Of Brains and Markets

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Outline

1 Setting the Scene
   - Of Markets . . .
   - Geometric Brownian Motion
   - Stepping Back – A Gedanken-Experiment
   - . . . and Brains

2 Analysis
   - Generating Functionals
   - Separation of Time-Scales — Stationarity

3 Results

4 Inference

5 Summary
Aim of the talk

Attempt to rationalise phenomenology of market dynamics.

Not prediction!
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Stylized Facts of Market Dynamics

- Fat tailed (leptocurtic) return distributions
- Fast decorrelation of asset returns
- Slow decorrelation of absolute returns
- Long range correlations of volatility (volatility clustering).
Of Markets . . .

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S&P 500 return distributions

(Gopikrishnan et al PRE, 1999)
Stylized Facts of Market Dynamics

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Auto-correlations of returns and absolute returns

(Gopikrishnan et al. PRE, 1999)
Of Markets . . .

- **Stylized Facts of Market Dynamics**
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![Daily S&P 500 Logarithmic Returns](chart.png)
Geometric Brownian Motion

- Geometric Brownian motion model (GBM)

\[ \text{d}S_i(t) = S_i(t)[\mu_i \text{d}t + \sigma_i \text{d}W_i(t)] \]

exhibiting

- Gaussian log-return distributions
- log-prices follow diffusive motion with drift
- no correlations of volatility

- Is the “harmonic oscillator” of Financial Mathematics.
- Is at the heart of the Black-Scholes option pricing method.
- Does not reproduce the key empirical facts of market dynamics.
- Yet, with modifications still widely used in financial industry.
Fixes

- **Phenomenological**
  - Replace Brownian (Gaussian) increments in GBM by fat tailed increments (e.g. Lévy: Mantegna and Stanley, 1994)
  - Add evolution of volatilities ⇒ ARCH/GARCH/stoch. volatility (Engle, 1982; Engle and Bollerslev, 1986; Heston, 1993)

- ... 
  - Typically single asset descriptions; no systemic perspective.

- **Agent based models**
  - e.g. Minority Game (Challet and Zhang, 1997)
  - Percolation models (Stauffer et al 1998, Cont and Bouchaud 2000)
  - Ising models of interacting agents (Iori, 1999; Da Silva Stauffer 2001)

- ... 
  - All need fine-tuning of parameters to reproduce stylized facts.

- **Somehow unsatisfactory.**
Stepping Back – A Gedanken-Experiment

- **Question**
  Can we, just by looking at the basic structure of the problem of describing market dynamics, obtain guidance about fundamental properties any good model of market dynamics should have?

- To answer this question, let us perform a Gedanken-Experiment. It runs like this:

- Suppose I knew everything about markets, and when I say this, I mean really everything!
Stepping Back – A Gedanken-Experiment

- I would write down the complete set of dynamical equations describing all processes governing a market.
  
  (basic economic laws, influence of supply and demand, effect of regulatory frameworks, psychology of traders, financial positions of trading agents, laws of order book dynamics, \ldots).

- Suppose that I would integrate out all degrees of freedom from my equations, except prices of assets traded in the market.

- Which properties would the reduced model necessarily have?

- It would
  
  - exhibit interactions between prices
  - exhibit a non-Markovian dynamics

\[ \Rightarrow: \] Formulate the simplest model with these properties.

Starting Point — GBM

- Recall GBM

\[ dS_i(t) = S_i(t)\left[\mu_i dt + \sigma_i dW_i(t)\right] \]

- Transform to log-prices \( u_i(t) = \log\left[\frac{S_i(t)}{S_{i0}}\right] \). Gives

\[ du_i(t) = I_i dt + \sigma_i dW_i(t) \]

with (lto) \( I_i = \mu_i - \frac{1}{2} \sigma_i^2 \).
A Minimal Model of Interacting Prices – iGBM

Generalization

\[ d u_i(t) = I_i dt + \sigma_i dW_i(t) \]
\[ + \left[ -\kappa_i u_i(t) + \sum_j J_{ij} \bar{g}_j(t) + \sigma_0 u_0(t) \right] dt , \]

\[ \bar{g}_j(t) = \int_0^t M(t - s) g(u_j(s)) \]

\[\Leftrightarrow\] interacting geometric Brownian motion model (iGBM),

with

- the \(\kappa_i\) producing a mean reversion effect,
- the \(J_{ij}\) describing strengths of interactions between assets,
- the \(g = g(\cdot)\) denoting non-linear functions (e.g. sigmoid) describing the nature of the feedback,
- the \(u_0(t)\) assumed to be a slow process describing the evolution of macro-economic conditions (model as slow OU process)
iGBM and Neural Networks — Brains and Markets

- iGBM and Neural Networks

\[ d u_i(t) = I_i dt + \sigma_i dW_i(t) \]
\[ + \left[ -\kappa_i u_i(t) + \sum_j J_{ij} \bar{g}_j(t) + \sigma_0 u_0(t) \right] dt. \]

\[ \bar{g}_j(t) = \int_t^t M(t - s) g(u_j(s)) \]

- Describes dynamics of a network of graded response neurons, with
  - the \( u_i \) denoting trans-membrane voltages,
  - the \( \kappa_i \) describing leakage across the membrane,
  - the \( J_{ij} \) denoting synaptic couplings,
  - the \( g(\cdot) \) being nonlinear functions (e.g. sigmoid) which describe the firing-rate of the neuron as a function of trans-membrane voltage,
  - the \( I_i \) describing external signals.
  - the function \( u_0(t) \) representing the effect of neuro-modulators.
Brains and Markets

- What we know..
  - iGBM is spin-glass like model (soft SK) with time-varying magnetic fields.
  - For symmetric couplings Lyapunov function for noisless dynamics (Hopfield 1984)
  - ... with a very large number of meta-stable states (Waugh et al, 1990, Fukai and Shiino 1990)
  - Phase diagrams for Hebb-Hopfield couplings (RK, S. Bös, JL van Hemmen 1991+93)
  - Self-consistent analysis for non-symmetric couplings (Shiino & Fukai 1992)
  - For fully asymmetric couplings, noiseless dynamics is chaotic (Sompolinsky, Chrisanti & Sommers 1988, Molgedey, Schuchart & Schuster 1992)
  - Captures market phenomenology (RK, P. Neu 2008)

- Analysis so far: Markovian approximation; $\bar{g}_j(t) = g(u_j(t))$
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Analysis — Generating Functionals

- Use generating functionals \( n_i(t) = g(u_i(t)) \)

\[
Z[\ell|u_0] = \left\langle \exp \left\{ -i \int dt \sum_i \ell_i(t)n_i(t) \right\} \right\rangle,
\]

- Averaging over couplings maps problem onto a family of effective single node problems,

\[
\dot{u}_\vartheta(t) = -\kappa u_\vartheta(t) + I + J_0 m(t) + \sigma_0 u_0(t) \\
+ \alpha J^2 \int_0^t ds G(t, s) n_\vartheta(s) + \phi_\vartheta(t),
\]

with \( \vartheta \equiv (I, \kappa, \sigma) \) used as shorthand for site-random quanities. Here \( \phi_\vartheta \) is couloured noise with

\[
\left\langle \phi_\vartheta(t) \right\rangle = 0 \\
\left\langle \phi_\vartheta(t) \phi_{\vartheta'}(s) \right\rangle = \delta_{\vartheta,\vartheta'} \left[ \sigma^2 \delta(t - s) + J^2 q(t, s) \right].
\]

Order-parameters are coupled through a set of self-consistency equations.
Self-Consistency Equations

- Self-consistency equations, \((n_\vartheta(t) = g(u_\vartheta(t)))\)

\[
m(t) = \left\langle \frac{\langle n_\vartheta(t) \rangle_{\phi_\vartheta}}{\delta \phi(s)} \right\rangle_{\vartheta},
\]

\[
q(t, s) = \left\langle \frac{\langle n_\vartheta(t) n_\vartheta(s) \rangle_{\phi_\vartheta}}{\delta \phi(s)} \right\rangle_{\vartheta},
\]

\[
G(t, s) = \left\langle \frac{\langle n_\vartheta(t) \rangle_{\phi_\vartheta}}{\delta \phi(s)} \right\rangle_{\vartheta}.
\]

- Inner averages over noise \(\phi_\vartheta\) evaluated using path-integral techniques (with an action that is a functional of \(m, q, \text{ and } G\)).
Assume macro-economic process $u_0(t)$ changes slowly: e.g.

$$du_0 = -\gamma u_0 dt + \sqrt{2\gamma} dW_0 , \quad \gamma \ll 1 ,$$

...so that the system becomes statistically stationary at given $u_0$

Derive FPEs for stationary states $\Rightarrow u_\varphi$ OU process

$$m = \left\langle \left\langle g(\bar{u}_\varphi + \sigma_{u\varphi} x) \right\rangle_x \right\rangle_\varphi ,$$

$$q(\tau) = \left\langle \left\langle g(\bar{u}_\varphi + \sigma_{u\varphi} x) \ g(\bar{u}_\varphi + \sigma_{u\varphi} y) \right\rangle_{xy} \right\rangle_\varphi ,$$

$$\chi = \left\langle \left\langle g'(\bar{u}_\varphi + \sigma_{u\varphi} x) \right\rangle_x \right\rangle_\varphi ,$$

$$\hat{C}(0) = \int_{-\infty}^{+\infty} d\tau \left[ q(\tau) - q \right] ,$$

in which $q = \lim_{\tau \to \infty} q(\tau)$, and $\chi$ is an integrated response.

For details, see K Anand, J Khedair, and RK, PRE 97 052312 (2018).
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System exhibits a glassy phase in large parts of parameter space (sufficiently small $J_0/J$, sufficiently small noise $\sigma_i \equiv \sigma$).

FM-SG boundary for $I \sim \mathcal{N}(0, \sigma^2_I)$, $u_0 = 0; J = 0.5$, $\alpha = 0.5$ $\sigma = 0.1$. 
Return Distributions

- Compute distribution of returns

\[ \Delta u_\vartheta \equiv u_\vartheta(t) - u_\vartheta(t') \]

in the quasi-stationary regime \( \gamma |t - t'| \ll 1 \).

- For individual \( u_\vartheta \) find

\[ \Delta u_\vartheta \sim \mathcal{N} \left( 0, \frac{\sigma^2}{\kappa} \left( 1 - e^{-\kappa |t-t'|} \right) \right). \]

- Time-scales (i) short: \( \kappa |t - t'| \ll 1 \), (ii) medium: \( \kappa |t - t'| = \mathcal{O}(1) \), (iii) long: \( \kappa |t - t'| \gg 1 \).

- Assuming the \( \kappa \) are \( \Gamma \)-distributed

\[ P(\kappa) = \frac{1}{\kappa_0 \Gamma(\nu)} \left( \frac{\kappa}{\kappa_0} \right)^{\nu-1} \exp(-\kappa/\kappa_0), \]

distribution of returns over \( \vartheta \)-ensemble (i.e. the market) long times:

\[ p(\Delta u) = \frac{\sqrt{\kappa_0}}{\sqrt{2\pi\sigma^2}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu)} \left( 1 + \frac{\kappa_0 (\Delta u)^2}{2\sigma^2} \right)^{-\left(\nu + \frac{1}{2}\right)}. \]

\( \Rightarrow \) fat power-law tails.
Return Distributions

- Distribution of returns

Distribution of returns for $\kappa_0 |t - t'| = 20$. $J_0 = J = \alpha = 0.5$, long time asymptotics (full line) and numerical evaluation (dashed), $\nu = 1$. 

$\text{Distribution of returns for } \kappa_0 |t - t'| = 20. J_0 = J = \alpha = 0.5$, long time asymptotics (full line) and numerical evaluation (dashed), $\nu = 1.$
Collective Pricing

- Quasi-stationary equilibrium log-prices $\bar{u}_{\vartheta}$ determined by collective effects

Distributions of equilibrium log-prices. Left: Non-interacting system Right: Interacting system. Narrow blue curves $\kappa = 0.5$, $u_0 = 0.1$, Wider set of curves: $\kappa = 0.2$ and $u_0 = 0.1$ for the nearly symmetric (black) curves; $u_0 = 0.5$ for the more asymmetric (red) curves. Overall $\Gamma$ distributed $\kappa$ with $\nu = 1$ and $\kappa_0 = 0.2$. Interacting system $J_0 = J = \alpha = 0.5$
Volatility Clustering and Metastability

- Embed attractors of known structure

\[
J_{ij} \rightarrow J_{ij} + \frac{1}{N} \sum_{\mu=1}^{p} \xi_{i}^{\mu} \xi_{j}^{\mu}
\]

\[
m_{\mu}(t) = \frac{1}{N} \sum_{i} \xi_{i}^{\mu} g(u_{i}(t))
\]

Top: changes of the market index for \( \Delta t = 25 \). Bottom: overlaps with three unbiased random patterns embedded in a system of \( N = 50 \) assets, with \( \gamma = 10^{-4} \).
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Simple ML Approach

- Use model to test inference algorithms and identify strengths/weaknesses
- In second step apply to real data (S & P 500)
- Log-likelihood

$$\mathcal{L} = \sum_{i,t} \frac{1}{2\sigma_i^2} \left[ \dot{u}_i - f_i(u(t)) \right]^2$$

with

$$f_i(u(t)) = -\kappa_i u_i + I_i + \sum_j J_{ij} g(u_j) + \sigma_0 u_0$$

Parameters $\theta = \{\kappa_i, I_i, J_{ij}\}$

- Use stochastic gradient descent or data batches to solve $\nabla_\theta \mathcal{L} = 0$.
  Second method gives linear equations with coefficients determined by various sample-correlations.
- Issues: (i) sampling noise, (ii) non-ergodicity of the dynamics.
ML equations require inversions of various correlation matrices that are estimated, sampling noise $\Rightarrow$ random Matrices. Shown are (left) spectra of estimated correlation matrices $C_{ij} = \langle \delta g(u_i) \delta g(u_j) \rangle$, compared with Marčenko Pastur law, (middle) corresponding scatter-plots of $\hat{J}$ vs. $J_{\text{true}}$ and (right) MP law and learnability of first principal component of $C$-matrix. Here $N = 125$, and $\alpha = N/T$. 

\[ \alpha = \frac{N}{T} = \frac{30}{36} \]
Issue (ii): Non-Ergodicity

- System dynamics is non-ergodic.
- Learning couplings requires to sample sufficiently many ergodic components.
- For fixed data sample size this depends on ergodic time-scale $\gamma^{-1}$.

(Left): Normalized error of couplings in gradient descent learning as function of number of iterations for various $\gamma$. (Right): Scatter plots of estimated vs initial couplings for a partially learnt situation ($\gamma = 0.001$) and a fully learnt situation ($\gamma = 0.01$). Parameters are $N = 125$ and $T = 10^4$.
Learning the interactions — Synthetic Data

Scatter plots of estimated vs true couplings. Parameters are and Left: partially learnt couplings ($\gamma = 10^{-4}$); right: fully learnt couplings ($\gamma = 0.1$). Parameters: $N = 100$, $T = 10^6$
Real-Data — Tentative

- Lots of issues (splits, discontinued trading, outlisting)
- So far no jumps in model; but can set up inference nonetheless: (jumps don’t affect estimates of couplings, drifts, mean-reversions)
- Inferred model reproduces some global properties of real data, such as distributions of log-returns, spectra of correlation matrices of return with reasonable accuracy

(Left): Spectrum of correlation matrix of true returns and of correlation matrix of returns generated from inferred model. 
(Right): distribution of normalized true log-returns compared with distribution generated from inferred model. $N = 200$, $\alpha = 0.03$. Define $u_0$ as projection of $u(t)$ on first principal component $v_1$ of $C^{uu} = (\langle \delta u_i \delta u_j \rangle)$. 
Real-Data — Market States?

(Top): Overlap of market state with 3 selected singular vectors of the inferred interaction matrix as a function of time for a 5y period. (Bottom): Concurrent changes of the index. The period includes two major restructurings overlapping with the Draghi speech 26/07/12 and with the flash crash of 24/08/15.
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Summary

- Argued
  - that market model formulated in terms of asset prices should exhibit interactions between prices, which exhibit memory.
  - simplest interacting generalization of GBM has structure of a NN
- Expect generally many meta-stable phases.
- Different susceptibilities within phases entail different volatilities.
- Find key properties of market dynamics in (semi-)quantitative fashion.
- Fat tailed return distributions, non-trivial equilibrium pricing distributions
- Clear relation between volatilities and meta-stable states.
- Started inference (synthetic and real data)
  - issues of sampling noise and non-ergodicity
  - real data reasonably well reproduced by simple inferred model
  - of use for risk-management?