# Exact Non-Equilibrium Fluctuation Dissipation Relations for Multi-Spin Observables in the Glauber-Ising Spin Chain

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**Abstract.** We investigate the relation between two-time, multi-spin, correlation and response functions in the non-equilibrium dynamics of the Glauber-Ising chain quenched to zero temperature. We find fluctuation-dissipation relations qualitatively similar to those reported in various glassy materials, but quantitatively dependent on the chosen observable. Our results can be understood by considering separately the contributions from large wavevectors, which are at quasi-equilibrium and obey the fluctuation dissipation theorem (FDT), and from small wavevectors where a generalized FDT holds with a non-trivial fluctuation-dissipation ratio $X^\infty$. For spin observables, reflecting critical aspects of the $T=0$ quench, we get $X^\infty = \frac{1}{2}$ while defect observables produce $X^\infty = 0$, revealing the underlying ordered phase. Our results suggest that the definition of an effective temperature $T_{\text{eff}} = T/X^\infty$ for non-equilibrated large length scales is generically possible in non-equilibrium critical dynamics.

## INTRODUCTION

The fluctuation-dissipation theorem (FDT) may be used to define the thermodynamic temperature $T$ of a system in equilibrium. An accumulating body of analytical and numerical results suggest a possibility to extend FDT – and thus the concept of temperature – to non-equilibrium systems. It amounts to the introduction of the fluctuation-dissipation ratio (FDR) $X(t,t_{\text{w}})$ via

$$R(t,t_{\text{w}}) = \frac{X(t,t_{\text{w}})}{T}\frac{\partial}{\partial t_{\text{w}}}C(t,t_{\text{w}}). \qquad (1)$$

Here $C(t,t_{\text{w}}) = \langle O(t)O(t_{\text{w}})\rangle - \langle O(t)\rangle\langle O(t_{\text{w}})\rangle$ denotes the two-time connected correlation function of the observable $O$. The complementary two-time response function is $R(t,t_{\text{w}}) = \delta\langle O(t)\rangle/\delta h(t_{\text{w}})|_{h=0}$, with $h$ the thermodynamically conjugate field to $O$. While in equilibrium one has $X \equiv 1$ according to FDT, it has been found that for non-equilibrium system, such as glassy, coarsening or critical systems, a non-trivial limit form $X(t,t_{\text{w}}) \to X(C(t,t_{\text{w}}))$ emerges at large times.

In the context of $p$-spin models, $T_{\text{eff}} = T/X$ has been shown to play the role of a time-scale dependent effective temperature [1]. Beyond mean-field models, however, the relevance of $X(t,t_{\text{w}})$ remains unclear. In particular the minimum requirement of *observable independence* of the FDR defined by (1) remains an open issue. To address this question we consider the dynamics of multi-spin observables in the one-dimensional Glauber-Ising chain and derive exact results for the FDR.

## MULTI-SPIN FUNCTIONS

The basis of our work is a novel, specifically tailored approach to solve the hierarchy of differential equations for correlation functions [2]. It essentially consists in first solving the inhomogeneous sub-systems of evolution equations for $k$-spin correlations. From that general expressions for arbitrary correlation functions are then constructed in a recursive manner. We also show in [2] that multi-spin two-time correlations

$$C^{(k,l)}_{\mathbf{i},\mathbf{j}}(t,t_{\text{w}}) = \langle \sigma_{i_1}(t)\cdots\sigma_{i_k}(t)\sigma_{j_1}(t_{\text{w}})\cdots\sigma_{j_l}(t_{\text{w}})\rangle \qquad (2)$$

and response functions

$$R^{(k,l)}_{\mathbf{i},\mathbf{j}}(t,t_{\text{w}}) = \left.\frac{\delta\langle\sigma_{i_1}(t)\cdots\sigma_{i_k}(t)\rangle}{\delta h_{\mathbf{j}}(t_{\text{w}})}\right|_{h_{\mathbf{j}}=0}, \qquad (3)$$

where $h_{\mathbf{j}}$ is the conjugate field to $\sigma_{j_1}\cdots\sigma_{j_l}$, satisfy formally equivalent evolution equations as one-time correlations. This allows us to obtain closed exact expressions for multi-spin two-time correlation and response functions after quenching the system from a random initial configuration to zero temperature.

## RESULTS

We focus on the FDR associated with the family of spin observables $O_{\text{s}} = \sum_i \varepsilon_i\sigma_i$ and defect observables $O_{\text{d}} = \sum_i \varepsilon_i\sigma_i\sigma_{i+1}$. The $\varepsilon_i$ are quenched random variables with

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zero mean $[\varepsilon_i]=0$ and translation invariant covariances $[\varepsilon_i\varepsilon_j]=q_{i-j}$; $[\cdot]$ denotes the average over the distribution of $\varepsilon$. By tuning the covariances between $q_n=\delta_{n,0}$ and $q_n=1$ the two-time connected correlation and response functions associated to $O_\mathrm{s}$, $O_\mathrm{d}$ as defined below (1) effectively interpolate between those for local, incoherent observables (individual spins and defects) and the ones for global, coherent observables (magnetisation and energy), respectively. Between the two extremes, we distinguish between short range $\sum_n |q_n|<\infty$ and infinite range $\sum_n |q_n|=\infty$ correlated fields.

Our results [3] show that varying $q_n$ has the same *qualitative* effect on the FDR for both spin and defect observables in the limit of large times: Incoherent observables produce an FDR that continuously crosses over from quasi-equilibrium $X=1$ at $t-t_\mathrm{w}\ll t_\mathrm{w}$ to its asymptotic value

$$X^\infty=\lim_{t_\mathrm{w}\to\infty}\lim_{t\to\infty}X(t,t_\mathrm{w}), \qquad (4)$$

whereas the FDR for coherent observables is constant at large times and coincides with $X^\infty$ (see Fig. 1). Any short range correlated field – apart from pathological cases – eventually leads to the same FDR as the incoherent observable. Infinite range correlated fields yield new intermediate limit forms for $X$ but again cross over from $X=1$ to the very same $X^\infty$.

We explain this robustness of $X^\infty$ by introducing a generic, length scale dependent FDT. This consists in spatially Fourier transforming the correlation and response functions (2) and (3) with respect to $i-j$ and linking them via (1) by the FDR $X(k;t,t_\mathrm{w})$ for modes with wave-vector $k$. In this notation the FDR for the observables $O_\mathrm{s}$, $O_\mathrm{d}$ may be written as

$$X(t,t_\mathrm{w})=\frac{\int \mathrm{d}k\,X(k;t,t_\mathrm{w})\,q(k)\,\frac{\partial}{\partial t_\mathrm{w}}C(k;t,t_\mathrm{w})}{\int \mathrm{d}k\,q(k)\,\frac{\partial}{\partial t_\mathrm{w}}C(k;t,t_\mathrm{w})}. \qquad (5)$$

Based on this representation we show in [3] that independently of the covariances $q_n$ the limit (4) is always dominated by the FDR $X(k=0;t,t_\mathrm{w})$. Therefore the definition of an effective temperature $T_\mathrm{eff}=T/X^\infty$ is generically possible for large, non-equilibrated length scales.

Interestingly it turns out that the *quantitative* values of $X^\infty$ are different for $O_\mathrm{s}$ and $O_\mathrm{d}$. For spin observables one finds $X_\mathrm{s}^\infty=1/2$ whereas defect observables produce $X_\mathrm{d}^\infty=0$. This ambiguity may be related to the fact that spin observables reflect the critical aspects of the $T=0$ quench while defect observables reveal the underlying ferromagnetic phase of the Glauber-Ising chain.

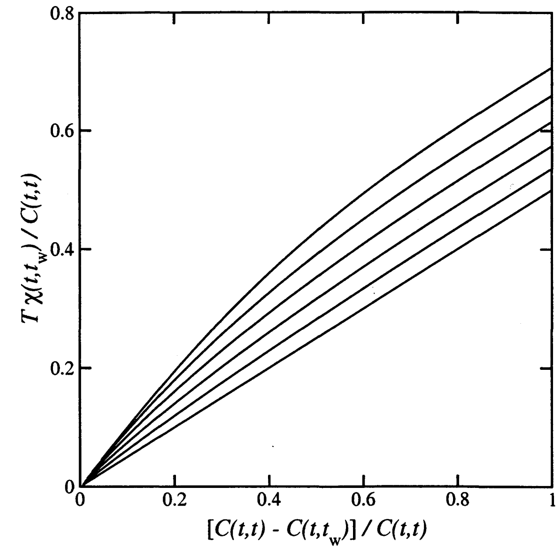


**FIGURE 1.** Long time fluctuation-dissipation limit plots for various spin observables $O_\mathrm{s}$ normalised by $C(t,t)$; here $\chi(t,t_\mathrm{w})=\int_{t_\mathrm{w}}^t \mathrm{d}\tau R(t,\tau)$. The slopes of the curves are given by the FDR $X(t,t_\mathrm{w})$. The top curve is obtained for a local spin, or any observable $O_\mathrm{s}$ with short range correlated fields, while the bottom curve corresponds to the magnetisation. Long range correlated fields that behave asymptotically like $q_n\sim|n|^{-\alpha}$ produce the intermediate curves for $\alpha=0.8,0.6,0.4,0.2$ from top to bottom, respectively. (Figure taken from [3])

## SUMMARY

We have investigated the robustness of fluctuation-dissipation relations in the Glauber-Ising spin chain at $T=0$ for a wide range of observables. While generally observable dependent, the FDT violations fall into well defined classes. All spin observables share the same $X_\mathrm{s}^\infty=\frac{1}{2}$ while $X_\mathrm{d}^\infty=0$ for defect observables. Our results suggest that the definition of an effective temperature $T_\mathrm{eff}=T/X^\infty$ is generically possible for non-equilibrated large length scales in critical systems.


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