Diagrams for dynamics: Some basics

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Abstract

I give a brief introduction to diagrammatic techniques that can be used to analyse dynamical systems defined by nonlinear Langevin equations. The main focus is on the simplest examples, although the methods extend easily to e.g. systems with quenched disorder.

1 Ito vs Stratonovich

Consider the linear Langevin equation

$$\dot{\phi}(t) = -\mu\phi(t) + h(t) + \eta(t) \tag{1}$$

where $h(t)$ is a small field used to probe linear response and η is zero mean (Brownian/thermal noise) with correlation function $\langle \eta(t)\eta(t')\rangle = 2T\delta(t-t')$. This can of course be integrated directly from an initial condition at time $t=0$

$$\phi(t) = \phi(0)e^{-\mu t} + \int_0^t dt'\, e^{-\mu(t-t')}[\eta(t') + h(t')] \tag{2}$$

and by averaging over the noise and, uncorrelated with it, $\phi(0)$, we get the correlation function

$$C(t,t') = \langle \phi(t)\phi(t')\rangle = \frac{T}{\mu}e^{-\mu|t-t'|} + \left(\langle \phi^2(0)\rangle - \frac{T}{\mu}\right)e^{-\mu(t+t')} \tag{3}$$

and the response function

$$R(t,t') = \frac{\partial \langle \phi(t)\rangle}{\partial h(t')} = \Theta(t-t')e^{-\mu(t-t')} \tag{4}$$

We will see below that in the diagrammatic perturbation theory for treating *nonlinear* Langevin equations, we will need the equal time response $R(t,t)$. This is a priori undefined though, since the response function has a step discontinuity at $t'=t$. Using the fact that the average over the distribution of η is with probability weight $P[\eta] \sim \exp[-(4T)^{-1}\int dt\, \eta^2(t)]$, and integrating by parts, we can write

$$R(t,t') = \left\langle \frac{\partial \phi(t)}{\partial \eta(t')}\right\rangle = \frac{1}{2T}\langle \phi(t)\eta(t')\rangle \tag{5}$$

and so our task is to define $\langle \phi(t)\eta(t)\rangle$, an equal time product of fluctuating quantities. There are two basic conventions for doing this, due to Stratonovich and to Ito.

1.1 Stratonovich convention

Idea: Physically, η is a noise process with nonzero correlation time, so we should really write $\langle \eta(t)\eta(t')\rangle = C_\eta(t-t')$ with C_η an even function, decaying quickly to zero for $|t-t'|$ greater than some small correlation time τ_0, and whose integral is $\int dt\, C_\eta(t) = 2T$. Then, setting the field h which we don't need at this point to zero:

$$\phi(t) = \phi(0)e^{-\mu t} + \int_0^t dt'\, e^{-\mu(t-t')}\eta(t') \tag{6}$$

so

$$\langle \phi(t)\eta(t)\rangle = \int_0^t dt'\, e^{-\mu(t-t')} C_\eta(t-t') = \int_{-\infty}^t dt'\, C_\eta(t-t') = T \tag{7}$$

where the second equality follows from the fast decay of C_η on macroscopic timescales of $\mathcal{O}(1/\mu)$ so that we can approximate $e^{-\mu(t-t')} = 1$. For $t' < t$ (more precisely, $t - t' \gg \tau_0$), on the other hand, all of the 'mass' of the correlation function is captured in the integration range; it therefore acts just like a δ-function and we get $\langle \phi(t)\eta(t')\rangle = 2T\exp[-\mu(t-t')]$ as expected from (4,5). So in the Stratonovich convention, the equal time value of the response function is *half* of that obtained in the limit $t' \to t-0$. It is therefore also called the *midpoint rule*; see below.

1.2 Ito convention

The Ito convention effectively assumes that the noise $\eta(t)$ acts "after $\phi(t)$ has been updated", so it sets $\langle \phi(t)\eta(t)\rangle = \lim_{t'\to t+0} \langle \phi(t)\eta(t')\rangle = 0$, and hence also $R(t,t) = 0$.

1.3 Discretization

We will later look at path integral representations of the dynamics and so need a discretization of the stochastic process $\phi(t)$. We'll now see that Ito and Stratonovich can be seen as corresponding to different discretization methods.
Let's discretize time $t = n\Delta$, with Δ a small time step eventually to be taken to zero, and write $\phi_n = \phi(t = n\Delta)$ and $h_n = h(t = n\Delta)$. The noise variables over the interval Δ are $\eta_n = \int_{n\Delta}^{(n+1)\Delta} dt\, \eta(t)$, with $\langle \eta_m \eta_n\rangle = 2T\Delta\delta_{mn}$. Then a suitable discrete version of (1) is

$$\phi_{n+1} - \phi_n = \Delta[(1-\lambda)(-\mu\phi_n + h_n) + \lambda(-\mu\phi_{n+1} + h_{n+1})] + \eta_n \tag{8}$$

for any $\lambda \in [0,1]$; we are here evaluating (the non-noise part of) the right hand side of (1) as a weighted combination of the values at the two ends of the interval Δ. It is easy to solve this linear recursion exactly: From

$$\phi_{n+1}[1 + \Delta\lambda\mu] = \phi_n[1 + \Delta(\lambda - 1)\mu] + \Delta[(1-\lambda)h_n + \lambda h_{n+1}] + \eta_n, \tag{9}$$

and setting

$$c = \frac{1 + \Delta(\lambda - 1)\mu}{1 + \Delta\lambda\mu} \tag{10}$$

we have

$$\begin{aligned} \phi_n &= \sum_{m=0}^{n-1} c^{n-m-1} \frac{\Delta[(1-\lambda)h_m + \lambda h_{m+1}] + \eta_m}{1 + \Delta\lambda\mu} && (11) \\ \langle \phi_n \rangle &= \frac{\Delta}{1 + \Delta\lambda\mu} \left\{ \lambda h_n + \sum_{m=1}^{n-1} \frac{c^{n-m-1}}{1 + \Delta\lambda\mu} h_m + c^{n-1}(1-\lambda)h_0 \right\} && (12) \end{aligned}$$

From this we can read off the response function $R_{nm} = \partial \langle \phi_n \rangle / \partial(\Delta h_m)$; setting $n = t/\Delta$, $m = t'/\Delta$ and taking $\Delta \to 0$ we then get for the continuous time response

$$R(t,t') = \begin{cases} 0 & \text{for} \quad t < t' \\ \lambda & \text{for} \quad t = t' \\ \exp[-\mu(t-t')] & \text{for} \quad t > t' \end{cases} \tag{13}$$

The value of λ only affects the equal time response; we see that $\lambda = 1/2$ gives the Stratonovich convention, while $\lambda = 0$ gives Ito. Note that in the discretization (8), $\lambda = 1/2$ corresponds to evaluating the change in ϕ at the midpoint of the interval between n and $n+1$ (hence "midpoint rule"). Ito ($\lambda = 0$) on the other hand just evaluates at the left point. Note that for different times, $t \neq t'$, both discretization schemes are equivalent as expected.
[Note: In the above discretization I have not maintained the relation between the response function and the $\langle \phi\eta \rangle$ correlator; this could be done by replacing $\eta_n \to (1-\lambda)\eta_n + \lambda\eta_{n+1}$, but isn't necessary for the following development, which will be simpler with the present version.]

1.4 Comments

The two conventions for multiplying equal-time fluctuations are, for the systems we will look at, simply different ways of describing the *same* time evolution $\phi(t)$. Cases where the noise strength is coupled to ϕ, such as in $\dot{\phi} = f(\phi) + g(\phi)\eta$, are more serious: A convention for the equal-time product $g(\phi)\eta$ has to be adopted, and the two conventions here actually give *different* stochastic processes $\phi(t)$; the corresponding Fokker-Planck equations differ by a nontrivial drift term.
For our simpler cases, Ito vs Stratonovich is basically a matter of taste. Stratonovich is the more 'physical' because it corresponds to a noise process η with small but nonzero correlation time; it also obeys all the usual rules for transformation of variables etc. Ito is more obvious from the discretized point of view (it's very much what you'd naively program in a simulation); we'll also see below that it can lead to technical simplifications, and mathematicians prefer it because it gives nice martingale properties.

2 Path integral

Now consider the nonlinear Langevin equation

$$\dot{\phi} = f(\phi) + h + \eta \tag{14}$$

and assume for simplicity that $\phi(0)$. It is straightforward to extend the formalism to systems with several components ϕ_i; the inclusion of distributions of initial values is discussed briefly in Sec. 5. Quenched disorder can also be treated: One averages the generating function (this is unproblematic since the latter is normalized) over the quenched disorder before performing the perturbative expansion.
We discretize as in (8), abbreviating $f_n = f(\phi_n)$:

$$\phi_{n+1} - \phi_n = \Delta[(1-\lambda)(f_n + h_n) + \lambda(f_{n+1} + h_{n+1})] + \eta_n \tag{15}$$

The plan now is to write down a path integral for this process and evaluate the effects of nonlinearities in $f(\phi)$ perturbatively. Let Δ be fixed for now and let M be the largest value of index n that we're interested in. Abbreviate $\phi = (\phi_1 \ldots \phi_M)$; then the relevant partition/generating function is

$$Z = \int d\phi \; P(\phi) \exp\left(i \sum_{n=1}^{M} \theta_n \phi_n\right) \tag{16}$$

from which averages follow by differentiation w.r.t. the θ's around the point $\theta = 0$. Z is normalized at $\theta = 0$ so we can average it over the thermal noise η (as well as quenched disorder if any).
The ϕ are, from (15), in one-to-one relation with the noise variables $\eta = \eta_0 \ldots \eta_{M-1}$. We know the distribution of the latter:

$$P(\eta) = (4\pi T\Delta)^{-M/2} \exp\left[-\frac{1}{4T\Delta} \sum_{n=0}^{M-1} \eta_n^2\right] \tag{17}$$

and so

$$P(\phi) = P(\eta) J(\phi) \tag{18}$$

where $J(\phi) = |\partial\eta/\partial\phi|$ is the Jacobian. Using (15) to express the η in terms of the ϕ,

$$\eta_n = \phi_{n+1} - \phi_n - \Delta[(1-\lambda)(f_n + h_n) + \lambda(f_{n+1} + h_{n+1})] \tag{19}$$

we thus have

$$\begin{aligned} Z \;=\; & (4\pi T\Delta)^{-M/2} \int d\phi \; J(\phi) \exp\left[i \sum_{n=1}^{M} \theta_n \phi_n + \right. \\ & \left. -\frac{1}{4T\Delta} \sum_{n=0}^{M-1} \left(\phi_{n+1} - \phi_n - \Delta[(1-\lambda)(f_n + h_n) + \lambda(f_{n+1} + h_{n+1})]\right)^2 \right] \end{aligned} \tag{20}$$

The square can be decoupled using conjugate integration variables $\hat{\phi} = \hat{\phi}_0 \ldots \hat{\phi}_{M-1}$:

$$
\begin{aligned}
Z &= \int \frac{d\phi\, d\hat{\phi}}{(2\pi)^M}\, J(\phi)\, \exp\left[i \sum_{n=1}^{M} \theta_n \phi_n + \right. \\
&\quad \left. \sum_{n=0}^{M-1} \left(-T\Delta \hat{\phi}_n^2 + i\hat{\phi}_n \left\{ -\phi_{n+1} + \phi_n + \Delta[(1-\lambda)(f_n + h_n) + \lambda(f_{n+1} + h_{n+1})] \right\} \right) \right]
\end{aligned}
\tag{21}
$$

Now it's time to evaluate $J(\phi)$. Consider the matrix $\partial\eta/\partial\phi$, remembering that the index for η runs from 0 to $M-1$, and that for ϕ from 1 to M. The diagonal elements $\partial\eta_n/\partial\phi_{n+1}$ are then, from (19), $1 - \Delta\lambda f'_{n+1}$ (with $f'_n = f'(\phi_n)$); the elements just below the diagonal are $\partial\eta_n/\partial\phi_n = -1 - \Delta(1-\lambda)f'_n$. All other elements are zero; in particular the matrix has all zeros above the diagonal and so its determinant is the product of the diagonal elements, giving

$$
J(\phi) = \prod_{n=1}^{M} (1 - \Delta\lambda f'_n) = \exp\left[\sum_{n=1}^{M} \ln(1 - \Delta\lambda f'_n) \right] = \exp\left[-\Delta\lambda \sum_{n=1}^{M} f'_n \right] \tag{22}
$$

where the last equality anticipates that Δ will be made small so that we can discard $\mathcal{O}(\Delta^2)$ terms in the exponent. Note that for the Ito convention ($\lambda = 0$), $J \equiv 1$ identically and this is one of the reasons for preferring Ito.
[The alternative to the above direct evaluation of $J(\phi)$ is to represent the determinant $|\partial\eta/\partial\phi|$ as an integral over Grassmann variables, leading to a supersymmetric formulation. Alexander may talk about this.]
With $J(\phi)$ evaluated, we have then

$$
\begin{aligned}
Z &= \int \frac{d\phi\, d\hat{\phi}}{(2\pi)^M}\, \exp\left[\sum_{n=1}^{M} (i\theta_n \phi_n - \Delta\lambda f'_n) + \right. \\
&\quad \left. \sum_{n=0}^{M-1} \left(-T\Delta \hat{\phi}_n^2 + i\hat{\phi}_n \left\{ -\phi_{n+1} + \phi_n + \Delta[(1-\lambda)(f_n + h_n) + \lambda(f_{n+1} + h_{n+1})] \right\} \right) \right]
\end{aligned}
\tag{23}
$$

Defining the average over a (normalized, complex valued) measure

$$
[\ldots] = \int \frac{d\phi\, d\hat{\phi}}{(2\pi)^M}\, \ldots\, \exp(-S) \tag{24}
$$

with the "action"

$$
-S = \sum_{n=0}^{M-1} \left(-T\Delta \hat{\phi}_n^2 + i\hat{\phi}_n \left\{ -\phi_{n+1} + \phi_n + \Delta[(1-\lambda)f_n + \lambda f_{n+1}] \right\} \right) - \Delta\lambda \sum_{n=1}^{M} f'_n \tag{25}
$$

one can also write

$$
Z = \left[\exp\left(i \sum_{n=1}^{M} \theta_n \phi_n + i \sum_{n=0}^{M-1} \hat{\phi}_n \Delta[(1-\lambda)h_n + \lambda h_{n+1}] \right) \right] \tag{26}
$$

From this representation one has, in particular (taking all derivatives at $\theta = h = 0$, and adopting the convention $\hat{\phi}_{-1} = 0$) the expressions for correlation and reponse functions

$$
C_{nm} = \langle \phi_n \phi_m \rangle = \frac{\partial}{i\partial\theta_n} \frac{\partial}{i\partial\theta_m} Z = [\phi_n \phi_m] \tag{27}
$$

$$
R_{nm} = \frac{\partial \langle \phi_n \rangle}{\partial(\Delta h_m)} = \frac{\partial}{\partial(\Delta h_m)} \frac{\partial}{i\partial\theta_n} Z = [\phi_n i\{(1-\lambda)\hat{\phi}_m + \lambda\hat{\phi}_{m-1}\}] \tag{28}
$$

It also follows that averages of any product of $\hat{\phi}$'s vanish (from the usual argument: $Z = 1$ for $\theta = 0$, whatever value the h's take). We can combine the expressions for correlation and response if we define new variables as

$$
\psi_{1n} = \phi_n \qquad (n = 1 \ldots M) \tag{29}
$$

$$
\psi_{2n} = i\{(1-\lambda)\hat{\phi}_n + \lambda\hat{\phi}_{n-1}\} \qquad (n = 0 \ldots M-1) \tag{30}
$$

Arranging these into a big vector $\psi = (\psi_{11}, \ldots, \psi_{1M}, \psi_{20}, \ldots, \psi_{2\,M-1})$,

$$[\psi\psi^{\mathrm{T}}] = \begin{pmatrix} C & R \\ R^{\mathrm{T}} & 0 \end{pmatrix} = G \tag{31}$$

and the calculation of G (the "propagator" in quantum field theory) will be our main goal. The fields θ and h have served their purpose and will be set to zero from now on.

3 Perturbation theory

To illustrate the perturbative expansion, consider a concrete example:

$$f(\phi) = -\mu\phi - \frac{g}{3!}\phi^3 \tag{32}$$

(the 3! factor is for later convenience). For $g = 0$, we recover a solvable linear Langevin equation; correspondingly, the action (25) then becomes quadratic. For $g \neq 0$, we therefore separate off this quadratic part and write

$$\begin{aligned} S &= S_0 + S_{\mathrm{int}} && (33) \\ -S_0 &= \sum_{n=0}^{M-1} \left(-T\Delta\hat{\phi}_n^2 + i\hat{\phi}_n \left\{ -\phi_{n+1} + \phi_n - \Delta\mu[(1-\lambda)\phi_n + \lambda\phi_{n+1}] \right\} \right) + \Delta\mu\lambda M && (34) \\ -S_{\mathrm{int}} &= i\Delta\left(-\frac{g}{6}\right) \sum_{n=0}^{M-1} \hat{\phi}_n[(1-\lambda)\phi_n^3 + \lambda\phi_{n+1}^3] - \left(-\frac{g}{2}\right)\Delta\lambda \sum_{n=1}^{M} \phi_n^2 && (35) \end{aligned}$$

Rearranging the first sum, the non-quadratic (or "interacting" in a field theory) part of the action can be written in the simpler form

$$-S_{\mathrm{int}} = -\frac{g}{6}\Delta \sum_{n=0}^{M-1} \psi_{2n}\psi_{1n}^3 + \frac{g}{2}\Delta\lambda \sum_{n=1}^{M} \psi_{1n}^2 \tag{36}$$

[I have dropped the last term, $\lambda\hat{\phi}_{M-1}\phi_M^3$ from the first sum; since (by causality) whatever we do with this last term doesn't affect any of the results for earlier times, we just have to make M larger than any times of interest for this omission to be irrelevant. Similarly, whether we have the sums start at $n = 0$ or $n = 1$ has a vanishing effect for $\Delta \to 0$, so in the following I will leave off the summation ranges.]

Now for $g = 0$ we have $S = S_0$, and a corresponding normalized *Gaussian* measure over the ψ for which we denote averages by $[\ldots]_0$ and the corresponding propagator by $G_0 = [\psi\psi^{\mathrm{T}}]_0$. For $g \neq 0$, our desired averages are then written as

$$[\ldots] = [\ldots \exp(-S_{\mathrm{int}})]_0 \tag{37}$$

Assuming g and hence S_{int} to be small, we thus arrive at the *perturbative expansion*

$$[\ldots] = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\ldots (-S_{\mathrm{int}})^k\right]_0 \tag{38}$$

which is a series expansion (in general asymptotic, non-convergent) in the nonlinearity parameter g. Each term in this series we can now evaluate using *Wick's theorem*: The average of any product of Gaussian random variables is found by summing over all possible pairings; symbolically (leaving out all indices etc)

$$[\psi\psi\ldots\psi]_0 = \sum_{\text{all pairings}} [\psi\psi]_0 \ldots [\psi\psi]_0 \tag{39}$$

Let's apply this to the simplest example: We know that $[1] = 1$, so this should be true to all orders in the expansion (38). Let's test it up to $\mathcal{O}(g)$:

$$\begin{aligned} [1] &= [1]_0 - [S_{\mathrm{int}}]_0 + \ldots && (40) \\ &= 1 + g\Delta \sum_n \left[-\frac{1}{6}\psi_{2n}\psi_{1n}^3 + \frac{1}{2}\lambda\psi_{1n}^2 \right]_0 && (41) \end{aligned}$$

Using Wick's theorem, the fourth-order average is

$$[\psi_{2n}\psi_{1n}\psi_{1n}\psi_{1n}]_0 = 3[\psi_{2n}\psi_{1n}]_0[\psi_{1n}\psi_{1n}]_0 = 3R^0_{nn}C^0_{nn} \tag{42}$$

(Why the 3? There are 3 different pairings between the 4 four variables—(1,2) & (3,4), (1,3) & (2,4), and (1,4) & (2,3)—but all give the same product of averages.) Here you see how the equal time response function appears in the formalism. Inserting the value $R^0_{nn} = \lambda$ that we found earlier, (41) thus becomes

$$[1] = 1 + g\Delta \sum_n \left(-\frac{1}{2}\lambda C^0_{nn} + \frac{1}{2}\lambda C^0_{nn}\right) = 1 + \mathcal{O}(g^2) \tag{43}$$

as it should be. We see here how the nontrivial determinant $J(\phi)$ that arises in the Stratonovich formalism ($\lambda = 1/2$) and appears as the term proportional to λ in the interaction part of the action (36) is essential for maintaining the correct normalization. Similar cancellations occur at all orders in g. Ito ($\lambda = 0$) is simpler here: The terms in the perturbation expansion of $[1] = 1 + \ldots$ all vanish *individually*, rather than just cancelling each other out.

4 Diagrams

4.1 Basics

Diagrams are just a pictorial way of keeping track of the various terms in the perturbation expansion (38), as evaluated using Wick's theorem. Having illustrated the equivalence between Ito and Stratonovic above, I stick to Ito ($\lambda = 0$) from now on, where the nontrivial part of the action is simply

$$-S_{\text{int}} = -(g/6)\Delta \sum_n \psi_{2n}\psi^3_{1n} \tag{44}$$

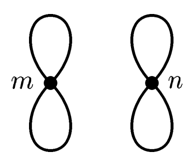
To illustrate the diagrammatic notation, consider again the expansion for the normalization factor

$$[1] = [1]_0 - [S_{\text{int}}]_0 + \frac{1}{2}[S^2_{\text{int}}]_0 + \ldots \tag{45}$$

We have dealt with the zeroth and first order terms above. Let's have a look at the second order term, which is

$$\frac{1}{2}(-g/6)^2\Delta^2 \sum_{m,n} \left[\psi_{2m}\psi^3_{1m}\psi_{2n}\psi^3_{1n}\right]_0 \tag{46}$$

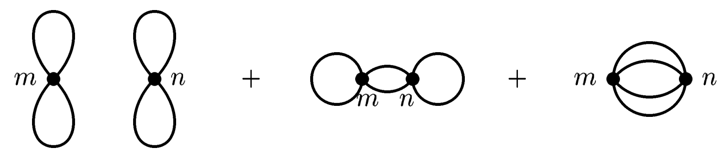
Represent each of the summed over time indices m and n by a vertex with four "legs" which symbolize the four ψ factors with the corrresponding index. Each vertex comes with a factor $-g/6$. Both of the time indices are summed over and the result multiplied by Δ; in the limit $\Delta \to 0$ these scaled sums of course become time integrals. The time indices are not fixed by our choice of observable to average, and are therefore called "internal" (we'll see external vertices in a moment). Now, having drawn the vertices, we can connect the legs in a number of ways; these represent the different pairings that Wick's theorem gives for the average in (46). E.g. the diagram where the legs from both vertices are connected to legs from the same vertex only



represents all the pairings where ψ_n's are connected to ψ_n's only, and ψ_m's to ψ_m's only. At each vertex there are three choices for pairings of this form, so this diagram has the value

$$\frac{1}{2}\left(-\frac{g}{6}\right)^2 \Delta^2 \sum_{m,n} 3[\psi_{2m}\psi_{1m}]_0[\psi_{1m}\psi_{1m}]_0 \times 3[\psi_{2n}\psi_{1n}]_0[\psi_{1n}\psi_{1n}]_0 = \frac{1}{2}\left\{\left(-\frac{g}{6}\right)\Delta \sum_n 3[\psi_{2n}\psi_{1n}]_0[\psi_{1n}\psi_{1n}]_0\right\}^2 \tag{47}$$

This illustrates an important fact: If a diagram separates into subparts which are not connected, its value is just the product of the two diagrams separately (apart from the overall factor of 1/2 here, which comes from the expansion of $\exp(-S_{\rm int})$).
The diagram above certainly doesn't exhaust all the Wick pairings of (46). So what are the other diagrams corresponding to (46)? We can have either two mn-pairings, one nn and one mm; or four mn pairings. Altogether, we'd therefore write (46) in diagrams as:



Let's write down the value of the last of these: Start with ψ_{2m}. From the diagram, this must be connected to a ψ on the other vertex, *i.e.* either ψ_{2n} or one of the three ψ_{1n}. In the first case, each of the remaining ψ_{1m} legs is connected to a ψ_{1n} leg; there are $3 \times 2 \times 1 = 6$ ways of making those connections (pairings). In the second case, the ψ_2 factor from the second vertex, ψ_{2n}, must be connected to one of the three ψ_{1m} vertices, and the remaining two ψ_{1m} and ψ_{1n} legs can be connected to each other in two ways. Thus the value of the diagram is:

$$\frac{1}{2}(-g/6)^2\Delta^2\sum_{m,n}\left\{6[\psi_{2m}\psi_{2n}]_0([\psi_{1m}\psi_{1n}]_0)^3 + 18[\psi_{2m}\psi_{1n}]_0[\psi_{1m}\psi_{2n}]_0([\psi_{1m}\psi_{1n}]_0)^2\right\} \qquad (48)$$

(Exercise: Evaluate the remaining second order diagram. As a check, count all pairings: The diagram just above dealt with $6+18=24$ pairings, while the first one corresponded to $3\times 3 = 9$ pairings. Since we have 8 ψ's in total, there are $7\times 5\times 3\times 1 = 8!/(4!2!^4) = 105$ pairings in total, so the remaining diagram must correspond to to $105-24-9=72$ pairings.)
In terms of diagrams, it is now quite easy to understand that $[1]=1$ (as it should) to all orders in g. Consider an arbitrary diagram in the expansion; let's assume it's connected (otherwise consider the subdiagrams separately). Now, within the Ito convention, the response function $[\psi_{2m}\psi_{1n}]_0$ is nonzero only for $n > m$ (for Stratonovich, the value for $n = m$ is also nonzero; for $n < m$ it is zero either way from causality). So to make the diagram nonzero, we have to connect the ψ_{2n_1} from a given vertex, with time index n_1, say, to the ψ_{1n_2} leg at *another* vertex, n_2. The ψ_{2n_2} from that vertex must in turn be connected to ψ_{1n_3} on another vertex and so on. Eventually, because we have a finite number of vertices, we must come back to our original vertex n_1. In the "ring" sequence $n_1, n_2, n_3, \ldots, n_1$ there are at least two time indices which are in the "wrong" order for the response function to be nonzero, so that the diagram contains at least one vanishing factor and thus vanishes itself.
Moral so far: Diagrams factorize over disconnected sub-diagrams; and (sub-)diagrams with only internal (summed over) vertices vanish [or, in the Stratonovich convention, cancel each other]. The latter are also called "vacuum diagrams" in field theory.

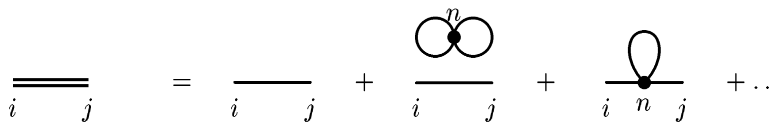
4.2 Diagrams for correlator/response; Dyson equation

Let's look at the diagrams for the propagator (correlator/response), $[\psi_{\alpha i}\psi_{\beta j}]$ where $\alpha, \beta \in \{1,2\}$. We now have two "external" vertices with fixed time indices i and j; the propagator is represented as a double line between these two vertices. (It is also called the "full" or "renormalized" propagator, in contrast to the "bare" propagator which is represented by the single lines in the diagrams and results from the pairings in Wick's theorem.) To zeroth order in g, full and bare propagator are obviously equal. To first order, we have, from (38),

$$G_{\alpha i,\beta j} = [\psi_{\alpha i}\psi_{\beta j}] = [\psi_{\alpha i}\psi_{\beta j}]_0 - [\psi_{\alpha i}\psi_{\beta j}S_{\rm int}]_0 + \ldots \qquad (49)$$

$$= [\psi_{\alpha i}\psi_{\beta j}]_0 - (g/6)\Delta\sum_n[\psi_{\alpha i}\psi_{\beta j}\psi_{2n}\psi_{1n}^3]_0 + \ldots \qquad (50)$$

In diagrams, we have two new contributions from the first order term, depending on whether we pair up ψ_i with ψ_j or with one of the ψ_n:



The first of these has a disconnected part with only internal vertices, however, so vanishes. The same argument applies to higher order diagrams as well: We only need to consider *connected* diagrams[1]. Thus, evaluating the surviving diagram,

$$G_{\alpha i,\beta j} = G^0_{\alpha i,\beta j} - (g/6)\Delta \sum_n \{3G^0_{\alpha i,1n}G^0_{1n,1n}G^0_{2n,\beta j} + 3G^0_{\alpha i,2n}G^0_{1n,1n}G^0_{1n,\beta j}\} \tag{51}$$

If we define a matrix Σ^1 by

$$\Sigma^1_{\gamma m,\delta n} = -(g/2)\Delta^{-1}\delta_{mn}G^0_{1n,1n}(\delta_{\gamma,1}\delta_{\delta,2} + \delta_{\gamma,2}\delta_{\delta,1}) \tag{52}$$

and agree to absorb a factor of Δ into matrix products, so that

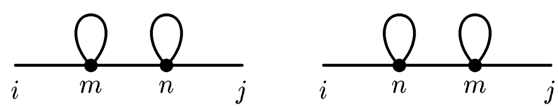
$$(AB)_{\alpha i,\gamma k} = \Delta \sum_{\beta,j} A_{\alpha i,\beta j}B_{\beta j,\gamma k} \tag{53}$$

then we can write (51) in matrix form simply as

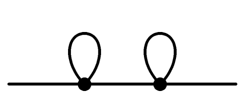
$$G = G_0 + G_0\Sigma^1 G_0 \tag{54}$$

The way factors of Δ are absorbed here ensures that matrix multiplications become time integrals in the natural way for $\Delta \to 0$, while the factor $\Delta^{-1}\delta_{mn}$ in Σ^1 becomes $\delta(t-t')$.

To first order in g, the above is the whole story for the propagator. Now look at the second order. Among the diagrams we have ones such as



These two only differ in how the internal vertices are labelled; since the latter are summed over, we can lump the two diagrams together into one *unlabelled* diagram. This just gives a factor of 2, which cancels exactly the prefactor $1/2!$ from the second order expansion of the exponential in (38). Again, the same happens at higher orders: At $\mathcal{O}(g^k)$ we have a prefactor of $1/k!$ but also k internal vertices which can be labelled in $k!$ different ways, so the unlabelled diagram has a prefactor of one. Thus, the unlabelled diagram



has the value (bearing in mind that the ψ_2 components at each of the internal vertices must be connected either to a ψ_1 leg at the other internal vertex, or to an external vertex; and

[1]Digression on equilibrium stat mech: Equilibrium averages can be evaluated by a diagrammatic expansion very similar to the one here. The main difference is that the partition function (whose perturbative expansion is the same as the one for [1] above) is not automatically normalized. Averages are thus written as $[\ldots] = [\ldots e^{-S_{\rm int}}]_0/[e^{-S_{\rm int}}]_0$ and the denominator, when expanded, ensures again that disconnected diagrams vanish. Equivalently, one can think of averages as derivatives of the *log* partitition function; it turns out that the perturbative expansion for $\ln Z$ consists of just the *connected* diagrams from the expansion of Z.

that there are three choices at each vertex for which pair of ψ_1's to connect to each other)

$$
\begin{aligned}
3\times 3\times(-g/6)^2\Delta^2\sum_{mn}\{ & G^0_{\alpha i,1m}G^0_{1m,1m}G^0_{2m,1n}G^0_{1n,1n}G^0_{2n,\beta j}+\\
& G^0_{\alpha i,1m}G^0_{1m,1m}G^0_{2m,2n}G^0_{1n,1n}G^0_{1n,\beta j}+\\
& G^0_{\alpha i,2m}G^0_{1m,1m}G^0_{1m,1n}G^0_{1n,1n}G^0_{2n,\beta j}+\\
& G^0_{\alpha i,2m}G^0_{1m,1m}G^0_{1m,2n}G^0_{1n,1n}G^0_{1n,\beta j}\}
\end{aligned}
\tag{55}
$$

Using the definition (52), one sees that in matrix form this is simply

$$G_0\Sigma^1 G_0\Sigma^1 G_0 \tag{56}$$

To make this simple form more obvious I have included in (55) the term in the second line which vanishes because it contains a zero G^0_{22} factor. We have an example here of a "one-particle reducible" (1PR) diagram, which can be cut in two by cutting just one bare propagator line (the one in the middle). The result illustrates that the value of such diagrams factorizes into the pieces they can be cut into; e.g. the diagram

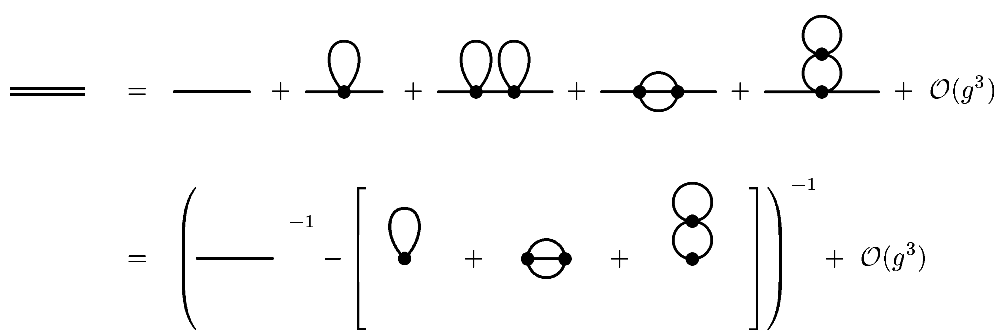
has the value $G_0\Sigma^1G_0\Sigma^1G_0\Sigma^1G_0$. If we sum up all the diagrams of this form, we get

$$G = G_0 + G_0\Sigma^1G_0\Sigma^1 + G_0\Sigma^1G_0\Sigma^1G_0 + G_0\Sigma^1G_0\Sigma^1G_0\Sigma^1G_0 + \ldots = [G_0^{-1}-\Sigma^1]^{-1} \tag{57}$$

or

$$G^{-1} = G_0^{-1} - \Sigma^1 \tag{58}$$

The inverses are relative to the appropriate unit element $\boldsymbol{I}$ for our redefined matrix multiplication, which has elements $I_{\alpha i,\beta j} = \Delta^{-1}\delta_{\alpha\beta}\delta_{ij}$. (So $A^{-1}A = \boldsymbol{I}$ means, because of the extra factor Δ in the matrix multiplication, that A^{-1} is Δ^{-2} times the conventional matrix inverse of A.)
The expression (58) is an example of a *resummation*: We have managed to sum up an infinite subseries from among all the diagrams for the propagator. What if we want to sum up *all* the diagrams (including *e.g.* the "tube" diagram above)? Again we can classify them into 1PI (one-particle irreducible) diagrams, which can't be cut in two by cutting a single line, and 1PR diagrams which factorize into their 1PI components. For example, to second order



More generally, if we denote the values of the different 1PI diagrams (defined analogously to Σ^1, *i.e.* without the external G_0 legs) by Σ^1, Σ^2, …then we can get all possible (1PI and 1PR) diagrams by "stringing" together all possible combinations of 1PI diagrams:

$$G = G_0\sum_{k=0}^{\infty}\sum_{i_1\ldots i_k}\Sigma^{i_1}G_0\cdots\Sigma^{i_k}G_0 = G_0\sum_{k=0}^{\infty}(\Sigma G_0)^k = [G_0^{-1}-\Sigma]^{-1} \tag{59}$$

where $\Sigma = \sum_{i=1}^{\infty} \Sigma^i$. Hence we see that the full propagator, with all diagrams summed up, can be written in the general form

$$G^{-1} = G_0^{-1} - \Sigma \tag{60}$$

where Σ, the so-called "self-energy" is the sum of all 1PI diagrams. Eq. (60) is called the *Dyson equation*. An alternative form that is often useful is

$$G_0^{-1} G = \boldsymbol{I} + \Sigma G \tag{61}$$

Let's write this out in terms of the separate blocks corresponding to correlation and response functions: We have

$$G_0 = \begin{pmatrix} C_0 & R_0 \\ R_0^{\mathrm{T}} & 0 \end{pmatrix} \qquad \Rightarrow \qquad G_0^{-1} = \begin{pmatrix} 0 & (R_0^{-1})^{\mathrm{T}} \\ R_0^{-1} & -R_0^{-1} C_0 (R_0^{-1})^{\mathrm{T}} \end{pmatrix} \tag{62}$$

and G^{-1} has the same structure, so that also the 11 block of Σ must vanish,

$$\Sigma = \begin{pmatrix} 0 & \Sigma_{12} \\ \Sigma_{12}^{\mathrm{T}} & \Sigma_{22} \end{pmatrix} \tag{63}$$

(This can also be shown diagrammatically: A nonzero contribution to Σ_{11} would correspond to diagrams where the internal vertices that make connections to the two external vertices do so via ψ_1 legs. This leaves all the ψ_2 legs to be connected amongst the internal vertices, and then the same argument as for vacuum diagrams can be applied.) Writing out (61) we have thus

$$\begin{aligned} (R_0^{-1})^{\mathrm{T}} R^{\mathrm{T}} &= \boldsymbol{I} + \Sigma_{12} R^{\mathrm{T}} && (64) \\ 0 &= 0 && (65) \\ R_0^{-1} C - R_0^{-1} C_0 (R_0^{-1})^{\mathrm{T}} R^{\mathrm{T}} &= \Sigma_{12}^{\mathrm{T}} C + \Sigma_{22} R^{\mathrm{T}} && (66) \\ R_0^{-1} R &= \boldsymbol{I} + \Sigma_{12}^{\mathrm{T}} R && (67) \end{aligned}$$

(where we have abused the notation by writing $\boldsymbol{I}$ also for the nonzero $M \times M$ sub-blocks of the original $2M \times 2M$ matrix $\boldsymbol{I}$). The first and last of these equations are both equivalent to

$$R^{-1} = R_0^{-1} - \Sigma_{12}^{\mathrm{T}} \tag{68}$$

implying that Σ_{12}^{T} acts like a self-energy for the reponse function. Rearranging, the components of the Dyson equation reduce to

$$\begin{aligned} R_0^{-1} R &= \Sigma_{12}^{\mathrm{T}} R + \boldsymbol{I} && (69) \\ R_0^{-1} C &= \Sigma_{12}^{\mathrm{T}} C + [R_0^{-1} C_0 (R_0^{-1})^{\mathrm{T}} + \Sigma_{22}] R^{\mathrm{T}} && (70) \end{aligned}$$

Using (68), the last equation can also be solved explicitly for C as

$$C = R[R_0^{-1} C_0 (R_0^{-1})^{\mathrm{T}} + \Sigma_{22}] R^{\mathrm{T}} \tag{71}$$

[Note: In the above I've defined Σ to be *plus* the sum of all 1PI diagrams. I think in field theory Σ is generally defined with the opposite sign.]

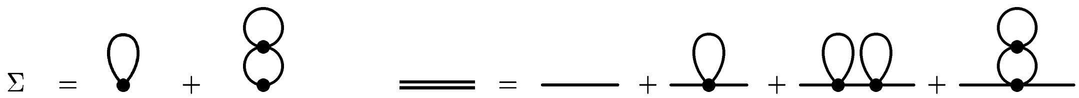
4.3 Self-consistency and mode coupling theory

Of course in general we cannot sum up all the diagrams for the self-energy, and must make some approximation. In (58) we used just the lowest order diagram to approximate Σ. Can we easily improve the approximation? Yes, if we replace G_0 in the expression for Σ by the full propagator G; diagrammatically,

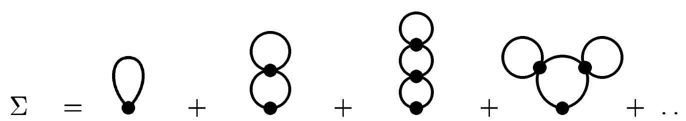
$\Sigma \quad =$ 

Which diagrams does this correspond to? This is easiest to find out order by order in g. To first order, Σ and G are

Re-inserting G into Σ we get its form to second order and therefore also G



and then we can iterate to get



So the simple operation of replacing G_0 by G effectively sums up an infinite series of "tadpole" diagrams in the expansion for the self-energy; of course the resulting Dyson equation (60) is correspondingly more complicated to solve. This idea can be applied to any low order expansion of the self-energy. It gives the exact results for many mean-field models (which means that the diagrams that haven't been taken into account are negligible in the thermodynamic limit); in other cases it is called "self-consistent one-loop" or "mode coupling" approximation. (And for p-spin models the form of the Dyson equation is indeed the same as for the simplified mode coupling theory for supercooled liquids.)

5 Example

Let's apply the formalism to our model (32). We want to take a distribution over the initial values $\phi(0)$ into account. But this only takes a small extension of the formalism: If the initial distribution is a zero mean Gaussian, we can simply include the average over $\phi(0)$ in the unperturbed average $[\ldots]_0$; the measure is still Gaussian, so we can apply all of the above formalism except that the values of C_0 and R_0 (the components of the bare propagator) are affected by the presence of uncertainty in the initial condition. (If the distribution has non-Gaussian parts, we include the Gaussian part as above and the remainder is put into the nontrivial part of the action S_{int}, giving a new kind of vertex in the diagrams.)
Now let's write down the Dyson equation for our model. Having derived all relations previously so that they have the obvious limits for $\Delta \to 0$, we work straight away with continous times. The bare response function is, from (4),

$$R_0(t, t') = \Theta(t - t')e^{-\mu(t-t')} \tag{72}$$

The inverse of R_0 is the operator $\partial/\partial t + \mu$ (since when applied to R_0 it gives $\delta(t - t')$). The other quantity we need to write down the Dyson equation (70) is $R_0^{-1}C_0(R_0^{-1})^{\mathrm{T}}$. To find this, it is easiest to start from the fact that

$$\phi(t) = \int_0^\infty dt_1\, R_0(t, t_1)[\eta(t_1) + \phi(0)\delta(t_1)] \tag{73}$$

The lower boundary of the integral here is meant as $0 - \epsilon$; the same applies to all integrals that follow. Averaging the product $\phi(t)\phi(t')$ gives

$$C_0(t, t') = \int_0^\infty dt_1\, dt_2\, R_0(t, t_1)[2T\delta(t_1 - t_2) + \langle \phi^2(0) \rangle\, \delta(t_1)\delta(t_2)]R_0(t', t_2) \tag{74}$$

so $C_0 = R_0 M R_0^{\mathrm{T}}$ with $M(t_1, t_2)$ given by the square brackets; thus

$$(R_0^{-1} C_0 (R_0^{-1})^{\mathrm{T}})(t, t') = 2T\delta(t - t') + \left\langle \phi^2(0) \right\rangle \delta(t)\delta(t') \tag{75}$$

Now we can write down the Dyson equation (70):

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mu\right) R(t, t') &= \int_0^\infty dt''\, \Sigma_{12}(t'', t) R(t'', t') + \delta(t - t') \\ \left(\frac{\partial}{\partial t} + \mu\right) C(t, t') &= \int_0^\infty dt''\, \Sigma_{12}(t'', t) C(t'', t') \\ &\quad + \int_0^\infty dt''\, [2T\delta(t - t'') + \left\langle \phi^2(0) \right\rangle \delta(t)\delta(t'') + \Sigma_{22}(t, t'')] R(t', t'') \end{aligned} \tag{76}$$

To first order in g, the self-energy is, from (52),

$$\Sigma_{12}(t, t') = -(g/2)\delta(t - t') C_0(t, t) \qquad \Sigma_{22}(t, t') = 0 \tag{77}$$

Within this approximation, the Dyson equation becomes

$$\left(\frac{\partial}{\partial t} + \mu\right) R(t, t') = -(g/2) C_0(t, t) R(t, t') + \delta(t - t') \tag{78}$$

$$\left(\frac{\partial}{\partial t} + \mu\right) C(t, t') = -(g/2) C_0(t, t) C(t, t') + 2T R(t', t) + \left\langle \phi^2(0) \right\rangle \delta(t) R(t', 0) \tag{79}$$

From the first equation,

$$R(t, t') = \Theta(t - t') \exp\left[\mu(t - t') - (g/2) \int_{t'}^{t} dt''\, C_0(t'', t'')\right] \tag{80}$$

and with this the second equation for C can also be solved (compare (71)) to give

$$C(t, t') = \left\langle \phi^2(0) \right\rangle R(t, 0) R(t', 0) + 2T \int_{t'}^{t} dt''\, R(t, t'') R(t', t'') \tag{81}$$

For the simplest case where C_0 is time-translation invariant, corresponding to $\left\langle \phi^2(0) \right\rangle = C_0(t, t) = T/\mu$, we see that the effect of g in the response function R is just to replace $\mu \to \mu + gT/(2\mu)$. At long times the effect on C is similar though at short times C will not be time-translation invariant.

If me make our first order (one-loop) approximation self-consistent, the only change is to replace C_0 by C in (77) and correspondingly in the Dyson equation, so that

$$R(t, t') = \Theta(t - t') \exp\left[-\mu(t - t') - (g/2) \int_{t'}^{t} dt''\, C(t'', t'')\right] \tag{82}$$

This has a simple interpretation: It corresponds to replacing our original nonlinear force term (32) by

$$f(\phi) \approx -\mu\phi - \frac{g}{3!} 3 \left\langle \phi^2 \right\rangle \phi \tag{83}$$

with $\left\langle \phi^2(t) \right\rangle = C(t, t)$ to be determined self-consistently. [The non-self-consistent version instead sets $\left\langle \phi^2(t) \right\rangle = C_0(t, t)$.] Assuming that we can find a solution with $C(t, t) = \alpha =$ constant, we get for C by inserting (82) into (81) and setting $\beta = \mu + \alpha g/2$

$$C(t, t') = \left\langle \phi^2(0) \right\rangle e^{-\beta(t+t')} + \frac{T}{\beta}[e^{-\beta|t-t'|} - e^{-\beta(t+t')}] \tag{84}$$

For $\left\langle \phi^2(0) \right\rangle = T/\beta$ we indeed get a time-translationally invariant $C(t, t') = (T/\beta)\exp(-\beta|t-t'|)$. This has $C(t, t) = T/\beta$ and so the self-consistent equation determining α is

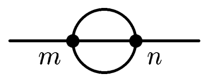
$$\frac{T}{\beta} = \alpha \qquad \Rightarrow \qquad T = \alpha(\mu + \alpha g/2) \tag{85}$$

(and the solution, correct to first order in g, can be written as $\alpha = T/[\mu + gT/(2\mu)]$). We see that for our particular model the self-consistent approximation gives a more sensible result than the "vanilla" first order approximation: It allows a time-translation invariant solution for both C and R

$$R(t,t') = \Theta(t-t')e^{-(\mu+\alpha g/2)(t-t')} \tag{86}$$

$$C(t,t') = \frac{T}{\mu+\alpha g/2}e^{-(\mu+\alpha g/2)|t-t'|} \tag{87}$$

which also obeys FDT, $R(t,t') = (1/T)(\partial/\partial t')C(t,t')$ for $t > t'$.
What if we want to improve the approximation to the self-energy further? The systematic approach is to include the lowest-order diagram not so far taken into account. We have the only first-order diagram already; the second-order "tadpole" diagrams are also taken into account through self-consistency. The only missing second order diagram is therefore the tube diagram



Let's work out what contribution to Σ this gives: Revert temporarily to discrete time notation and label the left and right vertex m and n, respectively. The elements $\Sigma_{1m,2n}$ of Σ_{12} correspond to those pairings where a ψ_{1m} leg from vertex m is attached to an external vertex on the left, and the ψ_{2n} leg from vertex n attached to an external vertex on the right. Internally (among the remaining legs) we thus have two $\psi_{1m}\psi_{1n}$ pairings and one $\psi_{2m}\psi_{1n}$ pairing. To work out the prefactor of the diagram, note that there are three choices for the externally attached ψ_{1m}; there are three choices for which of the ψ_{1n} to pair up with ψ_{2m}; and two more choices for how to make the two remaining $\psi_{1m}\psi_{1n}$ pairings. Thus, the diagram gives for Σ_{12}

$$\Sigma_{1m,2n} = 3 \times 3 \times 2(-g/6)^2(C^0_{mn})^2 R^0_{nm} = (g^2/2)(C^0_{mn})^2 R^0_{nm} \tag{88}$$

For Σ_{22}, we have both the ψ_{2m} and ψ_{2n} legs attached externally, and 6 choices for how to connect the three ψ_{1m} and ψ_{1n} legs internally, giving

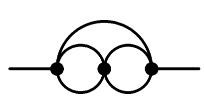
$$\Sigma_{2m,2n} = 6(-g/6)^2(C^0_{mn})^3 = (g^2/6)(C^0_{mn})^3 \tag{89}$$

We can again sum an infinite series of additional diagrams by replacing bare (C_0, R_0) by full quantities (C, R) here. Reverting to continuous time notation and including the first-order contribution in Σ_{12}, we thus get for the self-energy in the self-consistent two-loop approximation

$$\Sigma_{12}(t,t') = -(g/2)\delta(t-t')C(t,t) + (g^2/2)C^2(t,t')R(t',t) \tag{90}$$

$$\Sigma_{22}(t,t') = (g^2/6)C^3(t,t') \tag{91}$$

Intriguingly, when these are inserted into (76) we get (apart from the $\mathcal{O}(g)$ term in Σ_{12}, which is relatively trivial because it just renormalizes μ) the exact Dyson equation for a p-spin spherical spin glass with $p = 4$. A similar correspondence also holds for general p (with $f(\phi) = -\mu\phi + g\phi^{p-1}/(p-1)!$). In fact it has been shown for quite general (non-disordered) models that the *approximate* mode coupling equations can also be obtained as the *exact* equations for a suitably constructed mean-field model.
A final comment: Up to the order which we have considered, the free energy components $\Sigma_{12}(t,t')$ and $\Sigma_{22}(t,t')$ are simple *functions* of the correlation and response functions. This is not normally true once higher order diagrams are taken into account: For example, if we went to third order in g we would have to include the diagram



in the self energy (all other third order diagrams are automatically included by self-consistency). This now has one internal vertex whose time index is not fixed by the two time indices that the self-energy carries. It therefore gives a contribution to $\Sigma_{..}(t,t')$ which has an integral over this "internal" time; *e.g.* for Σ_{22} we get a contribution

$$\Sigma_{22}(t,t') \sim C(t,t') \int dt''\, C(t,t'')C(t',t'')[R(t,t'')C(t',t'') + R(t',t'')C(t,t'')] \qquad (92)$$

Diagrams of even higher order contain additional time integrals; in general, the self-energy is therefore a *functional* of the response and correlation functions. In fact, some people use this property to define what they mean by a mode coupling approximation: It is the self-consistent approximation which contains just those low-order diagrams that still leave the self-energy as a simple function of C and R.