A geometric Jacquet-Langlands correspondence for paramodular Siegel threefolds

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Thank the organisers!

1 Introduction

1.1 Modular Forms

Let $\mathfrak{h} := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ denote the complex upper half plane and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ act via

$$\gamma \cdot z = \frac{az+b}{cz+d}.$$

Definition 1.1.1. *Let $f : \mathfrak{h} \to \mathbb{C}$ be an holomorphic function, then we will call f weakly modular of weight k and level $\Gamma \subset SL(2, \mathbb{Z})$ if for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$ we have*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

We can interpret such f as sections of a certain line bundle ω^k on the quotient $Y_\Gamma := \Gamma \setminus \mathfrak{h}$, which is a (non-compact) Riemann surface. We call f a *modular form* if f extends to a section of ω^k on the compactification Y_Γ and a *cusp form* if moreover f vanishes at all the cusps.

Example 1.1.2. Consider the modular form given by the q-expansion

$$f(z) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11})^2,$$

where $q = e^{2\pi i z}$. This is a cusp form of weight 2 and level $\Gamma = \Gamma_0(11) \subset SL(2, \mathbb{Z})$, given by the matrices that are upper triangular mod 11. Then one can show that $f(z)\, \mathrm{d}z$ is a holomorphic differential on $Y_{\Gamma_0(11)}$.

Hecke operators act on modular forms and are they defined either explicitly in terms of q-expansions or in terms of Hecke correspondences on the modular curves Y_Γ. These curves have an interpretation in terms of moduli spaces of elliptic curves, and so do these Hecke operators. For example $Y_{\Gamma_0(11)}$ is the (coarse) moduli space of elliptic curves E equipped with an order 11 subgroup of $E[11]$.

Remark 1.1.3. The space of modular forms of a fixed weight and level is finite dimensional. However, it is a priori not easy to explicitly write down a basis (of normalised eigenforms) and to compute their q-expansions. In the next section we will meet a more finitary object (quaternionic modular forms) and we will see that they are intimately linked with modular forms.

1.2 Eichler's correspondence and quaternionic modular forms

Let $D/\mathbb{Q}$ be a definite quaternion algebra (this means that $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$, the Hamilton quaternions). Then the class set of D is a finite set, which we can write adelically as

$$D^\times \setminus D(\mathbb{A}^\infty)^\times / \hat{\mathcal{O}}_D^\times.$$

A quaternionic modular form is, roughly speaking, a function on this finite set. More generally if $U \subset D(\mathbb{A}^\infty$ is a compact open subgroup and V is a $\mathbb{Q}$-linear representation of D, then an algebraic modular form of weight V and level U is a $D^\times(\mathbb{Q})$-equivariant locally constant function

$$D(\mathbb{A}^\infty)^\times / U \to V.$$

When $V = \mathbb{Q}$ is the trivial representation and $U = \hat{\mathcal{O}}_D^\times$ then these are just functions from the (finite) class group of D to $\mathbb{Q}$. In general this space is always finite dimensional and is acted on in a natural way by Hecke operators.

Remark 1.2.1. A geometric way of thinking about this definition is that V defines a 'local system' on the finite set $D^\times \setminus D(\mathbb{A}^\infty)^\times / U$ and that the space of quaternionic modular forms is given by its global sections. Hecke operators then come from correspondences acting on this finite set.

Eichler's correspondence (not to be confused with the Eichler-Shimura correspondence) links spaces of modular forms with spaces of algebraic modular forms on certain quaternion algebras D. From a modern perspective, this is one of the first known instances of the Jacquet-Langlands correspondence, which is a correspondence between automorphic representations of Gl_2 and automorphic representations of $D^\times$.

Theorem 1.2.2 (Eichler)**.** *Let B be the unique definite quaternion algebra of discriminant p. Then for $k \geq 2$ there is a Hecke-equivariant injection*

$$\phi : S_k[\Gamma_0(p)]^{new} \to \mathcal{A}_k(B^\times, \hat{\mathcal{O}}_B^\times),$$

which is surjective for $k \geq 3$ (the cokernel is one-dimensional when $k = 2$).

Example 1.2.3. When $k = 2$ and $p = 11$ then $S_2[\Gamma_0(11)]^{\mathrm{new}}$ is one-dimensional, spanned by the modular form of Example 1.1.2. It turns out that in this case $\mathcal{A}_2(B^\times, \mathcal{O}_B^\times)$ is two-dimensional, spanned by the image of ϕ and by the constant algebraic modular forms.

1.3 Serre's geometrical interpretation

The main observation of this section is the following: A supersingular elliptic curve over $\overline{\mathbb{F}}_p$ has endormorphism algebra given by B. Moreover, there is a unique such supersingular elliptic curve up to isogeny and this will allow us to write the supersingular locus of the moduli space of elliptic curves over $\overline{\mathbb{F}}_p$ as (remark about stacks vs coarse spaces etc):

$$B^\times \setminus B(\mathbb{A}^\infty)^\times / \hat{\mathcal{O}}_B^\times.$$

To be precise, if we choose a base supersingular elliptic curve E, then we can identify all the other elliptic curves as lattices inside the adelic Tate module of E (up to local equivalences). The adelic double coset above precisely classifies such equivalence classes of lattices.

The next ingredient we need is the ℓ-adic Galois representation associated to modular forms:

Theorem 1.3.1 (Deligne)**.** *Given a normalised eigenform $f \in S_k[\Gamma_0(N)]$ there is a unique ℓ-adic Galois representations $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_2(\overline{\mathbb{Q}_\ell})$ such that:*

- *ρ_f is unramified outside ℓN and such that for $p \nmid \ell N$*
- *For primes $p \nmid \ell N$ the trace of the element $\rho_f(\mathrm{Frob}_p)$ is given by the algebraic number a_p (the p-th Fourier coefficient of f).*

The upshot of this theorem is that ρ_f knows everything about f and vice versa. Moreover, we can show that (up to Eisenstein things)

$$H^1_{\text{ét}}(Y_0(N), \overline{\mathbb{Q}}_\ell) \cong \bigoplus_f \rho_{f,\ell},$$

where f runs over a basis of normalised eigenforms of $S_2[\Gamma_0(N)]$.

1.4 Picard-Lefschetz

In order to construct Eichler's correspondence geometrically, we want to consider the characteristic p fiber of $Y_{\Gamma_0(p)}$. Why? Because the cohomology of the generic fibre contains $\rho_{f,\ell}$ for f that are newforms at p, and because the geometry of the special fiber is related to the adelic double coset for the definite quaternion algebra B.

However, $Y_{\Gamma_0(p)}$ does not have good reduction, but its bad reduction is well understood. DRAW THE STANDARD PICTURE, explain the link to the supersingular locus. Explain that we expect to see 'vanishing cycles' in the generic fibre for every singular point (DRAW THE SYMPLECTIC PICTURE). Let $Y/\mathbb{Z}_p$ be the integral model with generic fibre Y_η and special fibre Y_s, then we get a long exact sequence (modulo some lies)

$$0 \to H^1(Y_{\overline{s}}, \overline{\mathbb{Q}}_\ell) \to H^1(Y_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to \mathcal{A}_k(B^\times, \mathcal{O}_B^\times) \to H^2(Y_{\overline{s}}, \overline{\mathbb{Q}}_\ell) \to H^2(Y_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to 0$$

We know that $H^2(Y_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ is one-dimensional ($Y_{\overline{\eta}}$ is a smooth curve) and that $H^2(Y_{\overline{s}}, \overline{\mathbb{Q}}_\ell)$ is two-dimensional (because $Y_{\overline{s}}$ has two irreducible components). Moreover we can check (inertia act by Dehn twists along vanishing cycles) that only the ramified Galois representations $\rho_{f,\ell}$ map nontrivally to $\mathcal{A}_k(B^\times, \mathcal{O}_B^\times)$.

Remark 1.4.1. Explain why this produces Eichler's correspondence and ASK FOR QUESTIONS!

2 Siegel modular forms of genus 2

Siegel modular forms of genus 2 are a natural generalisation of modular forms and the theory is obtained by:

- Replacing $\mathfrak{h}$ with the Siegel upper half space
$$\mathfrak{h}^2 := \{z = x + iy \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) | z \text{ symmetric and } y \text{ positive definite}\};$$
- replacing $\mathrm{Gl}_2(\mathbb{R})$ acting on $\mathfrak{h}$ by $\mathrm{GSp}_4(\mathbb{R})$ acting on $\mathfrak{h}^2$ via (where a, b, c, d are 2×2 block matrices)
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$
- Replacing the automorphy factor $(cz + d)^k$ by the automorphy factors
$$\mathrm{Sym}^k(cz + d) \otimes \det{}^j(cz + d)$$
for $k \geq 0, j \geq 0$.

2.1 Ibukiyama's correspondence

Ihara first predicted that there should be a genus 2 analogue of Eichler's correspondence. He essentially defined algebraic modular forms for the group $G = GU_2(D)$ of skew-Hermitian similitudes of D^2(with D a definite quaternion algebra), defined L-functions for them and suggested that they should be related to the L-functions of Siegel modular forms. It was his student Ibukiyama that later stated precise conjectures about this relation, which we will state below. Let $S_{k,j}[K(p)]$ denote the space of Siegel modular forms of weight k, j and level $K(p)$, where $K(p)$ is the paramodular group given by

$$K(N) = \left\{ g \in \mathrm{GSp}_4(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

Let $\mathcal{A}^G_{k,j}[K_2(p)]$ denote the space of algebraic modular forms for G of weight k, j and level $K_2(p)$, where $K_2(p)$ is an analogue of the paramodular group.

Conjecture 1 (Ibukiyama)**.** *For $k \geq 0$ and for $j \geq 3$ there is an injective map*

$$S_{k,j}[K(p)]^{new} \hookrightarrow \mathcal{A}^G_{k,j}[K_2(p)],$$

which is Hecke-equivariant for the prime-to-p Hecke operators.

Remark 2.1.1. Ibukiyama proved that for most k, j, the dimensions of the left hand side equals the dimension of the right hand side by finding explicit dimension formulae.

Remark 2.1.2. This conjecture can be used to do explicit computations with Siegel modular forms, we give some examples: Lassina Dembele describes an algorithm for computing Hecke eigenvalues of algebraic modular forms for G, which by the conjecture computes Hecke eigenvalues of genus 2 Siegel modular forms. These sorts of computations are important because they can lead to new conjectures (c.f. Lansky-Pollack).

Theorem 2.1.3 (-)**.** *The conjecture is true (and we can say much more)*

2.2 Geometry of paramodular Siegel threefolds

Let $\mathcal{A}_{2,p}$ be the Siegel threefold of level $K(p)$, or in other words, the moduli space of abelian surfaces A with a polarisation λ of degree p. Then one can show that the Galois representations associated normalised eigenforms in $S_{k,j}[K(p)]$ 'occur' in

$$H^3(\mathcal{A}_{2,p,\overline{\mathbb{Q}}}, \mathbb{V}).$$

Moreover, we have the following results on the geometry of $\mathcal{A}_{2,p}$ over $\overline{\mathbb{F}}_p$

Theorem 2.2.1 (Yu)**.** *The special fiber $\mathcal{A}_{2,p,\overline{\mathbb{F}}_p}$ has isolated singularities, given precisely by those pairs (A, λ) such that $\ker \lambda \cong \alpha_p \times \alpha_p$ (so the corresponding abelian surfaces are automatically superspecial). Moreover, the set Σ of singular points is canonically in bijection with the adelic double coset*

$$G(\mathbb{Q}) \setminus G(\mathbb{A}^\infty)/K_2(p),$$

i.e., a certain set of equivalence classes of skew-Hermitian D-modules.

Yu moreover shows that the singular points are ordinary double points, so that we can once again apply the Picard-Lefschetz formula. We then get a long-exact sequence

$$insert$$

Note that we do not know very much about $H^4(\mathcal{A}_{2,p,\overline{\mathbb{F}}_p}, \mathbb{V})$ and most of the work goes into showing that this isn't too large. A geometric argument will actually show that the image of α can be identified with

$$H^4(\mathcal{A}_{2,\overline{\mathbb{F}}_p}) = H^4(\mathcal{A}_{2,\overline{\mathbb{Q}}_p})$$

and here we can apply cohomological vanishing results of Faltings (for $k > 0, j > 3$).

Remark 2.2.2. Unfortunately $\mathcal{A}_{2,p}$ is not proper, so one cannot, a priori, use the theory of nearby cycles. However results of Lan and Stroh allow us to 'pretend' that $\mathcal{A}_{2,p}$ is proper.

Remark 2.2.3. Another problem is that we still have an interpretation of α in terms of 'Dehn twists along the vanishing cycle', but we don't have very much control over the action of inertia. So along the way we also have to prove the weight-monodromy conjecture for

$$H^3(\mathcal{A}_{2,p,\overline{\mathbb{Q}}_p}, \mathbb{V}_{k,j}).$$