Contagion in an interacting economy

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Abstract

We investigate the credit risk model defined in Hatchett and Kühn [18] under more general assumptions, in particular using a general degree distribution for sparse graphs. Expanding upon earlier results, we show that the model is exactly solvable in the $N \to \infty$ limit and demonstrate that the exact solution is described by the message-passing approach outlined in Karrer and Newman [21], generalized to include heterogeneous agents and couplings. We provide comparisons with simulations in the case of a scale-free graph.

1 Introduction

Modern economies form complex, heavily interconnected ecosystems perhaps best epitomized by the extraordinary intricacies of supply chains, routinely involving hundreds of suppliers in dozens of countries. But as highlighted by Haldane and May [17], complexity is associated with greater systemic instability, and the crisis of 2007-2009 made it clear that proper analysis of systemic risk is needed. Accordingly, financial contagion and credit-risk modelling is a long-standing research subject [26] that has recently attracted renewed interest [16, 14, 20, 8, 25].

Credit events cluster in times of economic stress, resulting in large aggregate loss that are not captured by the risk rating (e.g. S&P) of individual institutions. As a result, attempts at regulatory controls in the spirit of the Basel II capital requirements should take into account the possibility of mutually dependent defaults in order to adequately model risk of large portfolios. Historical approaches in financial risk analysis include replacing the number of firms in a portfolio by a reduced effective number, or to condition the default probability on macro-economic indicators [22, 12]. Multi-factor Merton models correlate defaults by assuming that asset returns of different firms undergo correlated random walks [26, 15]. But while these models do take into account correlations, they are not causal and can thus fail to capture systemic fragility, i.e. the effect of the collapse of a single entity or group of entities on the entire network.

The physical perspective spurred by the development of econophysics is somewhat different, focusing more on system-wide risk through interactions between

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agents than on individual risks [13, 6]. In particular, there has been in the past decade an intensive research effort on the network structure of social and economic interactions, e.g. the structure of sexual contacts, of academic citations or of the internet [23, 31, 1, 11], showing that the number of neighbors or partners of an individual (or node) in most social networks follows a power-law distribution. This is sometimes explained as the result of a preferential attachment in the network formation process, as in the Barabasi-Albert model [4].

In this context, systemic risk is associated with contagious processes on the graph defined by economic interactions: given such processes, one is interested in the fraction of the network likely to be affected over a certain time horizon. Such an approach has been used for example by Gai and Kapadia [16, 7], modelling the banking network as a directed weighted network of exposures in which banks fail if they are overexposed to failed banks. Caccioli et al. [8] use a representation of the banking system is a bipartite network of assets and banks, while Hatchett and Kühn [18] investigate contatgion in networks of firms via general economic interactions which include but are not necessarily restricted to financial exposures. Contagion on networks, while a relatively heterogeneous set of phenomena, has nevertheless been largely successfully analyzed using tools from statistical mechanics and computer science such as message-passing [27, 28]. In the context of infection dynamics for example, Altarelli et al. [2] evaluate the efficiency of targeted immunization strategies; Moore and Newman [30] derive analytic solution for the susceptibility of a network to SIR-type epidemics.

We focus in this paper on the model developed in Hatchett and Kühn [18], which investigates how networks of economic interactions affect the system-wide default likelihood across economic cycles. That investigation provided an analytic solution of the model in the limit of "dilute yet large" connectivity, i.e. for networks where the typical number c of neighbors is large ($c \to \infty$), but small compared to the size of the system ($\frac{c}{N} \to 0$). Moreover, the analysis was limited to Erdös-Rényi random graphs rather than more realistic degree distribution.

The model has several attractive features: contagion dynamics is not exclusively driven by an initial shock. It reveals that systemic risk, while clearly dependent on system-wide distributions of exposures or connectivities, will be relatively insensitive to *individual* dependencies. The model provides a clear mechanism for default clustering, and (as we shall show in what follows) it is analytically tractable for a larger class of specifications than originally considered in [18]. Moreover it allows to recover other models such as the Centola-Macy [9] or Watts [32] models by taking adequate limits, making it a valuable toy model. Many of its parameters can also be inferred in principle by suitable rating procedures. Unlike other approaches however it does not look into the 'micro-structure' of contagion as generated e.g. by overlapping portfolios which are the main focus of Caccioli et al. [8], or by similarities in trading strategies. We believe, however, that it can be straightforwardly generalized to include such effects at least on a qualitative level.

The remainder of the paper is organized as follows: in section 2, we define our model and set up the formalism to analyse its dynamics. In section 3, we provide an analytic solution in the case where the underlying network is a tree. In the appendix we use a generating function analysis to demonstrate that this solution is the correct infinite-network limit for a configuration model random graph. In section 4, we provide numerical results, and summarize our findings in section 5.

2 Model definitions

Our model consists of weighted graph $G = (V, E)$ with edge weights (w_{ij}, w_{ji}) drawn according to some distribution $p_w(\{w_{ij}\})$. To each nodes we associate a threshold θ_i and a binary variable $n_{i,t} \in \{0, 1\}$ signifying whether the node is active ($n_{i,t} = 0$) or defaulted ($n_{i,t} = 1$).

In the context of modelling credit contagion, the threshold θ_i corresponds to the wealth-position of node i at the start of a risk-horizon, while a weight w_{ij} describes the impact of a default of node j on the wealth position of i [18]. The $n_{i,t}$ evolve according to the dynamical rule

$$n_{i,t+1} = n_{i,t} + (1 - n_{i,t})\Theta\left(\sum_j a_{ij} w_{ij} n_{j,t} - \eta_{i,t} - \theta_i\right) \tag{1}$$

where the noise variable $\eta_{i,t}$ has the form

$$\eta_{i,t} = \xi_{0,t} + \xi_{i,t} \ . \tag{2}$$

The ξ_{0t} are random variables corresponding to the correlation of the noise across the entire economy and thus represent global economic conditions, while the $\{\xi_{i,t}\}_{i\in G}$ are i.i.d. random variables (whose distribution may vary with time) corresponding to idiosyncratic noise. This decomposition of the noise corresponds to the minimal recommendations of the Basel II Accords [5]. $\mathbf{A} = \{a_{ij}\}$ is the adjacency matrix of the graph.

Written in this way the dynamics have $n_i = 1$ as an absorbing state: if $n_{i,t_0} = 1$ then for all $t > t_0$ we have $n_{i,t} = 1$. Our aim in this paper is to compute the fraction of defaulted nodes at a finite time horizon T, i.e to compute

$$m(T) = \frac{1}{N}\sum_i n_{i,T} \tag{3}$$

which, as in previous work [18], we will call the defaulted fraction.

Give the way the model is set up, the trajectories $\{n_{i,t}\}$ follow a Markovian dynamics: their joint probability factorizes as

$$P\left(\{n_{i,t}\}_{t=0,\cdots,T;\ i\in G}\right) = P(\{n_{i,0}\})\prod_{t=1}^{T} P\left(\{n_{i,t}\}_{i\in G} \,|\, \{n_{i,t-1}\}_{i\in G}\right) \ .$$

As the $\xi_{i,t}$ are independent, the transition probabilities $P(\{n_{i,t}\} \,|\, \{n_{i,t-1}\})$ factorize over the sites as

$$P(\{n_{i,t}\} \,|\, \{n_{i,t-1}\}) = \prod_i p(n_{i,t}|n_{i,t-1}, \{n_{j,t-1}\}_{j\in\partial_i}) \ ,$$

in which ∂i denotes the set of neighbors of node i. As long as a node i is active at time $t-1$, its transition probability $p(n_{i,t}|n_{i,t-1} = 0, \{n_{j,t-1}\}_{j\in\partial_i})$ depends on its local field

$$h_{i,t-1} = \sum_j a_{ij} w_{ij} n_{j,t-1} = \sum_{j\in\partial i} w_{ij} n_{j,t-1}$$

and on the threshold θ_i, i.e., it will be of the form

$$p(n_{i,t} = 1|n_{i,t-1} = 0, \{n_{j,t-1}\}_{j\in\partial i}) = W_{t-1}(h_{i,t-1} - \theta_i)$$

where W_{t-1} is a function whose exact form depends on the distribution of the noise $\eta_{i,t-1}$ at time $t-1$. Conversely, for nodes that are defaulted at time $t-1$, the single-site transition probabilities are independent of their local fields and are simply given by

$$p(n_{i,t}|n_{i,t-1}=1,\{n_{j,t-1}\}_{j\in\partial i})=\delta_{n_{i,t},1}\ ,$$

reflecting the irreversible nature of the dynamics in the sense described above (with $n_{i,t}=1$ as absorbing state).

Writing $\mathbf{n}_i=(n_{i,0},n_{i,1},\cdots,n_{i,T})$, we find it advantageous to parametrize the paths $\mathbf{n}_i$ by a single default time t_i defined as the time for which $n_{i,t<t_i}=0$ and $n_{i,t_i}=1$. Moreover we will assume $\{n_{i,0}\}_{i\in G}=0$ in the remainder of this article, and omit the dependence on initial conditions (which simply forbid the default time $t_i=0$).

Under these assumptions, we can write the path probabilities as default-time probabilities:

$$P\left(\{n_{i,t}\}\right)=P(\mathbf{n_0})P(\{t_i\}_{i\in G}\,|\,\{n_{i,0}\}_{i\in G})=\prod_i P(t_i|\theta_i,\mathbf{h}_i)\ ,$$

where

$$P(t_i|\theta_i,\mathbf{h}_i)=W_{t_i-1}\left(h_{i,t_i-1}-\theta_i\right)\prod_{s=0}^{t_i-2}\left[1-W_s\left(h_{i,s}-\theta_i\right)\right]\ , \tag{4}$$

and where we have introduced the notation $\mathbf{h}_i=\{h_{i,s}\}_{s=0,\cdots,T-1}$.

Additionally, we must include the special case where a node does not default within the time horizon T, corresponding to the paths $n_{i,t}=0$ for all $t\le T$. The probability of such a “survival” path of node i is given by

$$P(\text{Survival}|\theta_i,\mathbf{h}_i)=\prod_{s=0}^{T-1}\left[1-W_s\left(h_{i,s}-\theta_i\right)\right]\ . \tag{5}$$

In any sum over default times, the survival path will be implicitly included (it can be straightforwardly mapped onto a default time $t=T+1$ by setting $W_T=1$).

In the Gaussian case which will be our reference, we will take

$$W_t(x)=\Phi\left(x-\xi_{0,t}\right)\ . \tag{6}$$

We can remark at this point that the W_t need not be a c.d.f and can be arbitrary transition probabilities, e.g. they need not be monotone with respect to the local field. It is also possible to have the transition probabilities depend on the node's degree.

Bootstrap percolation is recovered by having $P(\mathbf{n_0})$ (or equivalently W_0) be a suitable seeding probability, taking deterministic couplings and wealth $w_{ij}=w_0$, $\theta=\theta_0$, taking the zero-noise limit and setting $\theta w_0^{-1}=c$ for a constant c corresponding to the number of defaulted (infected) neighbors needed to propagate default. For a Watts-type percolation, $\theta=k\theta_0$ should be considered instead with k being the node's degree.

3 Message-passing approach

The following approach is an extension of a method proposed by Newman and Karrer [21] and can be proven to be exact in the $N\to\infty$ limit by generating function analysis (see appendix A).

We consider a node i of degree k and the set neighboring node $\{j \in \partial i\}$. In order to compute node i's default probability at time t, we need the marginal default probability of every neighboring nodes j at all times $\tau < t$ knowing that node i only defaults at time t. But given the form of the dynamics, the marginal probabilities of defaulting at a time $t' < t$ do not depend on the specific time of the default t, only on the fact that it be posterior to the default time t'. Thanks to this, it is very easy to compute marginals by forward-integration.

Consider a specific instance of a weighted tree G, and a node i on this graph. We write $p_i(t_i)$ for the probability for i to default at t_i. If we now consider the neighbors of i, we can write that

$$p_i(t_i) = \sum_{\{\tau_j\}_{j\in\partial i}} p_i(t_i|\,\{\tau_j\}_{j\in\partial i}) \prod_{j\in\partial i} p_j(\tau_j|t_i)$$

where $p_j(\tau_j|t_i)$ is the probability that j defaults at τ_j conditional on i defaulting at t_i. The factorization of the neighbors' conditional probabilities is due to the underlying graph being a tree. In terms of the specification of the previous section, the conditional probability $p_i(t_i|\,\{\tau_j\}_{j\in\partial i})$ is simply given $P\left(t_i|\theta_i, \mathbf{h}_i\right)$ defined in (4).

Likewise, we can write

$$p_j(\tau_j|t_i) = \sum_{\{\tau_l\}_{l\in\partial j\setminus i}} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}) \prod_{l\in\partial j\setminus i} p_l(\tau_l|\tau_j)\ . \tag{7}$$

But we notice then that once a node has defaulted, the subsequent dynamics of its neighbors no longer influence it. In particular, the conditional probability $p_l(\tau_l|\tau_j)$ depends on τ_j only insofar as $\tau_j < \tau_l$. Hence,

$$\forall \tau_j > \tau_l, \qquad p_l(\tau_l|\tau_j) = p(\tau_l|\tau_l) \equiv \rho_l(\tau_l)\ . \tag{8}$$

Likewise, we notice that the conditional probabilities $p_i(t_i|\,\{\tau_j\}_{j\in\partial i})$ and $p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i})$ only depend on the neighbors' default times insofar as these precede their own, i.e.

$$p_i(t_i|\,\{\tau_j\}_{j\in\tau_i}) = p_i(t_i|\,\{\tau_j;\, \tau_j < t_i\})\ ,$$

and it is clear that

$$\sum_{\tau_l\geq\tau_j} p(\tau_l|\tau_j) = 1 - \sum_{\tau_l<\tau_j} p(\tau_l|\tau_j) = 1 - \sum_{\tau_l<\tau_j} \rho(\tau_l)\ .$$

Using these results we can take a new look at the equation for the conditional probability $p_j(\tau_j|t_i)$ and evaluate the r.h.s. of eq. (7), expressing it in terms of the

$\rho_l(\tau_l)$ for $\tau_l < \tau_j$. For all r in the neighborhood of j (i excepted), we have

$$\begin{aligned}
p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}) \prod_{l\in\partial j\setminus i} p_l(\tau_l|\tau_j) \\
&= \sum_{\tau_r<\tau_j} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}) p_r(\tau_r|\tau_j) \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j) \\
&\quad + \sum_{\tau_r\geq\tau_j} p_j\left(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}\right) p_r(\tau_r|\tau_j) \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j) \\
&= \sum_{\tau_r<\tau_j} p_j\left(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}\right) \rho_r(\tau_r) \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j) \\
&\quad + p_j\left(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}\right) \left[\sum_{\tau_r\geq\tau_j} p_r(\tau_r|\tau_j)\right] \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j) \\
&= \sum_{\tau_r<\tau_j} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}) \rho_r(\tau_r) \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j) \\
&\quad + p_j\left(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}\right) \left(1 - \sum_{\tau_r<\tau_j} \rho_r(\tau_r)\right) \prod_{l\in\partial j\setminus\{i,r\}} p_l(\tau_l|\tau_j)
\end{aligned}$$

Hence we can write

$$p_j(\tau_j|t_i) = \sum_{\{\tau_l\}_{l\in\partial j\setminus i}} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\setminus i}) \prod_{l\,|\tau_l<\tau_j} \rho_l(\tau_l) \prod_{l\,|\tau_l\geq\tau_j} \left[1 - \sum_{\tau_l<\tau_j} \rho_l(\tau_l)\right] , \quad (9)$$

whereas for $\tau_j < t_i$ we have

$$\rho(\tau_j) = \sum_{\{\tau_l\}_{l\in\partial j\setminus i}} p_j(\tau_j|\,\{\tau_l\}_{l\in\partial j\setminus i}) \prod_{l\,|\tau_l<\tau_j} \rho_l(\tau_l) \prod_{l\,|\tau_l\geq\tau_j} \left[1 - \sum_{\tau_l<\tau_j} \rho_l(\tau_l)\right] . \quad (10)$$

Replacing $p_j(\tau_j|\,\{\tau_l\}_{l\in\partial j\setminus i})$ by its more explicit version $P\Big(\tau_j|\,\theta_j, \sum_{l\in\partial j\setminus i} w_l\mathbf{n}_l\Big)$, this is expressed as

$$\rho_j(\tau_j) = \sum_{\{\tau_l\}_{l\in\partial j\setminus i}} P\Big(\tau_j|\,\theta_j, \sum_{l\in\partial j\setminus i} w_l\mathbf{n}_l\Big) \prod_{l\,|\tau_l<\tau_j} \rho_l(\tau_l) \prod_{l\,|\tau_l\geq\tau_j} \left[1 - \sum_{\tau_l<\tau_j} \rho_l(\tau_l)\right] . \quad (11)$$

This single-instance equation can then be averaged over the degree and wealth of the associated nodes, as well as the coupling strength, to obtain the typical behavior of the system in the infinite system size limit, $N \to \infty$. We note here that the neighbors' degree distribution is different from that of a node chosen at random: the probability for a neighbor to have degree k is $\frac{kp(k)}{\langle k\rangle}$. Thus the average of $\rho_j(\tau)$ is given by

$$\begin{aligned}
\rho(\tau) \equiv& \frac{1}{N} \sum_j \rho_j(\tau) \\
=& \sum_k \frac{kp(k)}{\langle k\rangle} \sum_{\tau_1,\cdots,\tau_{k-1}} \prod_{l|\tau_l<\tau} \rho(\tau_l) \prod_{l|\tau_l\geq\tau} \left[1 - \sum_{\tau'<\tau} \rho(\tau)\right] \left\langle P\left(\tau\Big|\,\theta, \sum_{l=1}^{k-1} w_l\mathbf{n}(\tau_l)\right)\right\rangle_{\theta,\{w_l\}}
\end{aligned} \quad (12)$$

where the average over the couplings $\langle \cdots \rangle_{\{w_l\}}$ is done over the marginal coupling distribution.

The resulting equation is forward-propagating in ρ, starting from $\rho(1) = \langle W_0(-\theta)\rangle_\theta$. The marginal default probability at time t, meanwhile, is given by

$$\begin{aligned} p(t) \equiv& \frac{1}{N}\sum_i p_i(t) \\ =& \sum_k p(k) \sum_{\tau_1,\cdots,\tau_k} \prod_{l|\tau_l<t} \rho(\tau_l) \prod_{l|\tau_l\geq t} \left[1 - \sum_{\tau'<t} \rho(\tau')\right] \left\langle P\left(t \middle| \theta, \sum_{l=1}^{k} w_l \mathbf{n}_l\right)\right\rangle_{\theta,\{w_l\}} \end{aligned} \tag{13}$$

The resulting numerical scheme is transparent and can be used to quickly compute default probabilities on an infinite tree. Assuming the probabilities ρ up to $t-1$ have already been computed, the procedure is:

- draw a degree k according to the neighbor degree distribution, and a wealth θ,
- draw $k-1$ interaction weights $\{w_l\}_{l=1,\cdots,k-1}$,
- draw $k-1$ default times according to the previously computed distribution $(\rho(1),\cdots,\rho(t-1), 1-\sum_{s<t}\rho(s))$,
- compute the resulting $P\left(t|\theta,\mathbf{h}\right)$,
- repeat the procedure $N_{sampling}$ times and average the results to give $\rho(t)$.

The marginal probability $p(t)$ can be computed in parallel by drawing a degree k according to $p(k)$, and drawing k interactions weights and default times (according to ρ). The defaulted fraction $m(T)$ is then given by

$$m(T) = \sum_{t=0}^{T} p(t) \ . \tag{14}$$

Using this method, we can rather easily correlate degree with wealth. It has been remarked in several contexts that message-passing works rather well even if the graph is only locally tree-like, and indeed we see very good agreement with simulations on finite samples.

4 Numerical results

In the following we present results of the above analysis for a stylized economy exhibiting mutual financial exposures which constitute a scale-free graph of dependencies.

In principle, three levels of analysis are available:

(i) using population dynamics to study equation (13)

(ii) simulating (large) single instances to solve equation (10)

(iii) stochastic simulations for a moderately large system sizes to check the validity of the theoretical analysis

Single-instances cavity equations have been studied in the case of bootstrap percolation by Altarelli et al [3] and in an epidemiolgical context by Lokhov et al [24] and will thus not be investigated here. Instead, we will focus on comparing the results of (i) and (iii).

Unless otherwise specified, we will use a truncated scale-free degree distribution, $p(k) \sim k^{-\gamma}$ with $\gamma = 3$, for $k \in [\![k_{min}, k_{max}]\!]$ with various values of $k_{min} \geq 1$, and $k_{max} = 100$. The wealth θ will be a Gaussian r.v. with mean $\theta_0 = 2.75$ and variance $\sigma_\theta = 0.3$ as used in [18]. We take $W_t(x) = \Phi(x)$ for all t, initially assuming neutral macro-economic condition (i.e. $\xi_{0,t} = 0$ in (6)). The couplings are taken to be Gaussian r.v. with mean $w_0 = 1$ and variance $\sigma_w = 0.5$. For simulations, we take for the network size $N = 10^3$. The time horizon is taken to be $T = 12$.

4.1 Initial acceleration

As shown in [18], we can qualitatively assess the interaction-induced increase in risk by looking at the discrete second derivative of the defaulted fraction at $t = 1$, $\Delta_1 = m_2 + m_0 - 2m_1 = p(2) - p(1)$. A positive initial acceleration of the fraction of defaulted firms can be seen as an indicator of destabilization of the economy through mutual exposures. Indeed, for a non-interacting system the initial acceleration is quickly found to be

$$\Delta_1 = -\langle W_0(-\theta) W_1(-\theta) \rangle < 0 \ .$$

In order to compare the results across networks with different mean degrees and with the high mean-degree results of previous works we plot the values of Δ_1 in the space of interaction parameters (w_0, σ_w). However, a higher degree means more liabilities and thus a possibly much higher likelihood of losses. Hence in order to make the results comparable between different degrees the coupling strength parameters are rescaled: as in [18] we take $w_{ij} = w_0 \langle k \rangle^{-1} + x_{ij} \sigma_w \langle k \rangle^{-1/2}$, with $x_{ij} \sim \mathcal{N}(0, 1)$. While the values of the $\Delta_1 = 0$ boundary depend on the details of the degree distribution, we find the theoretical predictions for large mean degree to agree remarkably well with previous results for high-connectivity Erdös-Renyí random graphs, even in the scale-free, finite connectivity case.

The domain boundaries are plotted in figure 1 in the scale-free case for $k_{min} = 1$ (i.e. $\langle k \rangle \simeq 1.4$), $k_{min} = 2$ ($\langle k \rangle \simeq 3.1$) and $k_{min} = 5$ ($\langle k \rangle \simeq 8.7$), and the Erdös-Renyí case with infinite connectivity is added for reference.

4.2 Macro-economic sensitivity

The Basel regulatory framework (Basel II and III) requires banks to take into account cyclical effects and macro-economic factors in their risk estimate, in the shape of a countercyclical capital buffer. It is thus worthwhile to investigate the role of cyclical effects on the default probability in our setting, highlighting once again the destabilizing effect of interactions. Assuming for simplicity that these cyclical effects to change slowly over the course of a year, we set $\xi_{0,t} = \xi_0$ in (6) to reflect macro-economic condition, and set ξ_0 to be a Gaussian r.v. (positive ξ_0 reflecting favorable, negative ξ_0 reflecting onfavorable conditions). We can then study the distribution $P(m_T)$ of the defaulted end-of-year fraction m_T induced by the distribution of ξ_0, the shape of the large m_T-tail giving an indication as to the vulnerability of the system to large-scale economic shocks. This is done in figure 2. The non-interacting case is added for comparison.

4.3 Interaction strength

In figure 3 we plot the time dependent defaulted fraction m_t at neutral macro-economic conditions ($\xi_{0,t} = 0$) for different values of the interaction strength w_0.

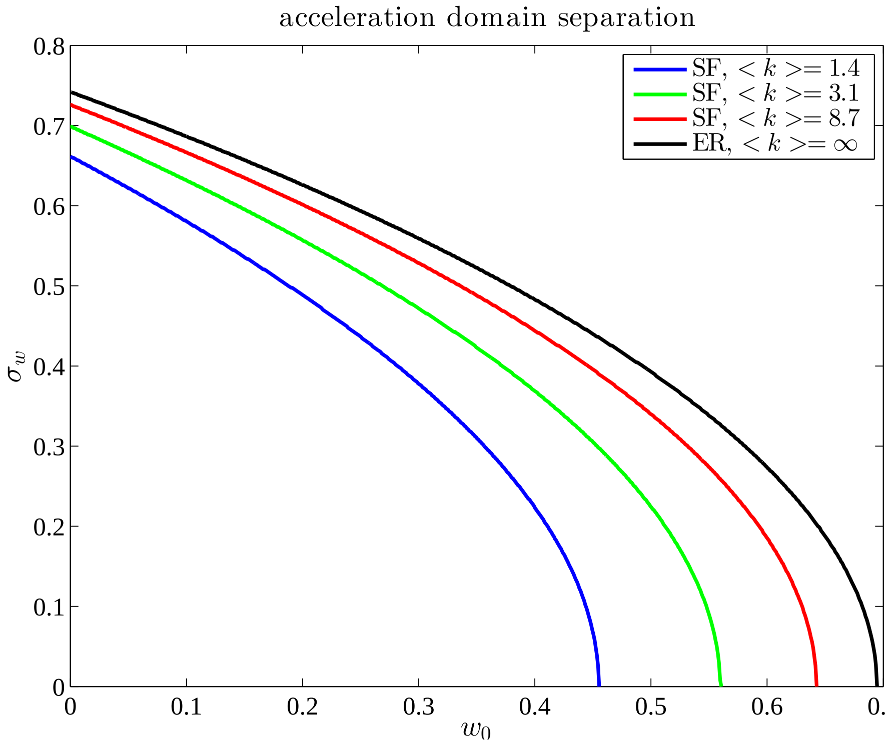


Figure 1: Acceleration domain boundaries for scale-free graphs with $\langle k \rangle \simeq 1.4$ (blue), $\langle k \rangle \simeq 3.1$ (green) and $\langle k \rangle \simeq 8.7$ (red), and the Erdös-Renyí graph with infinite connectivity (black). The external domain (large w_0, large σ_w) has positive acceleration, marking a destabilizing effect of interactions.

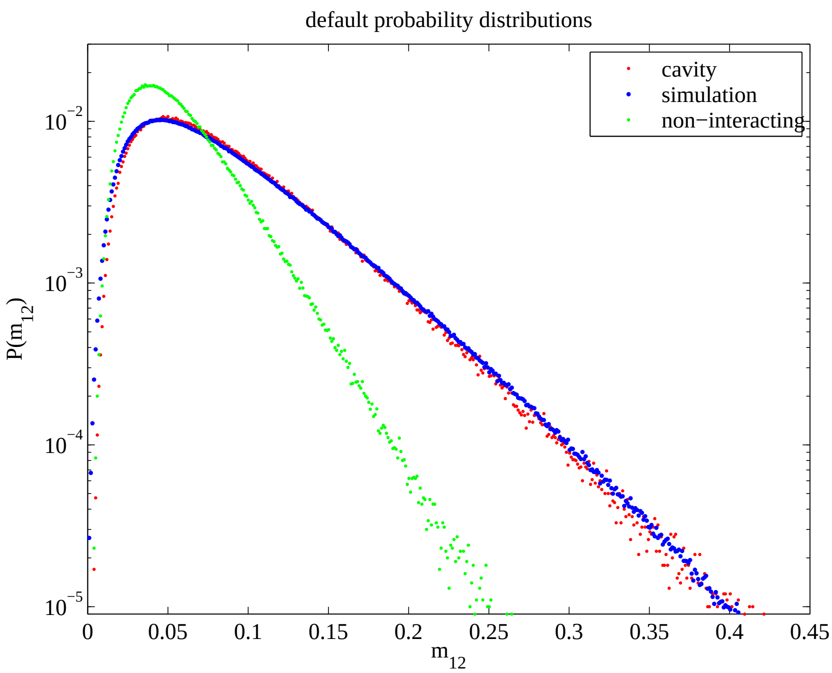


Figure 2: Default distribution for $\xi_0 \sim \mathcal{N}(0, 0.2)$: Simulations (blue) compared with cavity predictions (red) and with the non-interacting network (green).

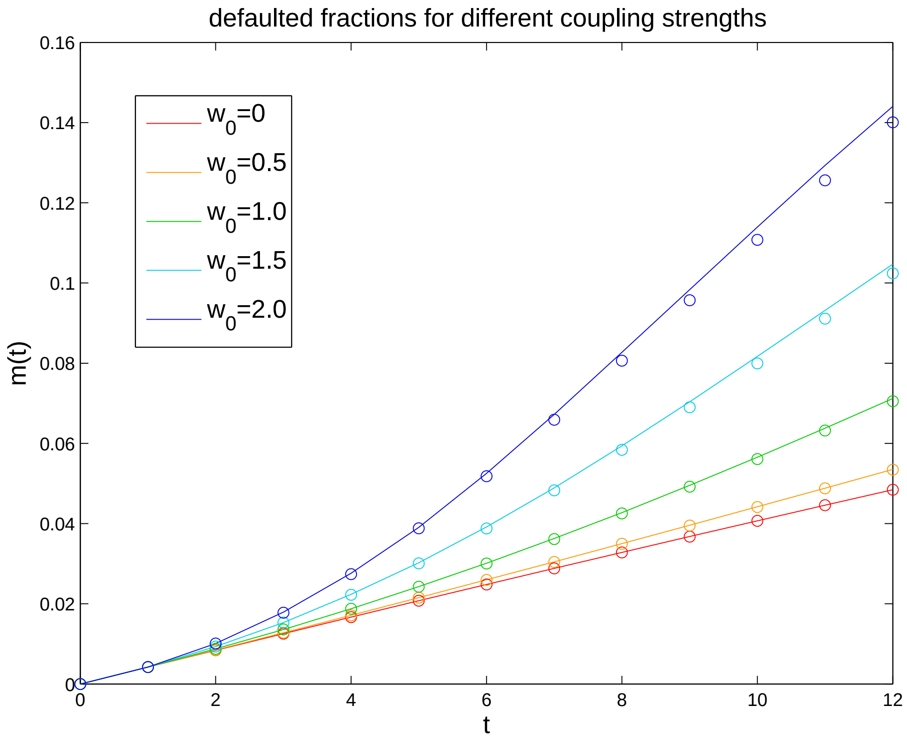


Figure 3: Defaulted fractions for different mean coupling strengths: simulation (circles) and cavity (solid lines).

We set σ_w to half the value of w_0, and average the simulation results over 250 graphs, with 500 runs on each graph.

As expected, the finite-size effects becomes larger as the interaction strength increases, which is to be expected: while the cavity method is exact on trees, the presence of loops strongly affects its performance. And while the graphs used are locally tree-like, for finite size systems there are still a large number of loops remaining. Since these loops only affect the dynamics when their constitutive nodes are defaulted, their effect is felt more strongly when the default rate is higher.

4.4 Extension: spillover

An important extension of the model is the inclusion of spillover effects, as induced by asset fire-sales. A fire-sale happens when a firm, short on liquidity, sells a large amount of assets in a short time (in the case of banks, it is a regulatory requirement to maintain a certain size of its liquid capital buffer). As a result of the sudden glut, asset prices will fall, diminishing the value of assets held by other firms.

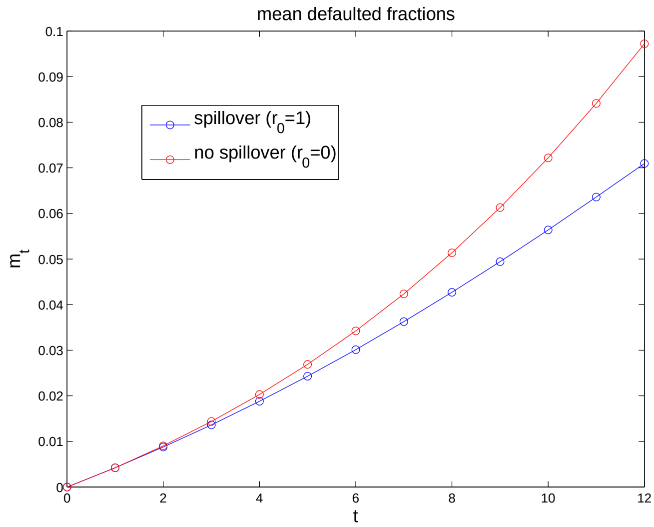
To implement this, we consider for simplicity a single class a of (tradeable) assets, representing a given (constant) fraction of every firm's wealth. When a firm's wealth falls below a certain fraction f_c of its initial wealth it enters a distressed state and sells asset a to maintain liquidity, resulting in a fall of the asset price. As a result, the value of the portfolio of every firm holding this asset falls, which we model by changing the firms' initial wealth by a factor $r(\mathbf{d}) = (1 + r_0\, d_t)^{-1}$ where d_t is the fraction of distressed firms at time t (defaulted firms are considered distressed as well), and r_0 a parameter that could describe market depth. Thus, a firm's wealth at time t becomes

$$\theta_{i,t} = r(d_{t-1})\,\theta_i - \sum_i w_{ij} c_{ij} n_{j,t} - \eta_{i,t}\ ,$$

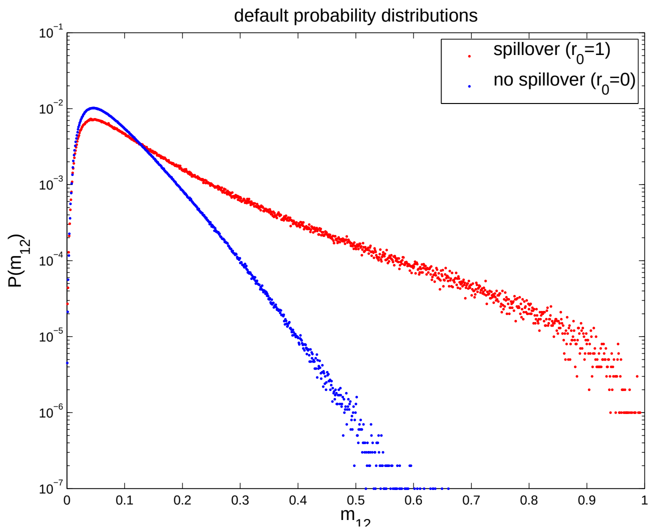
and default is triggered when $\theta_{i,t}$ falls below zero, while the firm enters a distressed state as soon as $\theta_{i,t} < f_c \theta_i$. This can be seen as implementing a simple version of the overlapping portfolio approach of Caccioli et al.[8] on top of the credit risk model, using only one asset class. Plotted below are the mean defaulted fraction in neutral macro-economic conditions ($\xi_0 = 0$, fig. 4a) and the probability distribution of defaulted fractions for random macro-economic conditions (fig. 4b), showing that spillover effects can dramatically increase the probability of large defaults. We take $f_c = 0.1$.

4.5 Network size

To check the convergence of the simulation results towards cavity results valid in the thermodynamic limit $N \to \infty$, we plot the relative error of the defaulted fraction at $T = 12$ compared to the cavity simulation $\epsilon = \frac{|m_{12}^N - m_{12}^\infty|}{m_{12}^\infty}$, for different network sizes. Numerical simulations are done on networks of size $N = 200$ to $N = 5000$. We average over 5000 graphs with 1000 runs on each graph (5000 for network sizes smaller than 10^3). The results are plotted in figure 5. Since the variance of the cavity results are orders of magnitude smaller than that of the simulations, the error bars represent the RMSE of the simulations relative to the average cavity result, scaled by the square root of the number of simulation samples.



(a) Mean defaulted fractions under neutral macro-economic conditions ($\xi_0 = 0$) for a model with spillover $r_0 = 1$ (red) compared to results without spillover (blue).



(b) End-of-year default probabilities with spillover $r_0 = 1$(red) compared to results without spillover (blue) for $\xi_0 \sim \mathcal{N}(0, 0.2)$.

Figure 4

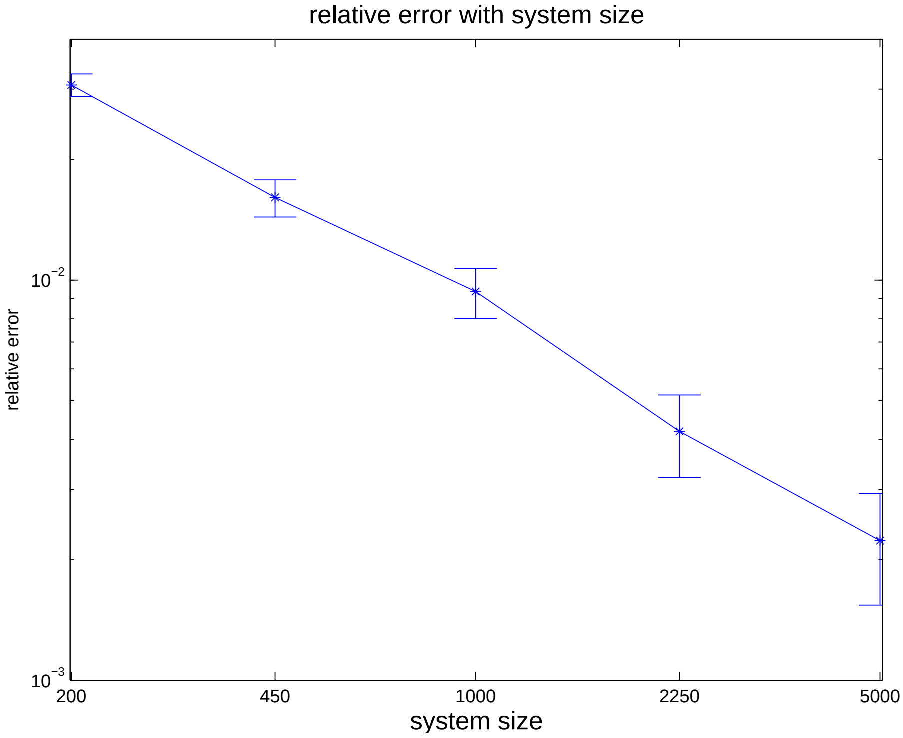


Figure 5: Log-log plot of the relative deviation of finite-size simulations from cavity predictions for different network sizes.

5 Conclusion

In this paper, we have studied the model of credit contagion introduced in [18], and have derived an analytic solution for the damage-spreading dynamics on generic locally tree-like sparse graphs, using both an extension of the Newman-Karrer method detailed in [21], and a generating functional analysis. Both approaches give rise to the same dynamical equations.

We have found the analytic predictions to be in good agreement with simulation results, both for average defaulted fractions and for distributions of end-of-year defaulted fractions induced by varying macro-economic conditions. Our preliminary results for a highly schematized szenario support the idea that spillover effects caused by asset fire-sales constitute a relevant driver of systemic instability, which appears to be at least as important as direct contagion.

Surprisingly, we find that the destabilizing effect of interactions as measured by the initial acceleration of default does not depend on the degree distribution in the large connectivity limit. This is due to a combination of the graph being locally tree-like, i.e. early defaults in the neighborhood of a node are independent, and the initial default probability being low. Taken together, this allows us to consider that only one among a node's k neighbors defaults in the first time step. In turns, this yields a contribution to the early default probabilities which only depends on the mean degree.

Unlike standard cavity equations for equilibrium problems, our solution does not require iteration until convergence and hence the only limiting numerical factor is the sampling effort required to achieve a desired precision.

On the analytical side, open questions remain. One question of interest is the computation of large-deviation functions, for which an annealed computation is possible using the current method but where quenched computations run into difficulties (e.g. we cannot, in the replica computation, factor the bond-disorder average).

The method exposed here can easily be extended to more general models. In particular, it is straightforward to add recovery scenarios where economic impacts incurred through deraults diminish with time. In this case, we can make an easy parallel to the SIR model. This extension will be dealt with in a separate paper.

6 Acknowledgements

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It is a pleasure to thank Luca Dall'Asta for enlightening discussions.

A Appendix

A.1 Generating Functional Analysis

We develop here the generating function analysis of the model described above. The analysis is very general and can be applied to almost any networks where nodes are coupled via local fields. The method is well understood and has been used in a variety of other studies [29, 10, 19], although the version presented here differs in that instead of introducing a path integral, we can limit ourselves to a finite (low) number of integration parameters.

The method proceeds as follows: considering a particular degree sequence and wealth configuration, we introduce a generating function for the evaluation of averages and correlation functions of observables related to contagion dynamics as described by eq. (1). Taking advantage of the sparseness of the network, we express this generating function in terms of an integral with an effective action, and we compute the integral to order N using a saddle-point approximation.

The initial part is simple: as is standard in field theory, we wish to compute a generating function for the correlations of observables $n_{i,t}$

$$G[\boldsymbol{\psi}|\mathbf{k}.\boldsymbol{\theta}] = \overline{\left\langle \exp\left\{ \sum_{i,t} \psi_{i,t} n_{i,t} \right\} \right\rangle} \tag{15}$$

for a given auxiliary field $\boldsymbol{\psi}$, where the $\langle \cdots \rangle$ denotes an average over the dynamics (the default times) for a given graph and wealth realization $(\mathbf{k}, \boldsymbol{\theta})$, while $\overline{(\cdots)}$ denotes an average over the graphs compatible with this realization.

Once such a function has been computed, the average value $\overline{\langle n_{i,t} \rangle}$ can be obtained using

$$\overline{\langle n_{i,t} \rangle} = \partial_{\psi_i} G[\boldsymbol{\psi}|\mathbf{k}, \boldsymbol{\theta}]|_{\boldsymbol{\psi}=0}$$

while correlation functions are given by higher derivatives, such as

$$\overline{\langle n_{i,t} n_{j,t'} \rangle} = \partial_{\psi_{i,t}\psi_{j,t'}} G[\boldsymbol{\psi}|\mathbf{k}, \boldsymbol{\theta}]\Big|_{\boldsymbol{\psi}=0} \ .$$

In our situation, we are primarily interested in the global average

$$m_t = \frac{1}{N} \sum_i \overline{\langle n_{i,t} \rangle} \ .$$

Since we use the generating functional method only as a vehicle to obtain a macroscopic description of the dynamics, we can in fact drop the source fields $\psi_{i,t}$ entirely in what follows.

Carrying out the average over the graphs, i.e. over their adjacency matrix $\mathbf{A}$ and couplings $\mathbf{w} = \{w_{ij}, w_{ji}\}_{(ij)}$, the generating function is expressed as

$$G[\mathbf{k}, \boldsymbol{\theta}] = \sum_{\mathbf{A}} \int \mathrm{d}\mathbf{w}\, P(\mathbf{A}, \mathbf{w}|\mathbf{k}) \sum_{\{t_i\}} \prod_i P\left(t_i|\theta_i, \mathbf{h}_i\right) \ .$$

We then use Dirac delta functions and their Fourier representations to 'extract' the depencence of the $P\left(t_i|\theta_i, \mathbf{h}_i\right)$ on the couplings $\mathbf{w}$ via the local-fields $\{\mathbf{h}_i\}$ to obtain

$$G[\mathbf{k}, \boldsymbol{\theta}] = \sum_{\mathbf{A}} \int \mathrm{d}\mathbf{w}\, P(\mathbf{A}, \mathbf{w}|\mathbf{k}) \int \prod_i \sum_{t_i} \int \frac{\mathrm{d}\mathbf{h}_i\, \mathrm{d}\hat{\mathbf{h}}_i}{(2\pi)^T} P\left(t_i|\theta_i, \mathbf{h}_i\right)$$
$$\times \exp\left\{ -i\hat{\mathbf{h}}_i \cdot \left(\mathbf{h}_i - \sum_j a_{ij} w_{ij} \mathbf{n}(t_j) \right) \right\} \ , \tag{16}$$

which, as we shall now see, allows the average over graphs to be performed.

A.2 Graph probabilities

When taking the average over graphs, even for tree-like graphs we have a number of graph ensembles to choose from. They fall into two broad categories: micro-canonical ensembles, where adjacency matrices are drawn so as to exactly reproduce a given degree sequence (with a prescribed degree distribution), and canonical models where links are randomly chosen such that degrees follow a given degree sequence only on average.

In the following derivation, we use a micro-canonical configuration model: we consider a "typical" (self-averaging) degree sequence $\mathbf{k}$, and we take a uniform probability on the graphs with this given degree sequence:

$$P(\mathbf{A}, \mathbf{w}|\mathbf{k}) \propto \prod_{(ij)} p_w(w_{ij}, w_{ji}) \delta_{a_{ij}, a_{ji}} \prod_i \delta_{k_i, \sum_j a_{ij}} \tag{17}$$

Tt is easier for our purpose to rewrite it as

$$P(\mathbf{A}, \mathbf{w}|\mathbf{k}) \propto \prod_{(ij)} p_w(w_{ij}, w_{ji}) \delta_{a_{ij}, a_{ji}} \left[\frac{\langle k \rangle}{N} \delta_{a_{ij},1} + \left(1 - \frac{\langle k \rangle}{N} \right) \delta_{a_{ij},0} \right] \prod_i \delta_{k_i, \sum_j a_{ij}}$$

where the extra factor is seen to be independent of the choice of $\mathbf{A}$ for all adjacency matries compatible with the chosen degree sequence [29], allowing to absorb it in the overall normalization. $\mathcal{N}$ of the distribution. We then use the Fourier decomposition of the Kronecker deltas to get

$$P(\mathbf{A}, \mathbf{w}|\mathbf{k}) = \frac{1}{\mathcal{N}} \int \prod_i \frac{\mathrm{d}\omega_i}{2\pi} e^{-i\omega_i k_i} \prod_{(ij)} p_w(w_{ij}, w_{ji}) \left[\frac{\langle k \rangle}{N} e^{i(\omega_i + \omega_j)} \delta_{a_{ij},1} + \left(1 - \frac{\langle k \rangle}{N} \right) \delta_{a_{ij},0} \right] \ .$$

The average over weighted graphs in eq. (16) factors with respect to the edges, so the generating function can be expressed as

$$G = \frac{1}{\mathcal{N}} \prod_i \sum_{t_i} \int \frac{\mathrm{d}\mathbf{h}_i\, \mathrm{d}\hat{\mathbf{h}}_i}{(2\pi)^T} \frac{\mathrm{d}\omega_i}{2\pi} e^{-i\omega_i k_i} P\left(t_i|\theta_i, \mathbf{h}_i\right) e^{-i\hat{\mathbf{h}}_i \cdot \mathbf{h}_i} \prod_{(ij)} D_{ij},$$

in which the individual edge contributions D_{ij} take the form

$$\begin{aligned}
D_{ij} = &\sum_{a_{ij}=0,1} \int \mathrm{d}w_{ij}\, \mathrm{d}w_{ji}\, p_w(w_{ij}, w_{ji}) \left[\frac{\langle k \rangle}{N} e^{i(\omega_i+\omega_j)} \delta_{a_{ij},1} + \left(1 - \frac{\langle k \rangle}{N} \right) \delta_{a_{ij},0} \right] \\
&\times \exp \left\{ a_{ij} \left[i w_{ij} \hat{\mathbf{h}}_i \cdot \mathbf{n}(t_j) + i w_{ji} \hat{\mathbf{h}}_j \cdot \mathbf{n}(t_i) \right] \right\} \\
= &\int \mathrm{d}w_{ij}\, \mathrm{d}w_{ji}\, p_w(w_{ij}, w_{ji}) \\
&\times \left\{ 1 + \frac{\langle k \rangle}{N} \left[e^{i(\omega_i+\omega_j)} \exp i \left\{ w_{ij} \hat{\mathbf{h}}_i \cdot \mathbf{n}(t_j) + w_{ji} \hat{\mathbf{h}}_j \cdot \mathbf{n}(t_i) \right\} - 1 \right] \right\}
\end{aligned} \tag{18}$$

We can carry out the integration over the edge weights, and as it turns out the integral is factorizable even if $p(w_{ij}, w_{ji})$ is not:

$$\left\langle e^{i w_{ij} \hat{\mathbf{h}}_i \cdot \mathbf{n}(t_j)} e^{i w_{ji} \hat{\mathbf{h}}_j \cdot \mathbf{n}(t_i)} \right\rangle_{w_{ij}, w_{ji}} = \left\langle e^{i w_{ij} \hat{\mathbf{h}}_i \cdot \mathbf{n}(t_j)} \right\rangle_{w_{ij}} \left\langle e^{i w_{ji} \hat{\mathbf{h}}_j \cdot \mathbf{n}(t_i)} \right\rangle_{w_{ji}} \tag{19}$$

It is intuitively clear why this should be the case: since a node once defaulted is not influenced by the subsequent defaults of its neighbors, the value of these couplings is irrelevant. Thus, if node i defaults first, whether w_{ji} follows the marginal distribution or the conditional $p(w_{ji}|w_{ij})$ is both irrelevant and impossible to determine, and we can assume the former.

From the formal point of view, this is due to a rather subtle point: for a given node i with default time t_i, while the fields $\mathbf{h}_i$ and $\hat{\mathbf{h}}_i$ have T components $(h_{i,0}, h_{i,1}, \cdots, h_{i,T-1})$, $P(t_i|\theta_i, \mathbf{h}_i)$ only depends on the first $t_i - 1$ components of $\mathbf{h}_i$. Therefore, for the remaining components, the integration over the $\{h_{i,s}\}_{s \geq t_i}$ yield $\delta(h_{i,s})$. In order to avoid cluttering the expressions we do not carry out this integration explicitly, but note that $\hat{h}_{i,s} = 0$ for $s \geq t_i$ implies

$$\exp i \left\{ w_{ij} \hat{\mathbf{h}}_i \cdot \mathbf{n}(t_j) + w_{ji} \hat{\mathbf{h}}_j \cdot \mathbf{n}(t_i) \right\} = \exp i \left\{ w_{ij} \sum_{t_j \leq s < t_i} \hat{h}_{i,s} + w_{ji} \sum_{t_i \leq s < t_j} \hat{h}_{j,s} \right\} ,$$

and thus only one among the pair (w_{ij}, w_{ji}) appears in the integral. This "dynamical factorization" is a crucial simplification. To simplify our expressions, we introduce

$$\chi(\hat{\mathbf{h}}, t) = \left\langle e^{i w \hat{\mathbf{h}} \cdot \mathbf{n}(t)} \right\rangle_w \tag{20}$$

A.3 Effective action

Combining averages over all edge-related parts, we obtain

$$\prod_{(ij)} D_{ij} = \prod_{(ij)} \left(1 + \frac{\langle k \rangle}{N} \left[\chi(\hat{\mathbf{h}}_i, t_j) \chi(\hat{\mathbf{h}}_j, t_i) e^{i(\omega_i+\omega_j)} - 1 \right] \right)$$

Assuming the graph to be sparse, i.e. $\langle k \rangle \ll N$, this can be rewritten in exponential form

$$\begin{aligned}
\prod_{(ij)} D_{ij} = &\exp \left\{ \frac{\langle k \rangle}{N} \sum_{(ij)} \left[\chi(\hat{\mathbf{h}}_i, t_j) \chi(\hat{\mathbf{h}}_j, t_i) e^{i(\omega_i+\omega_j)} - 1 \right] \right\} \\
= &\exp \left\{ \frac{\langle k \rangle}{2N} \sum_{i,j} \chi(\hat{\mathbf{h}}_i, t_j) \chi(\hat{\mathbf{h}}_j, t_i) e^{i(\omega_i+\omega_j)} - \frac{N \langle k \rangle}{2} \right\} .
\end{aligned} \tag{21}$$

We shall absorb the $\frac{N\langle k\rangle}{2}$ part into the normalization constant $\mathcal{N}$. Writing

$$P(\omega, t, \hat{\mathbf{h}}) = \frac{1}{N}\sum_i \delta(\omega-\omega_i)\delta_{t,t_i}\delta(\hat{\mathbf{h}}-\hat{\mathbf{h}}_i)$$

we can rewrite our previous expression (21) as

$$\prod_{(ij)} D_{ij} = \exp\left\{N\frac{\langle k\rangle}{2}\sum_{t,t'}\int \mathrm{d}\hat{\mathbf{h}}\,\mathrm{d}\hat{\mathbf{h}}'\int \mathrm{d}\omega\,\mathrm{d}\omega'\,\chi(\hat{\mathbf{h}},t')\chi(\hat{\mathbf{h}}',t)e^{i\omega}e^{i\omega'}P(\omega,t,\hat{\mathbf{h}})P(\omega',t',\hat{\mathbf{h}}')\right\}$$

We see that this form almost factorizes. We then introduce the quantity

$$\begin{aligned}
p(t|t') =& \int \mathrm{d}\hat{\mathbf{h}}\,\mathrm{d}\omega\,\chi(\hat{\mathbf{h}},t')e^{i\omega}P(\omega,t,\hat{\mathbf{h}}) \\
=& \int \mathrm{d}\hat{\mathbf{h}}\,\mathrm{d}\omega\,\frac{1}{N}\sum_i \delta(\omega-\omega_i)\delta_{t,t_i}\delta(\hat{\mathbf{h}}-\hat{\mathbf{h}}_i)\chi(\hat{\mathbf{h}},t')e^{i\omega} \\
=& \frac{1}{N}\sum_i \delta_{t,t_i}\chi(\hat{\mathbf{h}}_i,t')e^{i\omega_i}\ ,
\end{aligned}$$

and decouple sites by considering the $p(t|t')$ as integration variables, using auxiliary variables $q(t|t')$ to enforce their definition via Fourier representations of δ-functions:

$$1 = \int \frac{\mathrm{d}p(t|t')\,\mathrm{d}q(t|t')}{2\pi/N}\exp\left\{-iq(t|t')\left(Np(t|t') - \sum_i \delta_{t,t_i}\chi(\hat{\mathbf{h}}_i,t')e^{i\omega_i}\right)\right\}\ .$$

We notice that after the introduction of this integral, sites are effectively decoupled. As a result the generating function reads, to leading order in N,

$$G = \frac{1}{\mathcal{N}}\int\prod_{t,t'}\frac{\mathrm{d}p(t|t')\,\mathrm{d}q(t|t')}{2\pi/N}\exp\left\{N\left[G_1+G_2+G_3\right]\right\},$$

with

$$\begin{aligned}
G_1 =& \frac{\langle k\rangle}{2}\sum_{t,t'} p(t|t')p(t'|t)\ , \\
G_2 =& -i\sum_{t,t'} p(t|t')q(t|t')\ , \\
G_3 =& \frac{1}{N}\sum_i \ln Z_i\ ,
\end{aligned}$$

where

$$Z_i = \sum_t \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega k_i}P\left(t|\theta_i,\mathbf{h}\right)e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}}\exp\left\{ie^{i\omega}\sum_{t'}\chi(\hat{\mathbf{h}},t')q(t|t')\right\} \quad (22)$$

Using the self-averaging properties of the $(\mathbf{k},\theta)$ configuration, we can replace $\frac{1}{N}\sum_i Z_i(k_i,\theta_i)$ with $\langle Z(k,\theta)\rangle_{k,\theta}$ while only making an error of order $N^{-1/2}$. The function $G_1 + G_2 + G_3$ is the effective action of the problem.

A.4 Saddle-point

Now, considering the form of the integral, we are led in the $N \to \infty$ limit to consider a saddle-point approximation, which will lead us to replace to leading order the integral by its value at the saddle-point. The saddle-point equations

$$\frac{\partial}{\partial p(t|t')}\left(G_1 + G_2 + G_3\right) = 0$$

and

$$\frac{\partial}{\partial q(t|t')}\left(G_1 + G_2 + G_3\right) = 0$$

read

$$\langle k \rangle\, p(t'|t) = iq(t|t') \tag{23}$$

and

$$p(t|t') = \left\langle \frac{\int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} \int \frac{\mathrm{d}\omega}{2\pi} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} e^{-i\omega(k-1)} \chi(\hat{\mathbf{h}},t') \exp\left\{ie^{i\omega}\sum_{t'}\chi(\hat{\mathbf{h}},t')q(t|t')\right\}}{\sum_s \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T}\int \frac{\mathrm{d}\omega}{2\pi} P\left(s|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} e^{-i\omega k} \exp\left\{ie^{i\omega}\sum_{s'}\chi(\hat{\mathbf{h}},s')q(s|s')\right\}} \right\rangle_{k,\theta}$$

We can carry out the integration over ω, and using (23), this gives

$$p(t|t') = \left\langle \sum_k \frac{kp(k)}{\langle k \rangle} \frac{\int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \chi(\hat{\mathbf{h}},t') \left\{\sum_{t'}\chi(\hat{\mathbf{h}},t')p(t'|t)\right\}^{k-1}}{\sum_s \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(s|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\sum_{s'}\chi(\hat{\mathbf{h}},s')p(s'|s)\right\}^{k}} \right\rangle_{\theta} \tag{24}$$

We can interpret the $p(t|t')$ using the same method as in [29]: the $p(t|t')$ *at the saddle-point* appear as conditional probabilities, i.e. the probability that a node default at time t given that we know one of its neighbors has defaulted at time t'. Thus we must have $\sum_t p(t|t') = 1$, remembering that we are also considering in this sum the probability that the node has not defaulted within the finite risk horizon, $1, \ldots, T$.

Another way of looking at things is that we assume the normalization $\sum_t p(t|t') = 1$ and will show that such solutions are self-consistent and coincide with conditional probabilities in the message-passing solutions.

We now remember two things. First, in the integral in the numberator of (24) all components $\hat{h}_s$ for $s \geq t$ cancel, as was noted previously and exploited in eq. (19). Second, $\chi(\hat{\mathbf{h}}, t')$ is a function of the scalar product $\hat{\mathbf{h}} \cdot \mathbf{n}(t') = \sum_{s=t'}^{T-1} \hat{h}_s$. Therefore in the integral in (24) this sum is actually $\hat{\mathbf{h}} \cdot \mathbf{n}(t') = \sum_{s=t'}^{t-1} \hat{h}_s$, and therefore $\chi(\hat{\mathbf{h}}, t' \geq t) = \chi(0, t') = 1$.

Consequently $p(t|s \geq t) = p(t|t)$, e.g. for example $p(1|2) = p(1|1)$. This is clear from the interpretation of the $p(t|t')$ given above, since defaults of a neighbor after one's own default cannot influence the original node as was noted in previous sections. We thus write, as previously in section 3, $\rho(t) \equiv p(t|t' \geq t)$.

Using these observations, we can simplify our equations further. First we notice that with these conventions, in eq. (24) we have

$$\sum_{t'} \chi(\hat{\mathbf{h}}, t') p(t'|t) = \left(1 - \sum_{t'<t} \rho(t')\right) + \sum_{t'<t} \rho(t') \chi(\hat{\mathbf{h}}, t') \ ,$$

since, as was previously noted, all the components $\hat{h}_s$ vanish for $s \geq t$ and $\chi\left(\hat{\mathbf{h}}, t'\right)$ only depends on the components $\hat{h}_s$ for $s \geq t'$.

Second, we notice that the denominator

$$\sum_s \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(s|\theta, \mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\sum_{s'} \chi(\hat{\mathbf{h}}, s') p(s'|s)\right\}^k$$

in (24) is equal to 1.

Indeed, consider the simple situation where $T = 2$: we have three possible trajectories:

- a node defaults on the first time step ($t = 1$), corresponding to the term
$$\int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} W_0(-\theta) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} = W_0(-\theta)$$
in the denominator
- a node defaults on the second time step ($t = 2$), to which corresponds the term
$$\int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} \left(1 - W_0(-\theta)\right) W_1(h_1 - \theta) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\rho(1)\chi(\hat{\mathbf{h}}, 1) + (1 - \rho(1))\right\}^k$$
- a node does not default during the time horizon, to which corresponds the term
$$\int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} \left(1 - W_0(-\theta)\right) \left(1 - W_1(h_1 - \theta)\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\rho(1)\chi(\hat{\mathbf{h}}, 1) + (1 - \rho(1))\right\}^k$$
as in eq. (5)

Summing these terms, we find

$$\begin{aligned}
Z(k, \theta) =& W_0(-\theta) + \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} \left(1 - W_0(-\theta)\right) W_1(h_1 - \theta) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\rho(1)\chi(\hat{\mathbf{h}}, 1) + (1 - \rho(1))\right\}^k \\
&+ \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} \left(1 - W_0(-\theta)\right) \left(1 - W_1(h_1 - \theta)\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\rho(1)\chi(\hat{\mathbf{h}}, 1) + (1 - \rho(1))\right\}^k \\
=& W_0(-\theta) + \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^2} \left(1 - W_0(-\theta)\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{\rho(1)\chi(\hat{\mathbf{h}}, 1) + (1 - \rho(1))\right\}^k \\
=& W_0(-\theta) + \left(1 - W_0(-\theta)\right) = 1
\end{aligned}$$

and the reasoning can be extended without major difficulty to $T > 2$.

Finally, the saddle-point equations for the quantities $p(t|t')$ read

$$p(t|t') =$$

$$\left\langle \sum_k \frac{kp(k)}{\langle k \rangle} \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta, \mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \chi(\hat{\mathbf{h}}, t') \left\{\sum_{t''<t} \rho(t'')\chi(\hat{\mathbf{h}}, t'') + \left(1 - \sum_{t''<t} \rho(t'')\right)\right\}^{k-1} \right\rangle_\theta$$

while

$$\rho(t) = $$

$$\left\langle \sum_k \frac{kp(k)}{\langle k\rangle} \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{ \sum_{t''<t} \rho(t'')\chi(\hat{\mathbf{h}},t'') + \left(1 - \sum_{t''<t}\rho(t'')\right) \right\}^{k-1} \right\rangle_\theta \tag{25}$$

And the (marginal) default probabilities are given by

$$p(t) =$$

$$\left\langle \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \left\{ \sum_{t'<t} \rho(t')\chi(\hat{\mathbf{h}},t') + \left(1 - \sum_{t'<t}\rho(t')\right) \right\}^{k} \right\rangle_{k,\theta}$$

What is the connection between these equations and their message-passing equivalents ? Recall that

$$\chi(\hat{\mathbf{h}},t) = \left\langle e^{iw\hat{\mathbf{h}}\cdot\mathbf{n}(t)} \right\rangle_w ,$$

Thus upon expanding (25), we obtain

$$\begin{aligned}
\rho(t) =& \left\langle \sum_k \frac{kp(k)}{\langle k\rangle} \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \sum_{q=0}^{k} \binom{k}{q} \left(1 - \sum_{t'<t}\rho(t')\right)^{k-q} \right. \\
&\left. \times \sum_{\tau_1,\cdots,\tau_q \le t} \prod_{i=1}^{q} \chi(\hat{\mathbf{h}},t_i)\rho(\tau_i) \right\rangle_\theta \\
=& \left\langle \sum_k \frac{kp(k)}{\langle k\rangle} \int \frac{\mathrm{d}\mathbf{h}\,\mathrm{d}\hat{\mathbf{h}}}{(2\pi)^T} P\left(t|\theta,\mathbf{h}\right) e^{-i\hat{\mathbf{h}}\cdot\mathbf{h}} \sum_{q=0}^{k} \binom{k}{q} \left(1 - \sum_{t'<t}\rho(t')\right)^{k-q} \right. \\
&\left. \times \sum_{\tau_1,\cdots,\tau_q \le t} \int \prod_{i=1}^{q} \mathrm{d}w_i\, p_w(w_i)\rho(\tau_i) e^{iw_i\hat{\mathbf{h}}\cdot\mathbf{n}(\tau_i)} \right\rangle_\theta \\
=& \left\langle \sum_k \frac{kp(k)}{\langle k\rangle} \int \mathrm{d}\mathbf{h}\, P\left(t|\theta,\mathbf{h}\right) \sum_{q=0}^{k} \binom{k}{q} \left(1 - \sum_{t'<t}\rho(t')\right)^{k-q} \right. \\
&\left. \times \sum_{\tau_1,\cdots,\tau_q \le t} \int \prod_{i=1}^{q} \mathrm{d}w_i\, p_w(w_i)\rho(\tau_i) \delta(\mathbf{h} - \sum_{i=1}^{q} w_i\mathbf{n}(\tau_i)) \right\rangle_\theta \\
=& \sum_k \frac{kp(k)}{\langle k\rangle} \sum_{q=0}^{k} \binom{k}{q} \left(1 - \sum_{t'<t}\rho(t')\right)^{k-q} \\
& \times \sum_{\tau_1,\cdots,\tau_q \le t} \int \prod_{i=1}^{q} \mathrm{d}w_i\, p_w(w_i)\rho(\tau_i) \left\langle P\left(t|\theta, \sum_{i=1}^{q} w_i\mathbf{n}(\tau_i)\right) \right\rangle_\theta
\end{aligned}$$

which can be seen to be the same as (12).

References

[1] L. A. Adamic and B. A. Huberman. Power-law distribution of the world wide web. *Science*, 287(5461):2115–2115, 2000.

[2] F. Altarelli, A. Braunstein, L. Dall'Asta, J. Wakeling, and R. Zecchina. Containing epidemic outbreaks by message-passing techniques. *arXiv preprint arXiv:1309.2805*, 2013.

[3] F. Altarelli, A. Braunstein, L. Dall'Asta, and R. Zecchina. Large deviations of cascade processes on graphs. *Physical Review E*, 87(6):062115, 2013.

[4] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *science*, 286(5439):509–512, 1999.

[5] I. Basel. International convergence of capital measurement and capital standards: a revised framework, 2006.

[6] J.-P. Bouchaud. Crises and collective socio-economic phenomena: simple models and challenges. *Journal of Statistical Physics*, 151(3-4):567–606, 2013.

[7] F. Caccioli, T. A. Catanach, and J. D. Farmer. Heterogeneity, correlations and financial contagion. *Advances in Complex Systems*, 15(supp02), 2012.

[8] F. Caccioli, M. Shrestha, C. Moore, and J. D. Farmer. Stability analysis of financial contagion due to overlapping portfolios. *arXiv preprint arXiv:1210.5987*, 2012.

[9] D. Centola and M. Macy. Complex contagions and the weakness of long ties. *American Journal of Sociology*, 113(3):702–734, 2007.

[10] A. Coolen and S. Rabello. Generating functional analysis of complex formation and dissociation in large protein interaction networks. In *Journal of Physics: Conference Series*, volume 197, page 012006. IOP Publishing, 2009.

[11] P. Crucitti, V. Latora, and M. Marchiori. A topological analysis of the italian electric power grid. *Physica A: Statistical Mechanics and its Applications*, 338(1):92–97, 2004.

[12] D. Duffie and K. J. Singleton. Modeling term structures of defaultable bonds. *Review of Financial studies*, 12(4):687–720, 1999.

[13] S. N. Durlauf. Statistical mechanics approaches to socioeconomic behavior, 1996.

[14] D. Egloff, M. Leippold, and P. Vanini. A simple model of credit contagion. *Journal of Banking & Finance*, 31(8):2475–2492, 2007.

[15] A. Elizalde. Credit risk models ii: Structural models. *Available on www. defaultrisk. com/pp model*, 86, 2005.

[16] P. Gai and S. Kapadia. Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2120):2401–2423, 2010.

[17] A. G. Haldane and R. M. May. Systemic risk in banking ecosystems. *Nature*, 469(7330):351–355, 2011.

[18] J. Hatchett and R. Kühn. Effects of economic interactions on credit risk. *Journal of Physics A: Mathematical and General*, 39(10):2231, 2006.

[19] J. Hatchett, B. Wemmenhove, I. P. Castillo, T. Nikoletopoulos, N. Skantzos, and A. Coolen. Parallel dynamics of disordered ising spin systems on finitely connected random graphs. *Journal of Physics A: Mathematical and General*, 37(24):6201, 2004.

[20] S. Heise and R. Kühn. Derivatives and credit contagion in interconnected networks. *The European Physical Journal B*, 85(4):1–19, 2012.

[21] B. Karrer and M. E. Newman. Message passing approach for general epidemic models. *Physical Review E*, 82(1):016101, 2010.

[22] D. Lando. On cox processes and credit risky securities. *Review of Derivatives research*, 2(2-3):99–120, 1998.

[23] F. Liljeros, C. R. Edling, L. A. N. Amaral, H. E. Stanley, and Y. Åberg. The web of human sexual contacts. *Nature*, 411(6840):907–908, 2001.

[24] A. Y. Lokhov, M. Mézard, H. Ohta, and L. Zdeborová. Inferring the origin of an epidemic with dynamic message-passing algorithm. *arXiv preprint arXiv:1303.5315*, 2013.

[25] R. M. May and N. Arinaminpathy. Systemic risk: the dynamics of model banking systems. *Journal of the Royal Society Interface*, 7(46):823–838, 2010.

[26] R. C. Merton. On the pricing of corporate debt: The risk structure of interest rates*. *The Journal of Finance*, 29(2):449–470, 1974.

[27] M. Mezard and A. Montanari. *Information, physics, and computation.* Oxford University Press, 2009.

[28] M. Mézard and G. Parisi. The bethe lattice spin glass revisited. *The European Physical Journal B-Condensed Matter and Complex Systems*, 20(2):217–233, 2001.

[29] K. Mimura and A. Coolen. Parallel dynamics of disordered ising spin systems on finitely connected directed random graphs with arbitrary degree distributions. *Journal of Physics A: Mathematical and Theoretical*, 42(41):415001, 2009.

[30] C. Moore and M. E. Newman. Epidemics and percolation in small-world networks. *Physical Review E*, 61(5):5678, 2000.

[31] A. Vázquez. Statistics of citation networks. *arXiv preprint cond-mat/0105031*, 2001.

[32] D. J. Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771, 2002.