Pathwise volatility: Cox-Ingersoll-Ross initial-value problems and their fast reversion exit-time limits

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Motivation

- We do not fully-understand even the most basic processes, like OU and CIR, as their reversion speed tends to infinity
- ‘Rough’ models of volatility implicitly depend on infinitesimal reversionary timescales
- We therefore lack the knowledge to draw sensible comparisons between, say, the Heston and rough Heston processes
- I believe the ‘successes’ of rough models are presently overstated for this reason

- Goal: to *truly* understand fast-reverting processes (in particular CIR) and *explain* the result of Mechkov (2015)
Article summary

- In arXiv:1902.01673, the problem of understanding the CIR SDE solution as reversion tends to infinity is mapped, via a time-change equation (TCE), onto an IVP:

$$\text{SDE} : (C, C, \mathbb{W}) \rightarrow \text{TCE} : (C, C, \mathbb{P}) \rightarrow \text{IVP} : (C, C).$$

- This ultimately provides an example of Skorokhod’s representation theorem (weak to pathwise convergence)

- The IVP is of type

$$x' = f(t, x), \quad x(0) = 0, \quad f(t, x) = \varepsilon^{-1}(\omega(x) + t - x) + 1$$

with $\omega \in C$ and reversionary timescale $\varepsilon > 0$.

- Solutions $\varphi = \varphi_\varepsilon(\omega)$ correspond to time-averaged volatility trajectories, with $\partial_t \varphi$ corresponding to volatility trajectories.

- Despite no constraints on $\omega \in C$ (e.g. Lipschitz, Hölder), existence, uniqueness, bounds and limits are established.
In particular, pathwise exit-time limits for solutions $\varphi$ are established:
Article summary

Corollaries include:

- Pathwise convergence of the time-averaged CIR process (actually a time-changed version) to the inverse-Gaussian Lévy subordinator on the \((D, M_1)\) topology
- Finite dimensional convergence of the Heston process to the normal inverse-Gaussian process of Barndorff et al. (2001)
- Pathwise convergence of the Heston process to a limit on \((E, \mathcal{E})\); the ‘excursion’ topology of Whitt (2002, Ch.15)

- These results clarify that Heston's classical one-factor model of volatility can (for all practical purposes) generate explosive volatility surfaces (ATM skew, curvature, etc.), \textit{when it is parameterised to handle the relevant reversionary timescales:}
Essentially-explosive implied volatility surfaces