Letter

Level-set percolation of Gaussian random fields on complex networks

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We provide an explicit solution of the problem of level-set percolation for multivariate Gaussians defined in terms of weighted graph Laplacians on complex networks. The solution requires an analysis of the heterogeneous microstructure of the percolation problem, i.e., a self-consistent determination of locally varying percolation probabilities. This is achieved using a cavity or message passing approach. It can be evaluated, both for single large instances of locally treelike graphs, and in the thermodynamic limit of random graphs of finite mean degree in the configuration model class.

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Introduction. At its core, percolation describes a geometric phase transition, at which as a function of the relative density of existing links in a lattice or a graph—either by construction or after random removal of a subset of edges or vertices—the system either decomposes into a collection of finite clusters of contiguously connected vertices, or on the contrary exhibits a so-called giant connected component (GCC) that occupies a finite fraction of vertices in the large system limit [1]. Apart from an intrinsic interest in percolation transitions and the critical singularities associated with them, they are of relevance in many other contexts. For example, below the percolation threshold, where a system consists of a collection of finite isolated clusters, diffusive or hopping transport via edges of a graph is clearly impossible, and such systems would therefore be insulators. In a different context, the size of a susceptible-infected-recovered (SIR) epidemic can be mapped on the size of its GCC in a percolation problem where edge retention probabilities are given by probabilities of transmitting a disease before recovery [2–5]. This connection has been exploited to formulate effective vaccination strategies for diseases spreading through contact networks (e.g., Refs. [6–8]). More generally, percolation has been studied to assess the robustness of complex networks, both natural and artificial, against random failures of their components or against targeted attacks (e.g., Refs. [9–11]). At a more fundamental level, a nonpercolating system cannot support stable phases with spontaneous macroscopic long-range order at any nonzero temperature (see, for example, Ref. [12]), and it has been argued that gene regulatory networks would for the same reasons not be able to support multicellular life, if they were composed of small independent clusters of interacting genes [13,14].

There is a version of the percolation problem which is of a radically different nature than the case of independent Bernoulli percolation just described. It is concerned with the distribution of sizes of contiguous clusters, over which a random field exceeds a given level h. It can be brought to bear on the analysis of electron localization in amorphous solids [15,16], or of statistical properties of landscape topography [16,17], and it could be of use to assess risk of damage due to randomly varying levels of contaminants. It has been studied over continuous spaces ($\mathbb{R}^d$) [18–24], over lattices ($\mathbb{Z}^d$) [23,25–28], or over random graphs [29–36]. Because of correlations between values of a random field at different points in space the problem is *much* harder than that of independent Bernoulli percolation. In $\mathbb{Z}^d$, though not in the case studied in the present Letter, those correlations are in fact even *long range*. Early on, Molchanov and Stepanov [18,25] disproved the naive intuition according to which there would *always* exist a finite critical level h_c, above which all contiguous clusters for which the random field exceeds the level $h > h_c$ would be bounded, whereas for $h < h_c$ there would be an extensive giant connected component (referred to as hGCC in what follows) over which the random field exceeds the level h. While a range of important results about level-set percolation have been obtained over the years, including existence (e.g., Refs. [18,19,23–27,29]) and sharpness [23] of the percolation transition, uniqueness (e.g., Ref. [31]) and extensivity of the hGCC (e.g., Refs. [33,35,36]) in the percolating phase, as well as critical exponents of the transition on transient graphs (e.g., Refs. [34,36]), significant gaps remain. For example, despite recent intense activity and considerable progress in the study of level-set percolation on random graphs [29–36], explicit solutions of this problem that go beyond a characterization of the near critical regime on regular trees do to the best our knowledge still elude us, in contrast to the case of Bernoulli percolation on random graphs, for which full solutions are meanwhile textbook material (e.g., Refs. [37–39]). The purpose of the present Letter is to fill this gap.

Level-set percolation for Gaussian free fields on random graphs. We consider a (random) graph of N vertices on which a multivariate Gaussian field is defined via

$$P(\boldsymbol{x}) = \frac{1}{Z}\exp\left(-\frac{1}{2}\mu\sum_i x_i^2 - \frac{1}{4}\sum_{i,j}K_{ij}(x_i - x_j)^2\right), \quad (1)$$

with $\mu \geqslant 0$ and $K_{ij} = K_{ji} > 0$ for vertices of the network connected by an edge, and zero otherwise. In Eq. (1), the first sum in the exponential is over all N vertices of the graph, while in the double sum, both sums are over all N vertices of the graph. That double sum is indeed a quadratic form of a weighted graph Laplacian $\Delta^{(K)}$ with edge weights $\{K_{ij}\}$, i.e.,

$$\frac{1}{4}\sum_{i,j} K_{ij}(x_i - x_j)^2 = -\frac{1}{2}\sum_{i,j} \Delta^{(K)}_{ij} x_i x_j. \tag{2}$$

The positivity of the nonzero edge weights is needed to keep the Gaussian normalizable also in the $\mu = 0$ case, in which the field is referred to as massless. For the analysis of level-set percolation on random graphs, it turns out to be essential to be able to characterize its heterogeneous microstructure, i.e., the node-dependent probabilities of vertices in the graph to belong to the hGCC. This can be done by adapting an approach developed in Refs. [40,41]. It is based on cavity or message passing ideas specifically designed to analyze problems on locally treelike graphs. As we are only interested in heterogeneous percolation probabilities, a somewhat simpler version outlined, e.g., in Refs. [42,43], can be used.

For a node i to belong to the hGCC, the Gaussian field at i must itself exceed the specified level, i.e., $x_i \geqslant h$, *and* it must be connected to the hGCC via at least one of its neighbors. Introducing indicator variables $n_i \in \{0, 1\}$ which signify whether i is ($n_i = 1$) or is not ($n_i = 0$) in the hGCC, we require

$$n_i = \chi_{\{x_i \geqslant h\}}\left[1 - \prod_{j\in\partial i}\left(1 - \chi_{\{x_j\geqslant h\}} n_j^{(i)}\right)\right], \tag{3}$$

in which the characteristic function $\chi_{\{x_i\geqslant h\}}$ expresses the fact that the Gaussian field at i must itself exceed the specified level, i.e., $x_i \geqslant h$, while the second factor expresses the fact that i is connected to the hGGC via at least one neighbor. This requires that for at least one $j \in \partial_i$ the Gaussian field must exceed the specified level ($x_j \geqslant h$), *and* it must be connected to the hGCC via one of *its* neighbors other than i (on the cavity graph $G^{(i)}$ from which i and the edges connected to it are removed); this is expressed by the cavity indicator variable $n_j^{(i)}$ taking the value $n_j^{(i)} = 1$.

For the cavity indicator variable $n_j^{(i)}$ to take the value 1, it is required that on $G^{(i)}$ the node j is itself connected to the hGCC via at least one of its neighbors other than i. This entails that

$$n_j^{(i)} = 1 - \prod_{\ell\in\partial j\setminus i}\left(1 - \chi_{\{x_\ell\geqslant h\}} n_\ell^{(j)}\right). \tag{4}$$

Averaging Eqs. (3) and (4) over possible realizations of the Gaussian field $\boldsymbol{x}$ with the joint probability density function (PDF) described by Eq. (1) is facilitated by the fact that—conditioned on x_i—the averages over the $(\chi_{\{x_j\geqslant h\}} n_j^{(i)})_{j\in\partial i}$ in Eqs. (3) factor in j, if the graph in question is a tree, and that such factorization becomes asymptotically exact on locally treelike graphs in the thermodynamic limit. Analogous factorization is possible for averages over the $(\chi_{\{x_\ell\geqslant h\}} n_\ell^{(j)})_{\ell\in\partial j\setminus i}$ in Eqs. (4), when conditioned on x_j.

Performing the average over the Gaussian field $\boldsymbol{x}$ in this way, we obtain $g_i = \mathbb{E}_{\boldsymbol{x}}[n_i] = \mathbb{E}_{x_i}[\mathbb{E}_{\boldsymbol{x}}[n_i|x_i]]$ from Eq. (3) as

$$g_i = \mathbb{E}_{x_i}\left[\chi_{\{x_i\geqslant h\}}\left(1 - \prod_{j\in\partial i}\left(1 - \mathbb{E}_{\boldsymbol{x}}\left[\chi_{\{x_j\geqslant h\}} n_j^{(i)}\middle| x_i\right]\right)\right)\right] \tag{5}$$

by factorization of conditional expectations. Here, we use the notation $\mathbb{E}_{\boldsymbol{x}}[\cdot]$ to denote an expectation evaluated over the multivariate Gaussian defined on the graph, and $\mathbb{E}_{x_i}[\cdot]$ to denote an expectation evaluated over the single node marginal of x_i, while $\mathbb{E}_{\boldsymbol{x}}[\cdot|\{\star\}]$ and $\mathbb{E}_{x_i}[\cdot|\{\star\}]$ denote analogous conditional expectations, conditioned with respect to the event $\{\star\}$ Then, using further conditioning, we have

$$\begin{aligned}\mathbb{E}_{\boldsymbol{x}}\left[\chi_{\{x_j\geqslant h\}} n_j^{(i)}\middle| x_i\right] &= \mathbb{E}_{x_j}\left[\chi_{\{x_j\geqslant h\}}|x_i\right]\mathbb{E}_{\boldsymbol{x}}\left[n_j^{(i)}\middle|\{x_j\geqslant h\}, x_i\right]\\ &= H_j(h|x_i) g_j^{(i)},\end{aligned} \tag{6}$$

where we have introduced

$$H_j(h|x_i) = \mathbb{E}_{x_j}\left[\chi_{\{x_j\geqslant h\}}\middle| x_i\right] \tag{7}$$

and

$$g_j^{(i)} = \mathbb{E}_{\boldsymbol{x}}\left[n_j^{(i)}\middle|\{x_j\geqslant h\}\right]. \tag{8}$$

In Eq. (8) we have used the fact that for conditional expectations of observables such as $n_j^{(i)}$ pertaining to the cavity graph $\mathbb{E}_{\boldsymbol{x}}[n_j^{(i)}|\{x_j\geqslant h\}, x_i] = \mathbb{E}_{\boldsymbol{x}}[n_j^{(i)}|\{x_j\geqslant h\}]$. Putting things together, we have

$$g_i = \rho_i^h\left(1 - \mathbb{E}_{x_i}\left[\prod_{j\in\partial i}\left(1 - H_j(h|x_i) g_j^{(i)}\right)\middle|\{x_i\geqslant h\}\right]\right), \tag{9}$$

with $\rho_i^h = \mathbb{E}_{x_i}[\chi_{\{x_i\geqslant h\}}]$.

Following an entirely analogous line of reasoning and using the same sequence of conditionings, we can evaluate the $g_j^{(i)}$ defined in Eq. (8) by evaluating the conditional average of $n_j^{(i)}$ using Eq. (4), giving

$$g_j^{(i)} = 1 - \mathbb{E}_{x_j}\left[\prod_{\ell\in\partial j\setminus i}\left(1 - H_\ell(h|x_j) g_\ell^{(j)}\right)\middle|\{x_j\geqslant h\}\right]. \tag{10}$$

Equations (9) and (10) for g_i and $g_j^{(i)}$ can be evaluated, once single-site marginals and joint densities on adjacent sites of the Gaussian field defined by Eq. (1) are known; the latter are required to evaluate the conditional probabilities $H_j(h|x_i)$ defined in (7) [and similarly the $H_\ell(h|x_j)$ appearing in Eq. (10)], while the former are required to evaluate x_i expectations in Eqs. (9) and x_j expectations in Eqs. (10), respectively. They are obtained by their own cavity-type analysis, which has in fact been performed in Ref. [44] for single-site marginals of harmonically coupled systems on random graphs, and in Refs. [45,46] in the context of the spectral problem of sparse symmetric random matrices. All that is needed are the (Gaussian) single-site marginals $P_i(x_i)$ of $P(\boldsymbol{x})$, as well as the corresponding single-site cavity marginals $P_j^{(i)}(x_j)$ for $j \in \partial i$ on the cavity graph $G^{(i)}$, in terms of which joint densities on adjacent sites are easily obtained. Key identities needed in the analysis are reproduced in the Supplemental Material

[47]. Single-site marginals and single-node cavity marginals are fully characterized by their inverse variances (or precisions) ω_i and $\omega_j^{(i)}$, respectively. The latter are obtained by solving the system (4) of cavity self-consistency equations in the Supplemental Material [47]. $H_j(h|x_i)$ and the $H_\ell(h|x_j)$ can be expressed in closed form in terms of error functions, but the conditional x_i expectation of the product in Eq. (9) and similarly the conditional x_j expectation of the product in Eq. (10) will have to be evaluated numerically.

With all ingredients fully defined, Eqs. (10) constitute a set of coupled self-consistency equations for the $g_j^{(i)}$. They can be solved iteratively at given level h on large instances of locally treelike (random) graphs, starting from random initial conditions. Using the solutions, one obtains the node-dependent percolation probabilities g_i from Eqs. (9).

The value of the percolation threshold follows from a linear stability analysis of Eqs. (10). They are *always* solved by $g_j^{(i)} \equiv 0$. This solution becomes unstable, indicating the percolation transition, where the largest eigenvalue of the Hessian of the right-hand side of Eq. (10) evaluated at $g_j^{(i)} \equiv 0$ exceeds 1. The Hessian is a weighted version of a so-called nonbacktracking matrix, with nonzero elements

$$B_{(ij),(j\ell)} = \mathbb{E}_{x_j}[H_\ell(h|x_j)|\{x_j \geqslant h\}] \tag{11}$$

for $j \in \partial i$ and $\ell \in \partial j \setminus i$, and $B_{(ij),(k\ell)} = 0$ otherwise. Performing an appropriately adapted weakly nonlinear expansion of Eqs. (10) as in Ref. [41], one obtains site-dependent percolation probabilities to linear order in $h_c - h$ as

$$g_i \simeq \alpha(h_c - h) \sum_{j\in\partial i} v_j^{(i)}, \tag{12}$$

where $\boldsymbol{v} = (v_j^{(i)})$ is the Frobenius right eigenvector of corresponding to the largest eigenvalue $\lambda_{\max}(B)|_{h=h_c} = 1$ of the nonbacktracking matrix (11) evaluated at h_c, normalized such that $||v||_1 = 1$, and α is an amplitude given in Eq. (24) of the Supplemental Material, which also includes a derivation of the $O[(h_c - h)^2]$ contribution to g_i [47].

Thermodynamic limit. For random graphs in the configuration model class, i.e., graphs that are maximally random subject to a given degree distribution $p_k = \text{Prob}(k_i = k)$, one can analyze the level-set percolation problem in the thermodynamic limit of infinite system size. Assuming that a limiting probability law for the joint distribution of the cavity probabilities $g_j^{(i)}$ and the cavity precisions $\omega_j^{(i)}$ exists, probabilistic self-consistency compatible with the self-consistency equations (10) for the $g_j^{(i)}$ and with Eqs. (4) of the Supplemental Material [47] for the $\omega_j^{(i)}$ entails a self-consistency equation for the joint PDF $\tilde{\pi}(\tilde{g}, \tilde{\omega})$, which is documented as Eq. (13) of the Supplemental Material. It is efficiently solved by a population dynamics algorithm. The limiting PDF of local percolation probabilities is then evaluated from its solution.

In Fig. 1, we present an example of a distribution of level-set percolation probabilities for a system with a fat-tailed degree distribution, which shows that the theoretical analyses agree very well with a numerical simulation. Simulations are, of course, affected by sampling fluctuations and by finite-size effects (creating details depending on the specific single realization of the generated random graph), while the single instance cavity analysis is only affected by finite-size effects. Further results, both for different systems and a range of values for the level h, can be found in the Supplemental Material [47]. Remarkably, as also documented in the Supplemental Material [47], for a massless Gaussian field the marginal node-dependent precisions ω_i turn out to be a very precise, although not exact, predictor for the node-dependent percolation probabilities $g_i = g_i(h)$ at a given level h, which appears to be independent of the graph type. This is particularly interesting as g_i are *much* harder to evaluate than ω_i. However, that almost perfect correlation is lost for fields with nonzero mass $\mu > 0$. Results for systems with nonuniform edge weights K_{ij} will be presented and discussed in an extended version of this Letter.

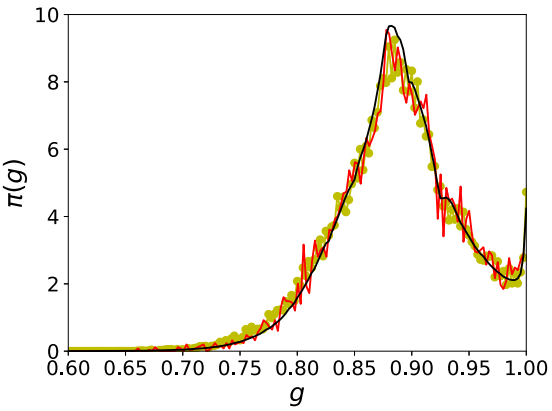

FIG. 1. Distribution $\pi(g)$ or local percolation probabilities for a random graph with power-law degree distribution $k \sim k^{-3}$ for $2 \leqslant k \leqslant 100$ at $h = -1$ and $\mu = 0.1$. We compare (i) results of a numerical simulation of a single instance of a graph of $N = 50\,000$ vertices, averaging over 5000 realizations of Gaussian field configurations to obtain the PDF of the g_i (yellow dots), with (ii) results of a single instance cavity analysis for the same graph (red solid line), and (iii) the result of an analysis in the thermodynamic limit (black solid line).

Random regular graphs. Specializing to random regular graphs (RRGs) with uniform couplings, more explicit results can be obtained. The key observation is that in the thermodynamic limit all nodes and all edges of the system are equivalent. Hence the self-consistency equation for the uniform cavity precisions on a RRG of degree c (or cRRG) reads

$$\tilde{\omega} = \mu + (c-1)\frac{K\tilde{\omega}}{K+\tilde{\omega}}. \tag{13}$$

This equation is solved by

$$\tilde{\omega}_\pm = \tfrac{1}{2}[\mu + K(c-2) \pm \sqrt{[\mu + K(c-2)]^2 + 4K\mu}], \tag{14}$$

the relevant (physical) solution being $\tilde{\omega} = \tilde{\omega}_+$. This entails a self-consistency equation for the uniform cavity percolation probabilities $g_j^{(i)} \equiv \tilde{g}$ of the form

$$\tilde{g} = 1 - \mathbb{E}_x[(1 - H(h|x)\tilde{g})^{c-1}|\{x \geqslant h\}], \tag{15}$$

in which $\mathbb{E}_x[\cdot]$ denotes an expectation over the generic single-node marginal of the RRG, for which $x \sim \mathcal{N}(0, 1/\omega)$, with

$$\omega = \mu + c\frac{K\tilde{\omega}}{K+\tilde{\omega}} \tag{16}$$

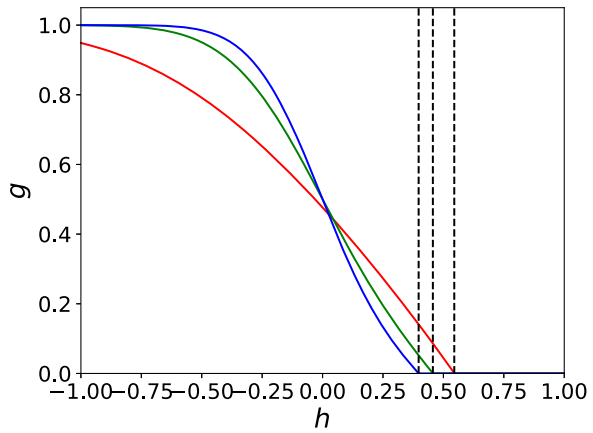

FIG. 2. Percolation probability g as a function of the level h for cRRGs with $c = 4$, 12, and 20. The steepness of the curves increases with c. Critical levels h_c as obtained from Eq. (17) are indicated as vertical dashed lines. For the three values of c shown here, they decrease with increasing c, and they agree perfectly with results of a numerical solution of Eq. (15).

the uniform single-node precision on the cRRG, while $H(h|x)$ is a conditional expectation of the type defined in Eq. (7), evaluated on the cRRG. Equation (15) is a simple scalar equation for $\tilde{g}$ which is easily solved numerically. It always has the trivial solution $\tilde{g} = 0$, which becomes unstable below a critical value h_c of the level h which follows from a linear stability analysis of Eq. (15) and is given as the solution of

$$(c-1)\mathbb{E}_x[H(h|x)|\{x \geqslant h\}] = 1. \tag{17}$$

From the solution of Eq. (15) at $h < h_c$ one obtains

$$g = \rho^h(1 - \mathbb{E}_x[(1 - H(h|x)\tilde{g})^c|\{x \geqslant h\}]), \tag{18}$$

with $\rho^h = \mathbb{E}_x[\chi_{\{x\geqslant h\}}]$ as the value of the percolation probability at level h. Percolation probabilities thus computed as functions of the level h are shown in Fig. 2 for cRRGs with uniform $K = 1$ and $\mu = 0$ for three different values of $c \geqslant 4$. For the range of c values shown, critical percolation thresholds h_c are decreasing with increasing c, but, as shown in Fig. 3 of the Supplemental Material, there is nonmonotonicity of h_c as a function of c in the range $c \in \{3, 4, 5\}$.

Summary and discussion. In this Letter we presented an explicit solution of the problem of level-set percolation of Gaussian free fields on locally treelike random graphs, both for finite large instances and in the thermodynamic limit of infinite system size for random graphs in the configuration model class with finite mean degree. The former requires the simultaneous solution of a set self-consistency equations for locally varying single-node cavity percolation probabilities $g_j^{(i)}$ and for the locally varying single-node cavity precisions $\omega_j^{(i)}$. The latter instead requires solving a nonlinear integral equation for their joint PDF $\tilde{\pi}(\tilde{g}, \tilde{\omega})$. Though we have restricted ourselves to a uniform mass parameter μ, such a restriction is not a matter of principle and can easily be relaxed. We found our results to be in excellent agreement with simulations. Simplifications are possible in the case of RRGs for which the uniform single-node cavity percolation probabilities $g_j^{(i)} \equiv \tilde{g}$ are obtained as solutions of a single scalar equation, from which in turn the uniform percolation probability g is easily evaluated using a single scalar equation. While our methods are nonrigorous, they are expected to be exact in the large system limit, a fact that should be amenable to rigorous proof.

The methods and heuristics used in the present Letter could be useful for the analysis of wider classes of level-set percolation problems. For instance, it is relatively straightforward to reformulate our methods to analyze level-set percolation of local fields for disordered Ising models defined on random graphs and thereby to level-set percolation for single-node magnetizations. Generalizing our methods to genuinely continuous multivariate *non-Gaussian* distributions, while in principle in reach of our methods, is *significantly harder* as it requires to replace the self-consistency equations for single-node cavity precisions by self-consistency equations for *distributions of entire effective single-node cavity potentials* (see Ref. [44]). It would also be interesting to investigate whether the approach of Ref. [41], which is capable of giving distributions of the sizes of *finite* clusters, both in the nonpercolating and in the percolating phase can be carried over to the present case of level-set percolation. Another, as yet unsolved, problem concerns the stability analysis of the integral equation (13) of the Supplemental Material, which could in principle allow one to obtain critical percolation levels h_c for configuration model networks directly in the thermodynamic limit. We hope to address some of these open problems in the near future.

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