Igusa varieties & Mantovan's formula

Pol van Hoften
pol.van_hoften@kcl.ac.uk

June 26, 2019

1 Introduction

In this talk we will study the geometry of the Newton stratification of the characteristic p fibre of a PEL Shimura datum at a prime of good reduction. Rapoport-Zink uniformisation describes the basic locus (the smallest Newton stratum) completely in terms of Rapoport-Zink spaces, i.e., as a quotient. In general if we choose a point x in a Newton stratum corresponding to an abelian variety with extra structures A, then the Rapoport-Zink space uniformises the locus of abelian varieties B such that A is isogenous to B via a p-power isogeny (an *isogeny leaf*). For example if we try to uniformise the ordinary locus of the modular curve this way, we only cover finitely many points. This suggests that the missing ingredient is a prime-to-p 'direction' of the Rapoport-Zink spaces.

The idea of Oort [Oor04] is to consider subsets (leaves) of Newton strata where the p-divisible group is constant. Equivalently if we fix an abelian variety A (with extra structures) then it is the subset consisting of abelian varieties B (with extra structures) that are prime-to-p isogenous to A (isogenies that commute with the extra structures). For example for the modular curve the whole ordinary locus is such a leaf. In general (if we go away from the modular curve), Newton strata are bigger than the leaves and are also not uniformised by Rapoport-Zink spaces. Oort shows in the case of Siegel modular varieties that every Newton stratum is a product of a leaf and an isogeny leaf. Mantovan generalised this to general PEL type Shimura varieties in [Man05] (see also [Man04]).

The reason that this is interesting has to do with computing the cohomology of Shimura varieties. Harris and Taylor use Igusa varieties in [HT01] in their proof of the local Langlands conjecture for GL_n (in their case the leaves are equal to the whole Newton stratum, which explains why there work predates Oort). Shin [Shi09; Shi10] uses Mantovan's work to describe the cohomology of Igusa varieties. The main point is to compare the Grothendieck-Lefschetz trace formula with Arthur's trace formula (don't ask me what this means precisely).

2 PEL type Shimura Varieties

Here we mostly follow Section 2 of [Man05], but see [Mil; Lan13]. Let p be a prime number (we probably need $p > 2$ to be safe), in this section we will discuss PEL type Shimura varieties that have good reduction 'at p'.

2.1 PEL data

We will fix the following types of data (PEL datum):

- A finite-dimensional simple algebra B over $\mathbb{Q}$ with a positive involution $*$;
- an $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B whose p-adic completion is a maximal order inside $B_{\mathbb{Q}_p}$;
- a finitely-generated (left) B-module V with a nondegenerate, alternating and $*$-Hermitian pairing $\langle -, - \rangle : V \times V \to \mathbb{Q}$

such that the following conditions hold (the PEL datum is 'unramified'):

- There is a lattice Λ in $V_{\mathbb{Q}_p}$ which is preserved by $*$ and self-dual under the pairing $\langle -, - \rangle$;
- the base change $B_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

Example 2.1.1. (Siegel modular varieties) Let $B = \mathbb{Q}$ with $\star = \mathrm{Id}$ and let $V = \mathbb{Q}^{2n}$ with the standard symplectic form

Example 2.1.2. (Shimura curves over $\mathbb{Q}$) Let $B/\mathbb{Q}$ be an indefinite quaternion algebra with canonical involution $\star$, let $V = B$ and let p be any prime such that $B_{\mathbb{Q}_p} \cong M_2(\mathbb{Q}_p)$.

Example 2.1.3. (Picard modular varieties) Let $B = E/\mathbb{Q}$ be an imaginary quadratic extension and let $\star$ equal complex conjugation. Let $V = \mathcal{E}^n$ with pairing defined by the matrix

$$\begin{pmatrix} & & 1_{b\times b} \\ & \epsilon 1_{(a-b)\times(a-b)} & \\ 1_{b\times b} & & \end{pmatrix},$$

where $0 \neq \epsilon \in \mathcal{O}_E$ satisfies $\bar{\epsilon} = -\epsilon$ and where a, b are nonnegative integers satisfying $a + b = n$. Here we can take p to be any prime number that is unramified in E.

2.2 Shimura data

Given a PEL datum as defined in the previous section, we will consider the $\mathbb{Q}$-algebra C defined as $\mathrm{End}_B(V)$ (the B-linear endomorphisms of V). This is a simple algebra over F (because it is a matrix algebra over a division algebra) and it has a adjoint involution $\#$ coming from the pairing $\langle -, - \rangle$. We define an algebraic group $G/\mathbb{Q}$ by its functor of points, for any $\mathbb{Q}$-algebra R we set

$$G(R) := \left\{ x \in (C \otimes_{\mathbb{Q}} R)^\times \mid x \cdot x^\# \in R^\times \right\}.$$

Equivalently, this can be described as (c.f. [Mil, pp. 82])

$$G(R) = \left\{ g \in \mathrm{Gl}_B(V \otimes_{\mathbb{Q}} R) \mid \exists \lambda \in R^\times \text{ s.t. } \langle gv, gw \rangle = \lambda \langle v, w \rangle \right\}.$$

Example 2.2.1. In the first example we find $G = \mathrm{GSp}_{2g}$.

Example 2.2.2. In the second example we have $G = B^\times$

Example 2.2.3. In our third example G is a unitary group of signature (a, b), so we write $G = GU(a, b)$ (but a and b do not determine the group G).

By Proposition 8.12 of [Mil] there exists a morphism of $\mathbb{R}$-algebras

$$\mathbb{C} \to C_{\mathbb{R}}$$

such that $h(\overline{z}) = h(z)^{\#}$ and such that the *symmetric* $\mathbb{R}$-valued form $\langle -, h(i)- \rangle$ on $V_{\mathbb{R}}$ is positive definite. This leads to a morphism

$$\mathbb{C}^{\times} \to C_{\mathbb{R}}^{\times}$$

which is pretty close to a morphism of algebraic groups $\mathbb{S} \to G_{\mathbb{R}}$, i.e., a (weak) Shimura datum.

Proposition 2.2.4 (Proposition 8.14 in [Mil])**.** *There exists a unique conjugacy class of morphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ that is a (weak) Shimura datum.*

The choice of h determines a decomposition of $V_{\mathbb{C}} = V_1 \oplus V_2$ as a $B_{\mathbb{C}}$-module. The complex representation V_1 of B is defined over a number field E, which is called the *reflex field*.

2.3 Shimura varieties

Now let $U^p \subset G(\mathbb{A}^{\infty,p})$ be a neat (c.f. [Lan13, Definition 1.4.1.8]) compact open subgroup. Then there is a smooth quasi-projective algebraic variety Y_U/E (the canonical model) such that

$$X_U(\mathbb{C}) = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}^{\infty})/U^p,$$

where X is the Hermitian symmetric space determined by h (it is $G^{\mathrm{ad}}(\mathbb{R})/\operatorname{stab}(h)$). We will now give a 'moduli description' of Y_U; Due to the failure of the Hasse principle for G, we will in general only produce a smooth quasi-projective moduli space M_U such that

$$Y_U \subset M_U$$

is an open and closed subscheme (so a disjoint union of connected components). Actually it will be convenient to directly give a moduli description of an integral model of M_U over $\mathcal{O}_{E,(p)} := \mathcal{O}_E \otimes Z_{(p)}$. Consider the set-valued functor $\mathcal{F}_{U^p}$ on the category of schemes over $\mathcal{O}_{E,(p)}$ which takes an $\mathcal{O}_{E,(p)}$-scheme T to the set of equivalence classes of quadruples $(A, \lambda, i, \overline{\mu})$, where

- A is an abelian scheme over T;
- $\lambda : A \to A^t$ is a prime-to-p polarisation;
- $i : \mathcal{O}_B \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a ring homomorphism such that the Rosati involution induces the involution $*$ on B and such that for all $b \in \mathcal{O}_B$

$$\det(b, \operatorname{Lie}(A)) = \det(b, V_1);$$

- $\overline{\mu}$ is a U^p level structure (c.f. [Lan13, Definition 1.3.7.1.])

Two such quadruples $(A, \lambda, i, \overline{\mu}), (A', \lambda', i', \overline{\mu}')$ are equivalent if there is an isogeny $f : A \to B$ such that the following diagrams commute

$$\begin{array}{ccc} B & = & B \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{f} & \operatorname{End}(A') \otimes_{\mathbb{Z}} \mathbb{Q} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda'} \\ A^t & \xleftarrow[f^t]{} & (A')^t \end{array}$$

and such that f takes $\overline{\mu}$ to $\overline{\mu}'$. We call an equivalence class of such quadruples an abelian scheme with extra structures, and we often drop λ, i and $\overline{\mu}$ from then notation. The moduli functor $\mathcal{F}_{U^p}$ is represented by a smooth quasi-projective scheme $\mathcal{X}_{U^p}$ over $\mathcal{O}_{E,(p)}$. If we vary U^p then the varieties $\mathcal{X}_{U^p}$ form a projective system endowed with an action of $G(\mathbb{A}^{\infty,p})$.

Now fix a prime $v \mid p$ of E, we let E_v denote the completion of E and let $k \cong \mathbb{F}_q$ denote the residue field of E_v. We define $X_k = X_{U^p}$ over k to be the reduction mod v of $\mathcal{X}_{U^p}$ and let A/X_k be the universal abelian scheme with extra structures. Choose an algebraic closure $\overline{k}$ of k and let $X = X_k \otimes_k \overline{k}$.

3 Newton Stratification

In this section we will quickly recall the Newton stratification of X. The idea is basically that there is a 'discrete invariant' associated to an abelian variety with extra structures (say over an algebraically closed field) and that the loci where this discrete invariant is constant should stratify X. When $G = \mathrm{GSp}_{2\mathrm{g}}$ then this discrete invariant is given by the Newton polygon, or equivalently the slopes of the isocrystal. In general this will be an element $b \in B(G, \mu^{-1})$ as in the previous talk (recall that $B(G, \mu^{-1})$ is a partially ordered set). For a point $x \in X(k)$ (with k algebraically closed) we denote $b(x) \in B(g, \mu^{-1})$ to denote the isomorphism class of the isocrystal with extra structures associated to b. Now let $b \in B(G, \mu^{-1})$ and define

Definition 3.0.1.

$$X^{[b]} := \{x \in X : b(\overline{x}) \leq b\}$$
$$X^{(b)} := \{x \in X : b(\overline{x}) = b\}.$$

Theorem 3.0.2 (Theorem 3.6 of [RR96])**.** *The subset $X^{[b]} \subset X$ is Zariski closed. This implies that*

$$X^{(b)} := \{x \in X : b(\overline{x}) = b\} = X^{[b]} \setminus \bigcup_{b' < b} X^{[b']}$$

is open in $X^{[b]}$ and is therefore locally closed

Remark 3.0.3. It is a Theorem of Hamacher [Ham15] (due to Oort [Oor00] in the Siegel case) that the closure of $X^{(b)}$ is equal to $X^{[b]}$. However, this uses the results that we will describe in the next section, so we cannot use his results!

Example 3.0.4. The Newton stratification on a Shimura curve has two strata: A one-dimensional 'ordinary locus' and a zero-dimensional 'supersingular locus'. (We are only considering primes p where the quaternion algebra splits, and so the p-divisible group G corresponding to the abelian surface will satisfy $G = H^2$ where H is a p-divisible group of height 2. We then call G ordinary if H is ordinary and G supersingular if H is supersingular.)

Example 3.0.5. If $G = \mathrm{GSp}_{2g}$ then the Newton strata can be described by Newton polygons and the partial order is given by the natural partial order on these polygons. For example if $g = 4$ then we have the following Newton strata (this is taken from example 8.1 in [Oor04]):

NP	ξ	f	$sdim(\xi)$	$c(\xi)$	$i(\xi)$
ρ	$(4,0)+(0,4)$	4	10	10	0
$f=3$	$(3,0)+(1,1)+(0,3)$	3	9	9	0
$f=2$	$(2,0)+(2,2)+(0,2)$	2	8	7	1
β	$(1,0)+(2,1)+(1,2)+(0,1)$	1	7	6	1
γ	$(1,0)+(3,3)+(0,1)$	1	6	4	2
δ	$(3,1)+(1,3)$	0	6	5	1
ν	$(2,1)+(1,1)+(1,2)$	0	5	3	2
σ	$(4,4)$	0	4	0	4

Here $\rho \geq (f=3) \geq (f=2) \geq \beta \geq \gamma \geq \nu \geq \sigma$ and $\beta \geq \delta \geq \nu$ and the number f denotes the p-rank of the abelian variety.

Example 3.0.6. S

Remark 3.0.7. Fixing a Barsotti-Tate group Σ with extra structures in an isogeny class $b \in B(G, \mu^{-1})$ defines a subset $C_\Sigma \subset X^(b)$ called a *leaf*. The Newton stratum $X^{(b)}$ will be the union of such leaves and we will define our Igusa varieties as certain covers of these leaves. However, for technical reasons, we cannot just work with an arbitrary Σ and we will spend the next section singling out a particularly nice class of such Σ's.

4 Slope Filtrations

Recall that Barsotti-Tate groups over an algebraically closed field k are classified by their Dieudonné-modules, which are free modules over $W(k)$. After inverting p, these become isocrystals over $W(k)[\frac{1}{p}]$, which were classified in terms of their slopes by Dieudonné and Manin. Below we will define the slopes of a Barsotti-Tate group directly, without appealing to this classification (which does not exist over more general bases).

4.1 Completely slope divisible Barsotti-Tate groups

Definition 4.1.1 ([OZ02]). *Let Σ/S be a Barsotti-Tate group with S of characteristic p. We say that Σ is* completely slope divisible *if there is a filtration by (Barsotti-Tate) subgroups*

$$0 = \Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_m = \Sigma$$

and integers $s, r_1, \cdots, r_m$ with $s \geq r_1 > r_2 > \cdots > r_m \geq 0$ such that the quasi-isogeny

$$p^{-r_i} F^s : \Sigma \to \Sigma^{(p^s)}$$

is an isogeny and such that

$$p^{-r_i} F^s : \Sigma_i/\Sigma_{i-1} \to (\Sigma_i/\Sigma_{i-1})^{(p^s)}$$

is an isomorphism. The rational numbers $\lambda_i = \frac{r_i}{s}$ are called the slopes of Σ; if Σ only has one slope λ we call Σ isoclinic *of slope λ. We will usually write Σ^i for the graded quotients Σ_i/Σ_{i-1}, these are isoclinic of slope λ_i by definition.*

Proposition 4.1.2. *Let k be an algebraically closed field of characteristic p. Then a Barsotti-Tate group Σ is completely slope divisible if and only if*

$$\Sigma = \bigoplus_i \Sigma_i$$

with the Σ_i isoclinic and defined over a finite field.

Remark 4.1.3. The notion of slope defined here agrees with the notion of slope of the isocrystal.

4.2 Existence of slope filtrations

If k is an algebraically closed field of characteristic of characteristic p that is not the algebraic closure of a finite field, then not every Barsotti-Tate group over k is completely slope divisible. However by the Dieudonné-Manin classification it is isogenous to a Barsotti-Tate group defined over a finite field and therefore isogenous to a completely slope divisible group. Something similar will be true for Barsotti-Tate groups Σ/S over nice bases S. However note that a completely slope divisible Barsotti-Tate group Σ/S has constant Newton polygon, i.e., that for all geometric points $\overline{x}$ of S the Barsotti-Tate group $\Sigma_{\overline{x}}$ has the same Newton polygon.

Theorem 4.2.1 (Corollary 2.2 of [OZ02]). *Let Σ/S be a Barsotti-Tate group over a Noetherian integral normal scheme of characteristic p. Then there exists an isogenty $\Sigma \to \Sigma'$ with Σ' completely slope divisible.*

We also record the following proposition which we will use in the next section:

Proposition 4.2.2 (Proposition 2.3 of [OZ02]). *Let S be an integral scheme of characteristic p with function field K and let Σ/S be a Barsotti-Tate group with constant Newton polygon. If Σ_K is completely slope divisible then so is Σ.*

5 Oort's Foliation and Igusa varieties

In this section we will define leaves, show that they are closed and smooth subsets of the Newton strata and then define Igusa varieties as covers of leaves associated to completely slope divisible Barsotti-Tate groups.

Proposition 5.0.1 (Proposition 1 of [Man05]). *Let $\Sigma/\overline{\mathbb{F}}_p$ be a Barsotti-Tate group with extra structures and let $b \in B(G, \mu^{-1})$ be its isogeny class. Define the* leaf

$$C_\Sigma := \{x \in X \mid \mathcal{G}_{\overline{x}} \cong \Sigma_{\overline{x}}\},$$

where $\mathcal{G}/X$ is the universal Barsotti-Tate group with extra structures and the isomorphisms are taken to commute with the extra structures. Then C_Σ is a closed subset of $X^{(b)}$ and it is smooth when given the induced reduced subscheme structure.

Proof. We start by recalling a Lemma of Oort, which we will use to prove that leaves are closed.

Lemma 5.0.2 (Theorem 2.2. of [Oor04]). *Let K be a field of characteristic p and let Σ/K be a Barsotti-Tate group. Let $S \to K$ be an excellent scheme (e.g. finite type over K) and let $\mathcal{G} \to S$ be a Barsotti-Tate group. Then the locus*

$$W_\Sigma(\mathcal{G}/S) := \left\{ s \in S \mid \text{there exists an algebraically closed field } k \supset k(s) \text{ such that } \Sigma \otimes_K k \cong \mathcal{G} \otimes_{k(s)} k \right\}$$

is closed.

If both Σ and $\mathcal{G}$ are equipped with extra (PEL) structures, then there is a subset

$$C_\Sigma(\mathcal{G}/S) \subset W_\Sigma(\mathcal{G}/S)$$

consisting of those points s where there exists an algebraically closed field $k \supset k(s)$ and an isomorphism $\Sigma \otimes_K k \cong \mathcal{G} \otimes_{k(s)} k$ commuting with the extra structures. Now we would like to show that $C_\Sigma(\mathcal{G}/S)$ is a closed subset of $W_\Sigma(\mathcal{G}/S)$. In fact we will show that it is a union of irreducible components of $W_\Sigma(\mathcal{G}/S)$. Let Z be an irreducible component of $W_\Sigma(\mathcal{G}/S)$ such that the generic point of Z lies in $C_\Sigma(\mathcal{G}/S)$. Then by Theorem 1.3 of [Oor04] there is a finite surjective morphism $T \to Z$ such that $\mathcal{G}_T$ is constant over T. It is clear that $\mathcal{G}_T$ must be then isomorphism to Σ_T, so

$$C_\Sigma(\mathcal{G}_T/T) = T.$$

Now note that the formation of $C_\Sigma(\mathcal{G}/S)$ commutes with base change because it is just a condition on geometric fibers. Therefore we find that

$$C_\Sigma(\mathcal{G}/Z)_T \cong T$$

and therefore $C_\Sigma(\mathcal{G}/Z) \cong Z$. We conclude that $C_\Sigma(\mathcal{G}/S)$ is the union of all the irreducible components Z such that the generic point $\eta_Z \in C_\Sigma(\mathcal{G}/Z)$, hence $C_\Sigma(\mathcal{G}/Z)$ must be closed.

Claim 5.0.3. Give C_Σ the induced reduced subscheme structure and let $x \in C_\Sigma$, then $\hat{\mathcal{O}}_{C_\Sigma,x}$ does not depend on x.

Proof of Claim. Let $x, y \in C_\Sigma$ with corresponding abelian varieties with extra structure X, Y and let A and B be the corresponding universal deformation rings (of deformations preserving the extra structures). By Serre-Tate theory and the isomorphism $X[p^\infty] \cong Y[p^\infty]$ we find that $A \cong B$. Furthermore the universal Barsotti-Tate group $\mathcal{H}$ over Spf A extends to a Barsotti-Tate group over Spec A by Lemma 2.4.4 of [Jon95]. Moreover the completed local rings of C_Σ at x and y are given by the quotient of R corresponding to $C_\Sigma(\mathcal{H}/\operatorname{Spec} R)$ which is closed by Lemma 5.0.2. In particular these complete local rings do not depend on x and y. □

Since C_Σ is reduced it is generally smooth which means there is a point $x \in C_\Sigma$ where the complete local ring is a power series ring. But by the claim this means that the complete local ring at every point is isomorphic to this power series ring, proving that C_Σ is smooth (and equidimensional). □

Let us now fix a completely slope divisible group $\Sigma/\overline{\mathbb{F}}_p$ with extra structures and write $\Sigma = \bigoplus_i \Sigma^i$ for its slope decomposition. Then the fact that

$$\operatorname{Isog}(\Sigma) = \prod_i \operatorname{Isog}(\Sigma^i)$$

tells us that the action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on Σ is given by an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on each Σ^i. The decomposition $\Sigma = \bigoplus_i \Sigma^i$ induces a dual decomposition $\Sigma^v = \bigoplus (\Sigma^i)^v$, where the slope of $(\Sigma^i)^v$ is $1 - \lambda_i$. The fact that Σ is polarised means that its Newton polygon symmetric, which means that for all i there exists a j such that $\lambda_i + \lambda_j = 1$. Moreover the polarisation

$$\lambda : \Sigma \to \Sigma^v$$

then induces isomorphisms $\Sigma^i \to (\Sigma^j)^v$ that commute with the extra structures. Now consider the leaf $C_b = C_\Sigma \subset X^{(b)}$ (here $b \in B(G)$ is the isogeny class of Σ).

Lemma 5.0.4. *Let $\mathcal{G}$ be the universal Barsotti-Tate group over C_b, then $\mathcal{G}$ is completely slope divisible.*

Proof. We can check this one connected component at a time, so we can assume that C_b is connected (is it not?). Since C_b is smooth we find that it is integral and so we can consider its function field K. It is clear that $\mathcal{G}_K$ is completely slope divisible (this can be checked over an algebraically closed extension, where $\mathcal{G}_K$ becomes isomorphic to Σ, by). Then Proposition 4.2.2 tells us that $\mathcal{G}$ is completely slope divisible. □

Let us write $0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k$ for this slope filtration (which is unique) and let us write $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$. It follows from the uniqueness of slope filtrations that $(\mathcal{G}^i)_x$ is isomorphic to Σ^i (compatible with extra structures) for all geometric points x of C_b.

Definition 5.0.5. *Let $J_{b,m} \to C_b$ be the scheme representing the functor* $\mathrm{Sch}/C_b \to \mathrm{Set}$ *given by sending S/C_b to the set of isomorphisms $\{\alpha_i : \Sigma^i_S[p^m] \to \mathcal{G}^i_S[p^m]\}_{i=1}^k$ of finite flat group schemes such that:*

- *They commute with the extra structures. In other words the map is $\mathcal{O}_{B_{\mathbb{Q}_p}}$-linear and commutes with the polarisation up to a scalar factor.*
- *They extend étale locally to any level $m' \geq m$.*

We call $J_{b,m} \to C_b$ the Igusa variety of level m.

Remark 5.0.6. The reason that we ask for the isomorphisms to extend étale locally to any level $m' \geq m$ is that there might exist endomorphisms $\Sigma^i \to \Sigma^i$ that don't commute with the extra structures but such that $\Sigma^i[p] \to \Sigma^i[p]$ does commute with the extra structures.

Let Γ_b be the group of automorphisms of Σ that commute with the extra structures. Then Γ_b clearly acts on $J_{b,m}$ via the quotient $\Gamma_b \to \Gamma_{b,m}$ (we quotient out by those automorphisms that are trivial on $\Sigma[p^m]$).

Proposition 5.0.7 (Proposition 3.3 of [Man04]). *For any $m \geq 1$, the Igusa variety $J_{b,m} \to C_b$ is finite étale and Galois with Galois group $\Gamma_{b,m}$. In other words $J_{b,m} \to C_b$ is an étale $\Gamma_{b,m}$-torsor.*

Proof. It is clear that the fibers of $J_{b,m}(K) \to C_b(K)$ are in bijection with the finite group $J_{b,m}$ for K algebraically closed. It is much harder to show that $J_{b,m} \to C_b$ is étale locally trivial, or that $\mathcal{G}^i[p^m] \cong \Sigma^i[p^m]$ étale locally. So let $x \in C_b$ be a point and let R be the strictly Henselian local ring of C_b at x (the inverse limit over all étale neighbourhoods of x), then it suffices to show that there is an isomorphism

$$\Sigma^i \times_{\overline{\mathbb{F}}_p} \operatorname{Spec} R \cong \mathcal{G}^i \times_{C_b} \operatorname{Spec} R,$$

commuting with the extra structures. Corollary 3.4 of [OZ02] tells us that there is an isogeny

$$\Phi : \Sigma^i \times_{\overline{\mathbb{F}}_p} \operatorname{Spec} R \to \mathcal{G}^i \times_{C_b} \operatorname{Spec} R,$$

that commutes with the extra structures. (To be precise the result says that any isogeny over the special fiber can be lifted to a quasi-isogeny over $\operatorname{Spec} R$. Checking that the resulting isogeny commutes with the extra structures means checking that certain equalities hold inside the ring of self-isogenies over $\Sigma / \operatorname{Spec} R$. Now we note that any self-isogeny of Σ is already defined over $\overline{\mathbb{F}_p}$ and so it suffices to check these equalities on the special fiber. So we just have to choose an isogeny $\Sigma^i \to \mathcal{G}^i$ on the special fiber that commutes with the extra structures, and such an isogeny exists because $x \in C_\Sigma$).

Let $d > 0$ be an integer such that the kernel of Φ is contained in $\Sigma^i_R[p^d]$. Then Φ determines an R-point g of the (reduced) Rapoport-Zink space $\overline{M}^{0,d}_{\Sigma^i}$ and if we can show that $g(\operatorname{Spec} R)$ is a single closed point, then it follows that $\mathcal{G}^i_R \cong \Sigma^i_R$. Note that $g(\operatorname{Spec} R)$ is contained in the subspace

$$Y = \left\{ t \in \overline{M}^{0,d}_{\Sigma^i} \mid \mathcal{H} \times \overline{k(t)} \cong \Sigma^i \times \overline{k(t)} \right\},$$

where $\mathcal{H}/\overline{M}^{0,d}_{\Sigma^i}$ is the universal Barsotti-Tate group. It follows from Lemma 2.6 of [Man04] that Y is a constructible subset of $\overline{M}^{0,d}_{\Sigma^i}$ and one can argue as in [Man04, pp.40] that Y is actually finite. Then the map $\operatorname{Spec} R \to Y$ has to factor through a single closed point because R is integral and Y is finite. □

Remark 5.0.8. There are projection maps $J_{b,m} \to J_{b,m'}$ for $m \geq m'$, given by restricting the isomorphisms to the $[p^{m'}]$-torsion subgroup. Moreover, these are equivariant for the action of Γ_b on $J_{b,m}$ and $J_{b,m'}$. This means that Γ_b acts on the tower $\{J_{b,m}\}_m$ and one can actually extend this action to the action of a certain monoid $\Gamma_b \subset S_b \subset J_b(\mathbb{Q}_p)$.

5.0.9 Truncated reduced Rapoport-Zink spaces

Let us quickly recall what the notation $\overline{M}^{n,d}_{\Sigma^i}$ means, it is the moduli space over $\overline{\mathbb{F}}_p$ whose R points are given by pairs (H, ρ) where

- H is a Barsotti-Tate group with extra structures over R;
- $\rho : \Sigma^i_R \to H$ is a quasi-isogeny that commutes with the extra structures,

such that $p^n \rho$ is an isogeny and such that $\ker p^n \rho \subset \Sigma^i[p^d]$. This is a finite type separated $\overline{\mathbb{F}}_p$-scheme.

6 Product Formula

6.1 Some vague intuition

The idea of the product formula is that one can move in 'two directions' on the newton stratum. More precisely given a point $x \in X^b$ corresponding to an abelian variety A with extra structures one can consider either

- The subset $\mathcal{Z}$ of X^b corresponding to abelian varieties B with extra structures that are prime-to-p isogenous to A. This subset is precisely the leaf $C_x = C_{A[p^\infty]}$.

- The subset $\mathcal{Y}$ of X^b corresponding to abelian varieties B with extra structures that are p-isogenous to A. This subspace is related to the Rapoport-Zink space $\overline{M}_{A[p^\infty]}$ (in fact it is p-adically uniformised by it by).

The moral of the story that I am about to tell is that these two subspaces are 'orthogonal' and so that roughly speaking $X^{(b)} \approx \mathcal{Z} \times \mathcal{Y}$.

6.2 Idea of the construction

Let Σ as before, then we are going to define maps

$$\pi_N : J_{b,m} \times \overline{M}_\Sigma^{n,d} \to X^b$$

indexed by positive integers m, n, d, N satisfying a certain admissibility condition. Let me set all parameters to infinity for a moment and describe what happens on points, then we will spend a while trying to make this work in families. We want to produce an abelian variety B with extra structures from the following data:

- (Igusa) An abelian variety A with extra structures equipped with an isomorphism $\alpha : \Sigma \to A[p^\infty]$ respecting the extra structures.
- (Rapoport-Zink) A Barsotti-Tate group H and an isogeny $\rho : \Sigma \to H$

Here is the construction: We take A and quotient out by $\alpha^{-1}(\ker \rho)$. It is now not so hard to see that we can reach every point in $X^{(b)}$ this way (all points of $X^{(b)}$ are isogenous to Σ after all).

Remark 6.2.1. There are two technical obstructions which make the construction not work in families

1. We don't actually work with isogenies $\rho : \Sigma \to H$ but only with quasi-isogenies. However we can restrict to the subspace $\overline{M}_\Sigma^{\infty,d}$ where $p^d\rho$ is an actual isogeny to fix this (but then our map will depend on d!). Furthermore we will have to restrict to the subspace $\overline{M}_\Sigma^{n,d}$ where $\ker p^d\rho \subset \Sigma[p^n]$ (so now our map also depends on n!)
2. We don't actually have isomorphisms $\Sigma[p^m] \to A[p^m]$, but only $\Sigma^i[p^m]$ to $A[p^m]^i$. We can split the slope filtration (for $\Sigma[p^m]$) by pulling back by Frobenius N times for $N \geq d/\delta B$ (so now our map will also depend on N).

6.3 Key Lemma

We start with an important Lemma

Lemma 6.3.1 (Lemma 8 of [Man05]). *Let G be a Barsotti-Tate group over a scheme S in characteristic p. Suppose that G is completely slope divisible with slope filtration*

$$0 \subset G_1 \subset \cdots \subset G_k = G,$$

and slopes $\lambda_1 > \cdots > \lambda_k$. Let q_i be the denominator of λ_i written in minimal form, let Q be the least common multiple of the q_i, and let $\delta = \min\{\lambda_1 - \lambda_2, \cdots, \lambda_{k-1} - \lambda_k\}$. Then for any $N \geq 0$ there is a *canonical isomorphism*

$$G^{(p^{NQ})}[p^{N\delta Q}] \cong \prod_{i=1}^{k} (G^i)^{(p^{NQ})}[p^{N\delta Q}]$$

that commutes with the extra structures.

Proof. This is Lemma 4.1 of [Man04]. The proof goes by induction on the number of slopes (the case $k = 1$ is vacuous). □

6.4 The actual construction

Now let m, n, d, N as above with $m \geq d$ and $N \geq d/\delta Q$, then we will construct a morphism:

$$\pi_N : J_{m,b} \times \overline{M}_{\Sigma}^{n,d} \to X^{(b)}$$

Equivalently, we will construct a natural transformation of the corresponding moduli problems: Let R be a $\overline{k}$-scheme and let $\phi : R \to J_{m,b} \times \overline{M}_{\Sigma}^{n,d}$, i.e., ϕ gives us:

- An abelian scheme $\mathcal{A}/\operatorname{Spec} R$ with extra structures with associated Barsotti-Tate group with extra structures $\mathcal{G} := A[p^\infty]$.
- Isomorphisms $\alpha_i : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$ that commutes with the extra structures.
- A Barsotti-Tate group $\mathcal{H}/\operatorname{Spec} R$ equipped with a quasi-isogeny $\rho : \Sigma_R \to \mathcal{H}$ that commutes with the extra structures. Moreover we know that $p^n\rho$ is an isogeny, that $\ker(p^n\rho) \subset \Sigma[p^d]$ and that $\Sigma[p^d] \subset \Sigma[p^m]$ since $m \geq d$.
- A canonical isomorphism, commutes with the extra structures,

$$\beta : \mathcal{G}^{(p^{NB})}[p^{N\delta Q}] \to \prod_{i=1}^{k} (\mathcal{G}^i)^{(p^{NB})}[p^{N\delta Q}]$$

Now let $I \subset \mathcal{A}$ be the following (finite flat) group scheme: Start with the kernel of $p^n\rho$, twists by Frobenius NB times, apply β and then apply $\prod_i \alpha^i$. The inequality $N \geq d/\delta Q$ makes it so that $\mathcal{G}[p^d] \subset \mathcal{G}[p^{N\delta Q}]$. In symbols we take

$$I := \prod_i \alpha_i^{-1} \left(\beta \left(\ker \rho \right)^{(p^{NB})} \right).$$

We define

$$\mathcal{B} := \mathcal{A}/I,$$

and we note that $\mathcal{B}$ naturally receives extra structures from A because I is stable under all the extra structures by construction [I am being purposefully vague here]. We have now defined an R-point of $X^{(b)}$ and so defined a morphism

$$\pi_N : J_{m,b} \times \overline{M}_{\Sigma}^{n,d} \to X^{(b)}.$$

6.5 Main result

We end by stating the main result of this talk:

Proposition 6.5.1 (Proposition 11 of [Man05])**.** *For any positive integers m, n, d, N with $m \geq d$ and $N \geq d/\delta B$ there exists a morphism*

$$\pi_N : J_{m,b} \times \overline{M}_{\Sigma}^{n,d} \to X^{(b)}$$

such that

1. *π_N is surjective for m, n, d, N sufficiently large (and $m \geq d, N \geq d/\delta B$) ;*
2. *π_N is quasi-finite;*
3. *π_N is proper.*

Proof.

1. Given a point $x \in X^{(b)}$, there are m, n, d, N sufficiently large (with $m \geq d$ and $N \geq d/\delta Q$) such that $\pi_N^{-1}(x)$ is nonempty. This follows from the fact that there is a p-power isogeny (commuting with the extra structures) from the abelian variety A_x to an abelian variety A_y, such that $y \in C_\Sigma$. In fact since $X^{(b)}$ is of finite type we can find a uniform m, n, d, N such that π_N is surjective. (Because π_N is proper the image is closed, so we can just take m, n, d, N sufficiently large so that the image contains all the generic points of irreducible components of $X^{(b)}$, since there are finitely many of those).
2. This is Proposition 4.5 of [Man04]. (Her proof there doesn't take into account extra structures, but the result with extra structures follows from it as the Rapoport-Zink spaces and Igusa varieties with extra structures are subsets of the ones without extra structures).
3. This is Proposition 4.8 of [Man04], the proof of which extends to the PEL case as claimed in the proof of Proposition 11 of [Man05]. The main idea is to use the valuative criterion for properness, but the proof is quite long so we do not include it.

□

References

[Jon95] A. J. de Jong. "Crystalline Dieudonné module theory via formal and rigid geometry". In: *Inst. Hautes Études Sci. Publ. Math.* 82 (1995), 5–96 (1996). ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1995__82__5_0.

[RR96] M. Rapoport and M. Richartz. "On the classification and specialization of F-isocrystals with additional structure". In: *Compositio Math.* 103.2 (1996), pp. 153–181. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1996__103_2_153_0.

[Oor00] Frans Oort. "Newton polygons and formal groups: conjectures by Manin and Grothendieck". In: *Ann. of Math. (2)* 152.1 (2000), pp. 183–206. ISSN: 0003-486X. DOI: 10.2307/2661381. URL: https://doi.org/10.2307/2661381.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*. Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton University Press, Princeton, NJ, 2001, pp. viii+276. ISBN: 0-691-09090-4.

[OZ02] Frans Oort and Thomas Zink. "Families of p-divisible groups with constant Newton polygon". In: *Doc. Math.* 7 (2002), pp. 183–201. ISSN: 1431-0635.

[Man04] Elena Mantovan. "On certain unitary group Shimura varieties". In: *Astérisque* 291 (2004). Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 201–331. ISSN: 0303-1179.

[Oor04] Frans Oort. "Foliations in moduli spaces of abelian varieties". In: *J. Amer. Math. Soc.* 17.2 (2004), pp. 267–296. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-04-00449-7. URL: https://doi.org/10.1090/S0894-0347-04-00449-7.

[Man05] Elena Mantovan. "On the cohomology of certain PEL-type Shimura varieties". In: *Duke Math. J.* 129.3 (2005), pp. 573–610. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-05-12935-0. URL: https://doi.org/10.1215/S0012-7094-05-12935-0.

[Shi09] Sug Woo Shin. "Counting points on Igusa varieties". In: *Duke Math. J.* 146.3 (2009), pp. 509–568. ISSN: 0012-7094. DOI: 10.1215/00127094-2009-004. URL: https://doi.org/10.1215/00127094-2009-004.

[Shi10] Sug Woo Shin. "A stable trace formula for Igusa varieties". In: *J. Inst. Math. Jussieu* 9.4 (2010), pp. 847–895. ISSN: 1474-7480. DOI: 10.1017/S1474748010000046. URL: https://doi.org/10.1017/S1474748010000046.

[Lan13] Kai-Wen Lan. *Arithmetic compactifications of PEL-type Shimura varieties*. Vol. 36. London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2013, pp. xxvi+561. ISBN: 978-0-691-15654-5. DOI: 10.1515/9781400846016. URL: https://doi.org/10.1515/9781400846016.

[Ham15] Paul Hamacher. "The geometry of Newton strata in the reduction modulo p of Shimura varieties of PEL type". In: *Duke Math. J.* 164.15 (2015), pp. 2809–2895. ISSN: 0012-7094. DOI: 10.1215/00127094-3328137. URL: https://doi.org/10.1215/00127094-3328137.

[Mil] J.S. Milne. *Introduction to Shimura Varieties*. URL: https://www.jmilne.org/math/xnotes/svi.pdf.