$



RSO on a Poissonian random graph with c = 4, and W = 1. Left: Spectral density and IPRs (N = 250, 500, 1000, and 2000. Right: Continuous DOS and its unfolding (k = 1, ..., 13)



RSO on a Poissonian random graph with c = 4, and W = 1. DOS and its unfolding (k = 0, ...5).

Summary

- Computed DOS of sparse random matrices using replica. For single instances see T. Rogers, et al PRE (2008).
- Techniques, ansätze etc inspired by previous work on Stat Mech of heterogeneous systems.
- Allows to disentangle pure point and continuous spectrum.
- Allows to compute local DOS unfolded according to coordination.
- Method is versatile (Poissonian and other degree distributions); Laplacians; discrete random Schrödinger operators; Anderson localisation.
- To do: asymmetric matrices (Rogers, Anand); modular& small world systems; eigenvector distributions, spectral correlations . . .