p-adic Modular Forms over Shimura Curves over $\mathbb{Q}$

by

Payman L Kassaei

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Signature of Author: ....
Department of Mathematics
May 14, 1999

Certified by ....
Steven Kleiman
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Richard Melrose
Professor of Mathematics
Chair, Departmental Committee on Graduate Students

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Abstract

In this thesis, we set up the basic theory of p-adic modular forms over Shimura curves over $\mathbb{Q}$, parallel to the classical case over modular curves. We define and study the structure of the spaces of p-adic modular forms with respect to certain quaternion algebras over $\mathbb{Q}$. We study the relation of these modular forms with classical quaternionic modular forms. We prove a canonical subgroup theorem for false elliptic curves. That enables us to define the Frobenius morphism of p-adic modular functions. We use rigid analytic geometry to give an alternative description of the p-adic modular forms and their Frobenius morphism. We use this to study the finiteness properties of the Frobenius morphism. We define the U operator of p-adic modular forms and study its continuity properties. We show that the U operator is completely continuous on the space of overconvergent p-adic modular forms. Finally, we use the Fredholm theory of U and rigid geometry to study the dimension of spaces of generalized eigenforms of U of a certain slope.

Thesis Supervisor: Steven Kleiman
Title: Professor of Mathematics

To My Parents

Mohammad and Nahid

Contents

1 Introduction

The theory of p-adic modular forms started with the work of J.P. Serre. Serre's motivation was to study congruences between modular forms in order to understand the congruence properties of special values of L-functions. He identified modular forms with their q-expansions and defined a p-adic modular form to be a limit of such q-expansions in the p-adic topology [15]. This was a natural and elementary definition which satisfactorily reflected the p-adic properties of modular forms.

N. Katz gave a modular and therefore more conceptual definition of Serre's p-adic modular forms (of integral weight)[11]. He defined p-adic modular forms of weight k and level N as functions on *ordinary* elliptic curves and level N structures. He also gave modular definitions of Hecke operators acting on the space of p-adic modular forms. The most important of the Hecke operators in this context is the analogue of the classical Atkin's U_p operator which is called the U operator. It takes a p-adic modular form with q-expansion $\sum_n a_n q^n$ to $\sum_n a_{np} q^n$. In studying the spectral theory of U operator Dwork introduced the notion of *growth condition*. Using this idea, Katz defined the space of p-adic modular forms with a specific growth condition. Let r be an element of the base ring R_0 (which is a complete d.v.r. of mixed characteristic $(0, p)$ with fraction field K and valuation normalized such that $ord(p) = 1$). p-adic modular forms of growth condition r are defined as certain functions on the set of elliptic curves with level structures which are not *too supersingular*, in the sense that the value of E_{p-1} (Eisenstein series of weight $p-1$) at them has p-adic valuation at most equal to that of r. If $r = 1$ (or is a unit), this amounts to saying that the value of the Hasse invariant (which equals the reduction mod p of the value of E_{p-1}) is nonzero at the reduction mod p of the elliptic curve, which is the same as being ordinary. So $r = 1$ corresponds to Serre's theory. p-adic modular forms with a non-unit growth condition are called *overconvergent* modular forms.

A major part of Katz's analysis of overconvergent modular forms was to prove that the space of these modular forms is preserved under the action of U. The importance of this result lies in the fact that the action of U on the space of overconvergent modular forms is *completely continuous*. This causes a remarkable difference in the shape of theory between the cases of overconvergent and convergent modular forms. The reason is that for completely continuous operators, one can use Serre's Fredholm theory which generalizes a lot of techniques from operator theory of finite dimensional vector spaces. This theory makes it possible to study the eigenforms of the U operator. R. Coleman has used this and techniques from rigid analytic geometry to analyze the spectral theory of U. He constructs rigid analytic families of overconvergent modular forms (in which the weight varies in a rigid space) and studies the action of U on these families. This way he is able to compare the operator theory of U for different weights. He also proves a criterion for deciding whether an overconvergent modular form is classical. He is therefore able to translate his results for classical modular forms, proving a weaker version of a conjecture of Gouvêa and Mazur. [6],[7].

The goal of this work is to set up a similar theory in the context of quaternionic modular forms over $\mathbb{Q}$. Let D denote a fixed quaternion algebra which is indefinite and splits at p. Unlike the classical case, these modular forms lack q-expansions and therefore we can not define p-adic modular forms in Serre's way. Since quaternionic modular forms of level N are sections of certain line bundles over a Shimura curve over $\mathbb{Q}$ which has a moduli theoretic description, we can follow Katz in defining the spaces of p-adic modular forms with growth condition in this context. This Shimura curve classifies abelian surfaces with level $V_1(N)$ structure which carry an action of $\mathcal{O}_D$, a fixed maximal order of D, and are called *false elliptic curves*. Although there is no Eisenstein series in this case, we show that the Hasse invariant can be lifted to a quaternionic modular form of weight $p-1$, which will play the role of E_{p-1} to measure supersingularity of false elliptic curves in our case. In fact it turns out that the theory is independent of choice of such a lifting (as long as we confine ourselves to false elliptic curves which are not "too supersingular"). This will be clear when we use rigid geometry to give a description of these modular forms as sections of certain line bundles over rigid analytic spaces which are obtained by removing supersingular discs of different sizes from the Shimura curve and are denoted by $X^D(r)$ (where r is the growth condition). Having defined p-adic modular forms, we devote our work to define the action of a U operator and study its properties.

As in the classical case the U operator will be defined using the *Frobenius morphism* of modular functions which is denoted by $Frob$. To define $Frob$, we'll need to prove the existence of a *canonical subgroup* for false elliptic curves which are not "too supersingular". The canonical subgroup is essentially a lifting of the kernel of Frobenius morphism of the reduction mod p of the false elliptic curve. We also show that it is possible to measure how supersingular the quotient of a false elliptic curve by its canonical subgroup is. Having done so, we can define $Frob$ on the modular level by *dividing by canonical subgroup*. The morphism $Frob$, corresponds to the classical V operator which takes $\sum_n a_n q^n$ to $\sum_n a_n q^{np}$.

We will define the U operator, essentially as a trace of $Frob$. To be able to do so, we will study the finiteness properties of $Frob$. Using Raynaud's formal approach to rigid geometry, we show that $Frob$ can be obtained as the pullback of a rigid analytic morphism between affinoid subdomains of the Shimura curve. This allows us to study properties of $Frob$ more effectively. We also follow Lubin's idea of the subtle analysis of the formal group of a false elliptic curve, to deduce results about the fibre of $Frob$ over a point, leading to a proof of étaleness of $Frob$. At this point we have done enough to be able to define U for convergent and overconvergent modular forms. We study the continuity properties of U. We will show that U is a continuous operator of K-Banach spaces which is completely continuous on the space of overconvergent modular forms.

Finally, we use the Fredholm theory of U and rigid geometry to study the dimension of generalized eigenforms of U of slope α as in [6]. An overconvergent modular form f is called a generalized eigenform of U of slope $\alpha \in \mathbb{Q}$, if there is a polynomial $Q(T) \in K[T]$ such that all of its roots (in $\bar{K}$) have valuation α and $Q(U)(f) = 0$. Let us denote by $d(k, N, \alpha)$, the dimension of space of overconvergent generalized eigenforms of slope α, level $V_1(N)$ and weight k with respect to D (where the growth condition r is understood). We prove that there exists a $1 > t > |p|^{1/(p+1)}$, such that whenever the growth condition r satisfies

$t > |r| > 1$, the following holds:

Theorem 1.1. *There exists an $M > 0$ depending only on D, α and N, such that if for integers k and k' we have $k \equiv k' \bmod p^M(p-1)$, then*

$$d(k, N, \alpha) = d(k', N, \alpha).$$

Moreover $d(k, N, \alpha)$ is uniformly bounded for all $k \in \mathbb{Z}$.

2 False Elliptic Curves

We refer to [8],[2],[5] for the following. Let D be an indefinite non-split quaternion algebra over $\mathbb{Q}$ with discriminant δ. Fix an isomorphism: $D \otimes \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R})$. Fix a maximal order $\mathcal{O}_D$ of D. For each prime q not dividing δ, we fix an isomorphism $\mathcal{O}_D \otimes \mathbb{Z}_q \xrightarrow{\sim} M_2(\mathbb{Z}_q)$. Note that there is an induced isomorphism $\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} M_2(\mathbb{Z}/N\mathbb{Z})$ for any $N > 0$ which is prime to δ. We also fix a prime number $p > 3$ which does not divide δ.

Let S be a scheme on which δ is invertible. A *false elliptic curve* over S (with respect to D) is a pair (A, i) where A is an abelian surface (i.e. an abelian scheme of relative dimension 2), and $i : \mathcal{O}_D \hookrightarrow End_S(A)$ is an injection of (noncommutative) rings with identity. In other words there is a faithful action of $\mathcal{O}_D$ on A/S.

Let $x \to x^\iota$ denote the canonical involution of D. Since $\mathbb{Q}(\sqrt{-\delta})$ splits D, there is a $t \in \mathcal{O}_D$ such that $t^2 = -\delta$. Define another involution $*$ on D by $x^* = t^{-1}x^\iota t$. Let (A, i) be a false elliptic curve over S. It can be shown that there is a unique principal polarization on A such that for any geometric point s of S, the corresponding Rosati involution on $End(A_s)$ induces $*$ on $\mathcal{O}_D$. So fixing t, any false elliptic curve admits a canonical principal polarization.

An *isogeny* $f : (A, i) \to (B, j)$ of false elliptic curves over S, is an isogeny $f : A \to B$ of abelian surfaces, such that $j(x) \circ f = f \circ i(x)$ for all $x \in \mathcal{O}_D$. If $f : (A, i) \to (B, j)$ is an isogeny of false elliptic curves, the dual isogeny $f^{\vee} : B^{\vee} \to A^{\vee}$ composed with the principal polarizations of A and B induces an isogeny of false elliptic curves $f^t : B \to A$. f^t is called the dual isogeny of f. The composite $f^t \circ f : A \to A$ is locally multiplication by an integer d. If this integer is constant on S we call it the *false degree* of f.

Let $N > 0$ be an integer prime to $p\delta$. let (A, i) be a false elliptic curve. By a *full level N structure* on (A, i) we mean an isomorphism

$$\alpha : (\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z})_S \xrightarrow{\sim} A[N]$$

of group schemes over S with left action of $\mathcal{O}_D$. Here $A[N]$ denotes the N-torsion points of A which inherits an action of $\mathcal{O}_D$, and $(\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z})_S$ is the constant group scheme on which $\mathcal{O}_D$ acts by left multiplication. A false elliptic curve with full level N structure is denoted by (A, i, α).

We also define the notion of an *arithmetic level N structure* (or a *level $V_1(N)$ structure*) on a false elliptic curve (A, i). It is an inclusion

$$\alpha_N : \mu_N \times \mu_N \hookrightarrow A[N]$$

of group schemes over S, which preserves the action of $\mathcal{O}_D$. The action of $\mathcal{O}_D$ on $\mu_N \times \mu_N$ is via the isomorphism $\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} M_2(\mathbb{Z}/N\mathbb{Z})$. A false elliptic curve with a level $V_1(N)$ structure is often denoted by (A, i, α_N).

Let $M > 0$ be an integer prime to N. Let H be a finite flat subgroup scheme of A of rank M which is stable under the action of $\mathcal{O}_D$. The quotient of A by H with the induced action of $\mathcal{O}_D$ is a false elliptic curve which we denote by $(A/H, i)$. Let α_N be a level $V_1(N)$ structure on (A, i). Let π denote the projection $A \to A/H$. Since $(M, N) = 1$, both π and π^t induce isomorphisms between $A[N]$ and $(A/H)[N]$. Define a level $V_1(N)$ structure on $(A/H, i)$ by $\alpha_N' = (\pi^t)^{-1} \circ \alpha_N$.

$$\begin{array}{ccc} \mu_N \times \mu_N & \xrightarrow{\alpha_N} & A[N] \\ & \searrow^{\alpha_N'} & \uparrow{\scriptstyle \pi^t} \\ & & (A/H)[N] \end{array}$$

We usually denote by $(A/H, i, \alpha_N)$ the quotient of (A, i, α_N) by H.

A *level Np structure with trivial character at p* (or a *level $V_1(N) \cap V_0(p)$ structure*) on a false elliptic curve (A, i) over S, is a pair (α_N, H) where α_N is a level $V_1(N)$ structure and H is a finite flat subgroup of A which has rank p^2 and is stable under the action of $\mathcal{O}_D$. A false elliptic curve with a level $V_1(N) \cap V_0(p)$ structure is denoted by (A, i, α_N, H).

3 Shimura Curves over $\mathbb{Q}$

We recall some basic facts about the moduli spaces of False elliptic curves with level structure. Our references are [8], [2], [5].

3.1 Shimura Curves of level $V_1(N)$

We continue to make the assumptions in previous section. In particular $p > 3$ is a prime number which does not divide δN. We will furthermore assume that $N \geq 4$. The functor $F : ((Schemes)) \to ((Sets))$ defined by:

$$F(S) = \{ \begin{array}{l} \textit{False elliptic curves with level } V_1(N) \\ \textit{structure } (A, i, \alpha_N) \textit{ up to isomorphism over } S \} \end{array}$$

is representable by a scheme $\mathbb{X}_1^D(N)$ which is smooth, proper and of relative dimension 1 over $\mathbb{Z}[1/N\delta]$. We will work over $\mathbb{Z}_p$ and will denote $\mathbb{X}_1^D(N) \otimes \mathbb{Z}_p$ again by $\mathbb{X}_1^D(N)$. There is a universal false elliptic curve with level $V_1(N)$ structure over $\mathbb{X}_1^D(N)$ which we will denote by $(\mathbb{A}_1^D(N), i, \alpha_N)$. Therefore for every false elliptic curve with level $V_1(N)$ structure, (A, i, α_N) over S, there exists a unique morphism of schemes over $\mathbb{Z}_p$, $f : S \to \mathbb{X}_1^D(N)$ such that

$$(A, i, \alpha_N) \xrightarrow{\sim} f^*(\mathbb{A}_1^D(N), i, \alpha_N)$$

We will here fix a choice of idempotents $e_1 \neq 1, 0$ and $e_2 = 1 - e_1$ in $M_2(\mathbb{Z}_p)$. Let $(A \xrightarrow{s} S, i)$ be a false elliptic curve over a $\mathbb{Z}_p$ scheme S. Then $s_*\Omega^1_{A/S}$ inherits an action of $\mathcal{O}_D^{opp}$ which tensored with the scalar action of $\mathbb{Z}_p$ gives an action of $M_2(\mathbb{Z}_p)$ on $s_*\Omega^1_{A/S}$. Define:

$$\underline{\omega}_{A/S} := e_2.s_*\Omega^1_{A/S}$$

$$\mathcal{H}_{A/S} := e_2.R^1 s_*\mathcal{O}_A.$$

If (A, i, α_N) over S is obtained via $f : S \to \mathbb{X}_1^D(N)$, then we have $\underline{\omega}_{A/S} = f^*\underline{\omega}_{\mathbb{A}_1^D(N)/\mathbb{X}_1^D(N)}$ and $\mathcal{H}_{A/S} = f^*\mathcal{H}_{\mathbb{A}_1^D(N)/\mathbb{X}_1^D(N)}$. We will usually drop the subscript and write $\underline{\omega}$ when the false elliptic curve is understood.

Theorem 3.1. *Let s denote the morphism $\mathbb{A}_1^D(N) \to \mathbb{X}_1^D(N)$.*

- $\mathcal{H} \xrightarrow{\sim} \underline{\omega}^{\otimes -1}$
- *The short exact sequence*

$$0 \to s^*\Omega^1_{S/\mathbb{Z}_p} \to \Omega^1_{A/\mathbb{Z}_p} \to \Omega^1_{A/S} \to 0$$

  induces an isomorphism

$$\underline{\omega} \to \Omega^1_{\mathbb{X}_1^D(N)/\mathbb{Z}_p} \otimes \mathcal{H}.$$

Proof. See [8]. □

Corollary 3.2. *Notation as above, we have:*

- *There is a Kodaira-Spencer isomorphism:* $\Omega^1_{\mathbb{X}_1^D(N)/\mathbb{Z}_p} \xrightarrow{\sim} \underline{\omega}^{\otimes 2}$.
- *The invertible sheaf $\underline{\omega}^{\otimes k}$ has degree $k(g-1)$ on geometric fibers of $\mathbb{X}_1^D(N)$ over $\mathbb{Z}_p$, where g denotes genus$(\mathbb{X}_1^D(N) \otimes_{\mathbb{Z}[1/N\delta]} \mathbb{C})$.*

We also recall the following result:

Proposition 3.3. *The number of supersingular points on $\mathbb{X}_1^D(N) \otimes \overline{\mathbb{F}}_p$ is given by $(p-1)(g-1)$.*

Proof. See [8]. □

3.2 Shimura Curves of Level $V_1(N) \cap V_0(p)$

The functor $F : ((Schemes/\mathbb{Z}_p)) \to ((sets))$ defined by:

$$F(S) = \{ \quad \begin{array}{l} False\ elliptic\ curves\ with\ level\ V_1(N) \cap V_0(p) \\ structure\ (A, i, \alpha_N, H)\ up\ to\ isomorphism\ over\ S \} \end{array}$$

is representable by a scheme $X^D(N,p)$, which is proper, flat and of relative dimension 1 over $\mathbb{Z}_p$. $X^D(N,p)$ is not smooth. Indeed the reduction modulo p of $\mathbb{X}^D(N,p)$ is a curve over $\mathbb{F}_p$ which consists of two isomorphic proper and smooth curves crossing transversally at the finitely many supersingular points. There is a universal false elliptic curve with level $V_1(N) \cap V_0(p)$ structure $(\mathbb{A}^D(N,p), i, \alpha_N, \mathbb{H})$ on $\mathbb{X}^D(N,p)$. The transformation of functors which corresponds to forgetting the finite flat subgroup of rank p^2, gives a finite flat morphism of degree $p+1$:

$$\mathbb{X}^D(N,p) \xrightarrow{\pi} \mathbb{X}_1^D(N).$$

We can define in a similar manner the invertible sheaves $\underline{\omega} = \underline{\omega}_{\mathbb{A}^D(N,p)/\mathbb{X}^D(N,p)}$. We have:

$$\pi^*(\underline{\omega}_{\mathbb{A}_1^D(N)/\mathbb{X}_1^D(N)}) = \underline{\omega}_{\mathbb{A}^D(N,p)/\mathbb{X}^D(N,p)}.$$

4 Modular Forms With Respect to D

Let p be a prime number not dividing δN. Let R_0 be a $\mathbb{Z}_p$-algebra.

4.1 Modular Forms of Level $V_1(N)$

A modular form with respect to D, of weight $k \in \mathbb{Z}$ and level $V_1(N)$ over R_0 is a rule which assigns to any (A, i, α_N, ω) where:

- R is an R_0-algebra,
- (A, i, α_N) is a false elliptic curve over R with level $V_1(N)$ structure,
- ω is a basis for $\underline{\omega}_{A/S}$,

an element $f(A, i, \alpha_N, \omega) \in R$ such that:

- $f(A, i, \alpha_N, \omega)$ only depends on the R-isomorphism class of (A, i, α_N, ω),
- The formation of $f(A, i, \alpha_N, \omega)$ commutes with arbitrary base change of R_0-algebras,
- For any $\lambda \in R^*$,

$$f(A, i, \alpha_N, \lambda\omega) = \lambda^{-k} f(A, i, \alpha_N, \omega).$$

Alternatively one can define a modular form with respect to D, of weight $k \in \mathbb{Z}$ and level $V_1(N)$ over R_0 to be a rule which assigns to any test object (A, i, α_N) consisting of a false elliptic curve with level $V_1(N)$ structure over an R_0-algebra R, a section $f(A, i, \alpha_N)$ of $\underline{\omega}_{A/R}^{\otimes k}$ over $Spec(R)$ such that:

- $f(A, i, \alpha_N)$ depends only on the isomorphism class of (A, i, α_N) over R,
- The formation of $f(A, i, \alpha_N)$ commutes with arbitrary change of base over R_0.

The two definition are related by:

$$f(A, i, \alpha_N) = f(A, i, \alpha_N, \omega)\omega^{\otimes k}.$$

Let us denote the space of modular forms of weight k and level $V_1(N)$ over R_0 by $S^D(R_0, N, k)$. Recall that $(\mathbb{A}_1^D(N) \otimes R_0, i, \alpha_N)$ is the universal test object over $\mathbb{X}_1^D(N) \otimes R_0$. From definition it is immediate that:

$$S^D(R_0, N, k) = H^0(\mathbb{X}_1^D(N) \otimes R_0, \underline{\omega}^{\otimes k}).$$

Notice that since we have no cusps on $\mathbb{X}_1^D(N) \otimes R_0$ all modular forms obtained this way are holomorphic.

4.2 Modular Forms of Level $V_1(N) \cap V_0(p)$

A modular form with respect to D, of weight $k \in \mathbb{Z}$ and level $V_1(N) \cap V_0(p)$ over R_0 is a rule which assigns to any (A, i, α_N, H) where:

- R is an R_0-algebra,
- (A, i, α_N, H) is a false elliptic curve over R with level $V_1(N) \cap V_0(p)$ structure,

a section $f(A, i, \alpha_N, H)$ of $\underline{\omega}_{A/R}^{\otimes k}$ such that:

- $f(A, i, \alpha_N, H)$ only depends on the R-isomorphism class of (A, i, α_N, H),
- The formation of $f(A, i, \alpha_N, H)$ commutes with arbitrary base change of R_0-algebras.

We denote the space of all modular form of weight $k \in \mathbb{Z}$ and level $V_1(N) \cap V_0(p)$ over R_0 with respect to D, by $S^D(R_0, (N, p), k)$. It is clear that:

$$S^D(R_0, (N, p), k) = H^0(X^D(N, p) \otimes R_0, \underline{\omega}^{\otimes k}).$$

There is a natural map:

$$S^D(R_0, N, k) \to S^D(R_0, (N, p), k),$$

which corresponds to pulling back a section of $\underline{\omega}^{\otimes k}$ via the projection $\mathbb{X}^D(N, p) \xrightarrow{\pi} \mathbb{X}_1^D(N)$.

5 The Hasse Invariant

Let R_0 be any $\mathbb{F}_p$-algebra. We define the *Hasse Invariant* (with respect to D) as a modular form of weight $p-1$ and level $V_1(1)$ over R_0 with respect to D. Let (A, i) be a false elliptic curve over R, an R_0-algebra. One can turn $A^{(p)}$ in an obvious manner into a false elliptic curve $(A^{(p)}, i)$. Let ω be a basis of $\underline{\omega}_{A/R}$ on $Spec(R)$. The Frobenius morphism $Frob_{A/R} : A \to A^{(p)}$ induces a homomorphism of R-modules:

$$Frob^*_{A/R} : H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \to H^1(A, \mathcal{O}_A)$$

Using the idempotents e_1 and e_2 described in section 2, we can decompose $H^1(A, \mathcal{O}_A) = e_1 H^1(A, \mathcal{O}_A) \times e_2 H^1(A, \mathcal{O}_A)$. The same can be done for $H^1(A^{(p)}, \mathcal{O}_{A^{(p)}})$.

By 3.1, we know $e_1 H^1(A, \mathcal{O}_A) \xrightarrow{\sim} e_1 H^0(A, \Omega^1_{A/R})^{\check{}}$. The existence of ω shows that $H^1(A, \mathcal{O}_A)$ (respectively $H^1(A^{(p)}, \mathcal{O}_{A^{(p)}})$) is a free R-module of rank 1 with basis $\eta = \check{\omega}$ (respectively $\eta^{(p)}$). Since $Frob_{A/R}$ commutes with the action of $\mathcal{O}_D$ on A, there is a homomorphism:

$$Frob^*_{A/R} : e_2 H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \to e_2 H^1(A, \mathcal{O}_A)$$

Define the Hasse invariant which we will denote by $\mathbf{H}$, as follows:

$$Frob^*_{A/R}(\eta^{(p)}) = \mathbf{H}(A, i, \omega).\eta$$

Let us compute the weight of $\mathbf{H}$. Replacing ω by $\lambda\omega$ amounts to replacing η by $\lambda^{-1}\eta$:

$$Frob^*_{A/R}((\lambda^{-1}\eta)^{(p)}) = Frob^*_{A/R}(\lambda^{-p}\eta^{(p)}) = \lambda^{-p} Frob^*_{A/R}(\eta^{(p)}) = \lambda^{1-p}\mathbf{H}(A, i, \omega).(\lambda^{-1}\eta)$$

which proves that $\mathbf{H}$ is of weight $p-1$.

We will show that if $R = k$ is a field and (A, i) is a false elliptic curve over k, then $\mathbf{H}(A, i) = 0$ if and only if A is a supersingular abelian surface. We know that A is supersingular if $Frob^*_{A/k} : H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \to H^1(A, \mathcal{O}_A)$ is a nilpotent homomorphism. Choose $g \in Gl_2(\mathbb{Z}_p)$ such that $g^{-1} e_1 g = e_2$. This is always possible because any two idempotents $\neq 0, 1$ in $M_2(\mathbb{Z}_p)$ are conjugate. Since the action of $Frob_{A/k}$ commutes with that of g, $Frob^*_{A/k} : e_i H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \to e_i H^1(A, \mathcal{O}_A)$ for $i = 1, 2$ are conjugate by g. Therefore $Frob^*_{A/k}$ is nilpotent on $H^1(A^{(p)}, \mathcal{O}_{A^{(p)}})$ iff it is nilpotent (and hence zero) on $e_2 H^1(A^{(p)}, \mathcal{O}_{A^{(p)}})$ which is equivalent to $\mathbf{H}(A, i) = 0$. We have proven the following:

Proposition 5.1. *If $p = 0$ in R_0, there is an $\mathbf{H} \in S^D(R_0, 1, p-1)$ which vanishes at (A, i, ω) exactly when A is supersingular.*

We will use this result later:

Lemma 5.2. *$\mathbf{H}$ has simple zeroes on $\mathbb{X}_1^D(N) \otimes \overline{\mathbb{F}}_p$.*

Proof. **H** is a section of $\underline{\omega}^{\otimes p-1}$ on $\mathbb{X}_1^D(N) \otimes \mathbb{F}_p$. By 3.2, $\underline{\omega}^{\otimes p-1}$ has degree $(p-1)(g-1)$ which is exactly the number of supersingular points on $\mathbb{X}_1^D(N) \otimes \overline{\mathbb{F}}_p$ by 3.3. So **H** vanishes at each supersingular prime exactly once. □

6 Lifting the Hasse Invariant

In the classical setting the Eisenstein series of weight $p-1$, $E_{p-1}(q)$, which is a modular form of weight $p-1$ and level 1 over $\mathbb{Z}_p$ gives a canonical lifting of the Hasse invariant to $\mathbb{Z}_p$ ($p > 3$). Because of the lack of q-expansions, there are no Eisenstein series in this context. However **H** can be noncanonically lifted to a modular form of weight $p-1$ and level $V_1(N)$ over $\mathbb{Z}_p$. This lifting will make it possible to define p-adic modular forms over our Shimura curves.

Lemma 6.1. *$H^1(\mathbb{X}_1^D(N), \underline{\omega}^k) = 0$ for $k \geq 3$.*

Proof. First note that by corollary 3.2, $H^1(\mathbb{X}_1^D(N) \otimes \overline{\mathbb{Q}}_p, \underline{\omega}^k) = H^0(\mathbb{X}_1^D(N) \otimes \overline{\mathbb{Q}}_p, \underline{\omega}^{2-k})\check{} = 0$ when $k \geq 3$. This shows that

$$H^1(\mathbb{X}_1^D(N), \underline{\omega}^k) \otimes \mathbb{Q}_p = 0$$

and therefore $p^M H^1(\mathbb{X}_1^D(N), \underline{\omega}^k) = 0$ for some $M \geq 0$. With the same reasoning we get $H^1(\mathbb{X}_1^D(N) \otimes \overline{\mathbb{F}}_p, \underline{\omega}^k) = 0$. Now,

$$0 = H^1(\mathbb{X}_1^D(N) \otimes \mathbb{F}_p, \underline{\omega}^k) = H^1(\mathbb{X}_1^D(N), \underline{\omega}^k) \otimes \mathbb{F}_p.$$

Therefore $pH^1(\mathbb{X}_1^D(N), \underline{\omega}^k) = H^1(\mathbb{X}_1^D(N), \underline{\omega}^k)$. But some power of p kills $H^1(\mathbb{X}_1^D(N), \underline{\omega}^k)$. This proves the lemma. □

Proposition 6.2. *There is an element E_{p-1} of $S^D(\mathbb{Z}_p, N, p-1)$ such that*

$$E_{p-1} \equiv \mathbf{H} \ \textit{mod}\ p.$$

Proof. Let $\underline{\omega}$ denote $\underline{\omega}_{\mathbb{A}_1^D(N)/\mathbb{X}_1^D(N)}$. The short exact sequence of $\mathcal{O}_{\mathbb{X}_1^D(N)}$-modules:

$$0 \to \underline{\omega}^{\otimes p-1} \xrightarrow{\times p} \underline{\omega}^{\otimes p-1} \to \underline{\omega}^{\otimes p-1} \otimes \mathbb{F}_p \to 0$$

gives rise to the long exact sequence of cohomologies:

$$\begin{array}{rl} 0 & \to H^0(\mathbb{X}_1^D(N), \underline{\omega}^{\otimes p-1}) \xrightarrow{\times p} H^0(\mathbb{X}_1^D(N), \underline{\omega}^{\otimes p-1}) \to H^0(\mathbb{X}_1^D(N) \otimes \mathbb{F}_p, \underline{\omega}^{\otimes p-1} \otimes \mathbb{F}_p) \to \\ & \to H^1(\mathbb{X}_1^D(N), \underline{\omega}^{\otimes p-1}) = 0 \qquad \textit{by lemma } 6.1 \end{array}$$

This shows that $\mathbf{H} \in H^0(\mathbb{X}_1^D(N) \otimes \mathbb{F}_p, \underline{\omega}^{\otimes p-1} \otimes \mathbb{F}_p)$ can be lifted to a section $E_{p-1} \in H^0(\mathbb{X}_1^D(N), \underline{\omega}^{\otimes p-1})$. □

Fix a lifting once and for all. We will later show that our theory is independent of the choice of E_{p-1}.

7 p-adic modular forms with respect to D

Let $p > 3$ be a prime number and R_0 a p-adically complete $\mathbb{Z}_p$-algebra. Let $r \in R_0$. Following Katz [11], we define the R_0-module $S^D(R_0, r, V_1(N), k)$ of p-adic modular forms over R_0 of growth condition r, level $V_1(N)$ and weight $k \in \mathbb{Z}$ (with respect to D):

Definition 7.1. *An element $f \in S^D(R_0, r, V_1(N), k)$ is a rule which assigns to any $(A, i, \alpha_N, Y, \omega)$ where:*

- *R is an R_0-algebra in which p is nilpotent,*
- *(A, i, α_N) is a false elliptic curve with level $V_1(N)$ structure over R,*
- *ω is a basis for $\underline{\omega}_{A/R}$,*
- *Y is an element of R such that $YE_{p-1}(A, i, \omega) = r$.*

an element $f(A, i, \alpha_N, Y, \omega) \in R$ such that:

- *$f(A, i, \alpha_N, Y, \omega)$ depends only on the isomorphism class of $(A, i, \alpha_N, Y, \omega)$ over R,*
- *The formation of $f(A, i, \alpha_N, Y, \omega)$ commutes with arbitrary base change of R_0 algebras in which p is nilpotent,*
- *For any $\lambda \in R^*$,*

$$f(A, i, \alpha_N, \lambda^{p-1}Y, \lambda\omega) = \lambda^{-k} f(A, i, \alpha_N, Y, \omega).$$

We may equivalently think of f as a rule which assigns to any r-test object (A, i, α_N, Y), where (A, i, α_N) is a false elliptic curve with level $V_1(N)$ structure over an R_0-scheme S on which p is nilpotent and Y is a section of $\underline{\omega}_{A/S}^{\otimes 1-p}$ such that $Y.E_{p-1} = r$, a section $f(A, i, \alpha_N, Y)$ of $\underline{\omega}_{A/R}^{\otimes k}$ over S such that:

- $f(A, i, \alpha_N, Y)$ depends only on the isomorphism class of (A, i, α_N, Y) over R,
- The formation of $f(A, i, \alpha_N, Y)$ commutes with arbitrary base change of R_0 algebras in which p is nilpotent.

It is easy to see that the two definition are linked by

$$f(A, i, \alpha_N, Y) = f(A, i, \alpha_N, Y, \omega)\omega^{\otimes k}.$$

From the definition it is evident that

$$S^D(R_0, r, V_1(N), k) = \varprojlim_{n \geq 1} S^D(R_0/p^n, r, V_1(N), k).$$

One can make the same definitions by allowing R to vary in p-adically complete R_0-algebras.

Any modular form f of weight k and level $V_1(N)$ with respect to D, determines an element $f^\sharp$ of $S^D(R_0, r, V_1(N), k)$ by defining:

$$f^\sharp(A, i, \alpha_N, Y) = f(A, i, \alpha_N)$$

8 Structure of $S^D(R_0, r, V_1(N), k)$

Lemma 8.1. *Let X be any scheme over R_0 and $\mathcal{L}$ a line bundle on X. Let $l \in H^0(X, \mathcal{L}\check{})$ and $r \in R_0$. The functor $F : ((Schemes/R_0)) \to ((Sets))$ defined by:*

$$F(S) = \{R_0\text{-}morphisms\ g : S \to X\ together\ with\ a\ section\ Y\ of\ g^*\mathcal{L}\ s.t.\ Y.l = r\}$$

is representable by $Spec_X(Symm(\mathcal{L}\check{})/<l - r>)$.

Proof. Cover X by affines $\{Spec(A_i)\}_{i \in I}$ such that there are isomorphisms $\mathcal{L}|_{Spec(A_i)} \xrightarrow{\sim} \mathcal{O}_{Spec(A_i)}$. Suppose that the transition functions with respect to this trivialization are $\{\alpha_{ij}\}_{i,j \in I}$ and l is given by l_i on $Spec(A_i)$ under the above isomorphisms. Let $g : S \to X$ be a morphism over R_0. To give a morphism $h : S \to Spec_X(Symm(\mathcal{L}\check{})/<l - r>)$ lifting g is equivalent to giving morphisms $h_i : g^{-1}(Spec(A_i)) \to Spec(A_i[T_i]/<l_iT_i - r>)$ for $i \in I$, compatible with the relations $T_i = \alpha_{ij}^{-1} T_j$ for $i, j \in I$. This is the same as giving for each $i \in I$ a section of $\mathcal{O}_S$ on $g^{-1}(Spec(A_i))$, Y_i, such that $Y_i\, g^*(\alpha_{ij}) = Y_j$ for $i, j \in I$ and $l_iY_i = r$, which is nothing but a section Y of $g^*\mathcal{L}$ on S with $Y.l = r$. □

If we let $X = \mathbb{X}_1^D(N) \otimes R_0$, $\mathcal{L} = \underline{\omega}^{\otimes p-1}$ and $l = E_{p-1}$ in the preceding lemma, we get the following corollary:

Corollary 8.2. *The functor $F : ((Schemes/R_0)) \to ((Sets))$ defined by:*

$$F(S) = \{\ \begin{array}{l} All\ quadruples\ (A, i, \alpha_N, Y)\ up\ to\ isomorphism\ over\ S \\ where\ (A, i, \alpha_N)\ is\ a\ false\ elliptic\ curve\ with\ level \\ V_1(N)\ structure\ over\ S\ and\ Y\ is\ a\ section\ of\ \underline{\omega}^{\otimes 1-p}\ on \\ S\ which\ satisfies:\ YE_{p-1} = r\ \ \} \end{array}$$

is representable by $\mathbb{Y}_r^D \otimes R_0$ where

$$\mathbb{Y}_r^D \otimes R_0 = Spec_{\mathbb{X}_1^D(N) \otimes R_0}(Symm(\underline{\omega}^{\otimes p-1})/<E_{p-1} - r>).$$

The universal object over $\mathbb{Y}_r^D \otimes R_0$ is $(\mathbb{B}_r^D \otimes R_0, i, \alpha_N, Y_r)$, where $(\mathbb{B}_r^D \otimes R_0, i, \alpha_N)$ is the pullback of $(\mathbb{A}_1^D(N) \otimes R_0, i, \alpha_N)$ under the natural projection $\mathbb{Y}_r^D \otimes R_0 \xrightarrow{\pi} \mathbb{X}_1^D(N) \otimes R_0$ and Y_r is the restriction of the canonical section of $\pi^*\underline{\omega}^{\otimes 1-p}$ on $Spec_{\mathbb{X}_1^D(N) \otimes R_0}(Symm(\underline{\omega}^{\otimes p-1}))$ to $\mathbb{Y}_r^D \otimes R_0$ (which is locally given by T_i). We often denote $\pi^*\underline{\omega}_{\mathbb{A}_1^D(N)/\mathbb{X}_1^D(N)}$ by $\underline{\omega}$ or sometimes by $\underline{\omega}_{\mathbb{B}_r^D \otimes R_0/\mathbb{Y}_r^D \otimes R_0}$ if we need to be more specific.

Proposition 8.3. *If p is nilpotent in R_0, there is a canonical isomorphism:*

$$S^D(R_0, r, V_1(N), k) \xrightarrow{\sim} H^0(\mathbb{X}_1^D(N) \otimes R_0, \bigoplus_{j\geq 0} \underline{\omega}^{\otimes k+j(p-1)} / < E_{p-1} - r >)$$

Proof. Note that the subsheaf$< E_{p-1} - r >$ is defined by considering $\bigoplus_{j\geq 0} \underline{\omega}^{\otimes k+j(p-1)}$ as a sheaf of modules over $\bigoplus_{j\geq 0} \underline{\omega}^{\otimes j(p-1)}$. Since p is nilpotent in R_0, any R_0-algebra is p-adically complete. Therefore an element of $S^D(R_0, r, V_1(N), k)$ is a function of test objects (A, i, α_N, Y) over all R_0-algebras. This means:

$$S^D(R_0, r, V_1(N), k) \xrightarrow{\sim} H^0(\mathbb{Y}_r^D \otimes R_0, \underline{\omega}^{\otimes k})$$

Now we note that in general if X is a scheme, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-algebras and $\mathcal{G}$ is a locally free $\mathcal{O}_X$-module, then $H^0(Spec_X(\mathcal{F}), \pi^*\mathcal{G}) = H^0(X, \mathcal{F} \otimes \mathcal{G})$, where π denotes the map $Spec_X(\mathcal{F}) \to X$. In our case we have:

$$\begin{aligned} H^0(\mathbb{Y}_r^D \otimes R_0, \underline{\omega}^{\otimes k}) &= H^0(Spec_{\mathbb{X}_1^D(N)\otimes R_0}(Symm(\underline{\omega}^{\otimes p-1})/{<}E_{p-1} - r{>}), \underline{\omega}^{\otimes k}) \\ &= H^0(\mathbb{X}_1^D(N) \otimes R_0, \textstyle\bigoplus_{j\geq 0} \underline{\omega}^{\otimes j(p-1)}/{<}E_{p-1} - r{>} \otimes \underline{\omega}^{\otimes k}) \\ &= H^0(\mathbb{X}_1^D(N) \otimes R_0, \textstyle\bigoplus_{j\geq 0} \underline{\omega}^{\otimes k+j(p-1)}/{<}E_{p-1} - r{>}) \end{aligned}$$

which proves the desired result. □

Note: Any section f of $\underline{\omega}^{\otimes k+j(p-1)}$ on $\mathbb{X}_1^D(N) \otimes R_0$ (i.e a modular form of weight $k+j(p-1)$ and level $V_1(N)$ over R_0) determines an element f' of $S^D(R_0, r, V_1(N), k)$. In fact working locally as in lemma 8.1, we see that if f is locally given by f_i, then f' is given by $f_i T_i^j$. This means that

$$f' = f.Y_r^j,$$

where Y_r can be viewed as an element of $S^D(R_0, r, V_1(N), 1-p)$ such that

$$Y_r(A, i, \alpha_N, Y) = Y.$$

When p is not nilpotent in R_0 we can give a description of p-adic modular forms as limits of classical modular forms. Despite of lack of q-expansions in this context, the following proposition provides a description of modular forms which is similar to Serre's definition in the classical setting. We assume R_0 to be a p-adically complete $\mathbb{Z}_p$ algebra flat over $\mathbb{Z}_p$.

Proposition 8.4. *Let R_0 be a p-adically complete $\mathbb{Z}_p$-algebra flat over $\mathbb{Z}_p$ and $r \in R_0$ not a zero divisor. Assume that either $k = 0$ or $k \geq 3$. Then the natural map:*

$$\varprojlim_{n\geq 0}\Big(\bigoplus_{j\geq 0} S^D(\mathbb{Z}_p, N, k + j(p-1))\Big) \otimes_{\mathbb{Z}_p} R_0/p^n \;/{<}E_{p-1} - r{>} \xrightarrow{\sim} S^D(R_0, r, V_1(N), k)$$

is an isomorphism. The above map is induced by taking inverse limit of the natural maps

$$\big(H^0(\mathbb{X}_1^D(N), \bigoplus_{j\geq 0} \underline{\omega}^{\otimes(k+j(p-1))}) \otimes R_0/p^n\big)/<E_{p-1}-r> \to S^D(R_0/p^n, r, V_1(N), k).$$

Proof. This is as in [11]. Let $\mathcal{F}$ denote the sheaf $\bigoplus_{j\geq 0} \underline{\omega}^{\otimes(k+j(p-1))}$ on $\mathbb{X}_1^D(N)$ and $\mathcal{F}_n$ denote $\mathcal{F} \otimes R_0/p^n$. There is a short exact sequence of sheaves on $\mathbb{X}_1^D(N)$:

$$0 \to \mathcal{F}_n \overset{E_{p-1}-r}{\longrightarrow} \mathcal{F}_n \to \mathcal{F}_n/<E_{p-1}-r> \to 0$$

Writing the long exact sequence of cohomology we get:

$$\begin{aligned} 0 \;&\to H^0(\mathbb{X}_1^D(N), \mathcal{F}_n) \overset{E_{p-1}-r}{\longrightarrow} H^0(\mathbb{X}_1^D(N), \mathcal{F}_n) \to H^0(\mathbb{X}_1^D(N), \mathcal{F}_n/<E_{p-1}-r>) \\ &\to H^1(\mathbb{X}_1^D(N), \mathcal{F}_n) \overset{E_{p-1}-r}{\longrightarrow} H^1(\mathbb{X}_1^D(N), \mathcal{F}_n) \to H^1(\mathbb{X}_1^D(N), \mathcal{F}_n/<E_{p-1}-r>) \;\to 0 \end{aligned}$$

First assume $k \geq 3$. Then by lemma 6.1 we know that $H^1(\mathbb{X}_1^D(N), \mathcal{F}) = 0$. This shows that

$$H^0(\mathbb{X}_1^D(N), \mathcal{F}_n) = H^0(\mathbb{X}_1^D(N), \mathcal{F}) \otimes R_0/p^n,$$
$$H^1(\mathbb{X}_1^D(N), \mathcal{F}_n) = 0.$$

Since $\mathbb{X}_1^D(N)$ is a curve, the latter implies that:

$$H^0(\mathbb{X}_1^D(N), \mathcal{F}_{n+1}) \to H^0(\mathbb{X}_1^D(N), \mathcal{F}_n)$$

is surjective. Thus in this case the H^0 terms in the above exact sequence form a short exact sequence of inverse systems in which the first one has surjective transition morphisms. Taking inverse limits we get:

$$\varprojlim_{n\geq 0}(H^0(\mathbb{X}_1^D(N), \mathcal{F}) \otimes R_0/p^n)\;/<E_{p-1}-r> \overset{\sim}{\longrightarrow} \varprojlim_{n\geq 0} H^0(\mathbb{X}_1^D(N), \mathcal{F}_n/<E_{p-1}-r>)$$

But $H^0(\mathbb{X}_1^D(N), \mathcal{F}_n/<E_{p-1}-r>) = S^D(R_0/p^n, r, V_1(N), k)$ and therefore we get the desired isomorphism.

Now assume that $k = 0$. Since $p-1 > 3$ we can again apply lemma 6.1 to deduce that $H^1(\mathbb{X}_1^D(N), \mathcal{F}) = H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)})$ which implies that:

$$H^0(\mathbb{X}_1^D(N), \mathcal{F}_n) = H^0(\mathbb{X}_1^D(N), \mathcal{F}) \otimes R_0/p^n$$
$$H^1(\mathbb{X}_1^D(N), \mathcal{F}_n) = H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/p^n$$

The above long exact sequence now becomes:

$$\begin{aligned} 0 \;&\to H^0(\mathbb{X}_1^D(N), \mathcal{F}) \otimes R_0/p^n \to H^0(\mathbb{X}_1^D(N), \mathcal{F}) \otimes R_0/p^n \to H^0(\mathbb{X}_1^D(N), \mathcal{F}_n/<E_{p-1}-r>) \\ &\to H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/p^n \overset{-r}{\longrightarrow} H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/p^n \\ &\to H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/<p^n, r> \to 0 \end{aligned}$$

To complete the proof we recall a lemma from [11]:

Lemma 8.5. *Let $0 \to K^0 \to K^1 \to K^2 \to ...$ be a long exact sequence in the category of inverse systems of abelian groups. Suppose that for all $j \neq j_0$ the inverse system K^j has surjective transition morphisms and that $\varprojlim K^{j_0+1} \to \varprojlim K^{j_0+2} \to \varprojlim K^{j_0+3}$ is exact. Then*

$$0 \to \varprojlim K^0 \to \varprojlim K^1 \to \varprojlim K^2 \to ...$$

is exact

Applying the lemma to our situation we get an exact sequence of inverse systems which has six terms. But the map:

$$\varprojlim_{n\geq 0} H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/p^n \xrightarrow{-r} \varprojlim_{n\geq 0} H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)}) \otimes R_0/p^n$$

is injective because $H^1(\mathbb{X}_1^D(N), \mathcal{O}_{\mathbb{X}_1^D(N)})$ is a finite free $\mathbb{Z}_p$ module and r is not a zero divisor in $\varprojlim_{n\geq 0} R_0/p^n \xrightarrow{\sim} R_0$. This breaks the exact sequence of six terms to two short exact sequences and the first short exact sequence gives the desired isomorphism as in the first part.

□

Note: As a result of the note after proposition 8.3, any $f \in H^0(\mathbb{X}_1^D(N) \otimes R_0, \underline{\omega}^{\otimes(k+j(p-1))})$ determines a p-adic modular form in $S^D(R_0, r, V_1(N), k)$ which is given by $f.Y_r^j$.

Let us examine the result of proposition 8.4 more closely when $r = 1$. Let us fix k and define the R_0 algebra

$$B := \bigoplus_{j\geq 0} S^D(R_0, k + j(p-1), N) \;/\!<\!E_{p-1} - 1\!> .$$

By proposition 8.4 $S^D(R_0, 1, V_1(N), k)$ is isomorphic to the p-adic completion of B which we call $\hat{B}$. Any element of $\hat{B}$ can be represented by an infinite formal sum:

$$\textstyle\sum_{n\geq 0} f_n \qquad f_n \in S^D(R_0, N, k + j(p-1))$$

in which $f_n \to 0$ as $n \to \infty$. On the other hand we have the relations $gE_{p-1} = g$ for any element of $\hat{B}$. Define the partial sums

$$F_n = \sum_{j=0}^{n} f_j$$

Clearly $F_n \to \sum_{n\geq 0} f_n$ as $n \to \infty$. Using the relations in $\hat{B}$ we deduce the following equation in $\hat{B}$:

$$F_n = \sum_{j=0}^{n} f_j E_{p-1}^{n-j} \in S^D(R_0, N, k + n(p-1))$$

This shows that any element of $S^D(R_0, 1, V_1(N), k)$ can be written as a p-adic limit of classical modular forms. This is where the similarity to Serre's definition of classical p-adic modular forms rises. However there are no q-expansions in this context and we can not give the counterpart of Serre's definition. In the following section we will discuss this issue more carefully and also for general r.

8.1 A Banach Basis for $S^D(R_0, r, V_1(N), k)$

As was explained in the previous section, we will study the structure of the Banach R_0 module $S^D(R_0, r, V_1(N), k)$ more closely by finding a Banach basis for this space consisting of classical modular forms. This description will shed more light on the relation between classical and p-adic modular forms with respect to D. We will continue to make the assumptions in proposition 8.4.

Lemma 8.6. *The injective homomorphism of finite free $\mathbb{Z}_p$ modules:*

$$S^D(\mathbb{Z}_p, N, k + n(p-1)) \stackrel{\times E_{p-1}}{\longrightarrow} S^D(\mathbb{Z}_p, N, k + (n+1)(p-1))$$

admits a section.

Proof. It is enough to show that the cokernel of the above homomorphism is finite and free over $\mathbb{Z}_p$. It is clearly finite over $\mathbb{Z}_p$. We will prove flatness. Assume that for some $f \in S^D(\mathbb{Z}_p, N, k+(n+1)(p-1))$ we have $pf = E_{p-1}g$ for some $g \in S^D(\mathbb{Z}_p, N, k+n(p-1))$. Reducing modulo p we get $\mathbf{H}\bar{g} = 0$ over $\mathbb{X}_1^D(N) \otimes \mathbb{F}_p$. Since $\mathbf{H}$ has a finite number of zeroes, we have $\bar{g} = 0$. This shows that $g = ph$ for some $h \in S^D(\mathbb{Z}_p, N, k + (n+1)(p-1))$. Thus $p(f - E_{p-1}h) = 0$ which implies $f = E_{p-1}h$ since $S^D(\mathbb{Z}_p, N, k + (n+1)(p-1))$ is flat over $\mathbb{Z}_p$. This proves that $S^D(\mathbb{Z}_p, N, k+(n+1)(p-1))/E_{p-1}S^D(\mathbb{Z}_p, N, k+n(p-1))$ is finite free over $\mathbb{Z}_p$. □

Let's fix such a section for all n, k. We get the corresponding decomposition:

$$S^D(\mathbb{Z}_p, N, k + (n+1)(p-1)) = E_{p-1}S^D(\mathbb{Z}_p, N, k + n(p-1)) \oplus B^D(N, k, n+1).$$

We also define:

$$B^D(N, k, 0) = S^D(\mathbb{Z}_p, N, k).$$

Over the base R_0 we define:

$$B^D(R_0, N, k, n) = B^D(N, k, n) \otimes R_0.$$

Corollary 8.7. *There is an isomorphism:*

$$\begin{array}{ccc} \bigoplus_{j=0}^n B^D(R_0, N, k, j) & \stackrel{\sim}{\longrightarrow} & S^D(R_0, N, k + n(p-1)) \\ \sum_{j=0}^n b_j & \mapsto & \sum_{j=0}^n E_{p-1}^{n-j} b_j, \end{array}$$

In particular if $f \in S^D(R_0, N, k+n(p-1))$ can be written as

$$f = \sum_{j=0}^{n} E_{p-1}^{n-j} b_j$$

then $f \equiv 0$ mod p^m iff $b_j \equiv 0$ mod p^m for $0 \le j \le n$.

We now define the Banach R_0 module $B^D_{rig}(R_0, N, k)$, whose elements are all formal sums

$$\textstyle\sum_{j \ge 0} b_j \quad b_j \in B^D(R_0, N, k, j),$$

whose terms tend to zero as $n \to \infty$. Indeed $B^D_{rig}(R_0, N, k)$ is the p-adic completion of $\bigoplus_{j \ge 0} B^D(R_0, N, k, j)$. We will show that all of the spaces $S^D(R_0, r, V_1(N), k)$ are isomorphic to this Banach R_0 module.

Theorem 8.8. *The inclusion of $B^D_{rig}(R_0, N, k)$ in the p-adic completion of $\bigoplus_{n \ge 0} S^D(R_0, N, k+n(p-1))$ induces an isomorphism:*

$$\begin{array}{rcl} B^D_{rig}(R_0, N, k) & \xrightarrow{\sim} & S^D(R_0, r, V_1(N), k) \\ \sum_{n \ge 0} b_n & \mapsto & \sum_{n \ge 0} b_n {Y_r}^n \end{array}$$

where Y_r is the p-adic modular form of weight $1-p$ whose value on any r-test object (A, i, α_N, Y) is Y.

Proof. Note that this map is defined via the isomorphism of proposition 8.4. The image of $\sum_{n \ge 0} b_n$ is represented by the same summation as an element of the p-adic completion of $H^0(\mathbb{X}_1^D(N) \otimes R_0, \bigoplus_{j \ge 0} S^D(R_0, N, k+j(p-1))/\langle E_{p-1} - r\rangle$. Also the note after proposition 8.4 shows that the image of any b_n in $S^D(R_0, r, V_1(N), k)$ is the p-adic modular form $b_n Y_r^n$ and hence the image of $\sum_{n \ge 0} b_n$ is given by

$$\sum_{n \ge 0} b_n {Y_r}^n.$$

To prove the injectivity assume for $\sum_{n \ge 0} b_n \in B^D_{rig}(R_0, N, k)$ we have:

$$\sum_{n \ge 0} b_n = (E_{p-1} - r) \sum_{n \ge 0} f_n,$$

where $f_n \in S^D(R_0, N, k+n(p-1))$ and $f_n \to 0$ as $n \to \infty$. By a simple induction we can show that:

$$r^{n+1} f_n = -\textstyle\sum_{j=0}^{n} r^j b_j E_{p-1}^{n-j} \qquad \forall n \ge 0$$

Let j_0 be the smallest index for which $b_{j_0} \ne 0$. From the above formula and by corollary 8.7, $f_n \equiv 0$ *mod* p^M iff $r^j b_j \equiv 0$ *mod* p^M for all $0 \le j \le n$. Now for any $M \ge 0$, there exists

$n \geq j_0$ such that $f_n \equiv 0 \ mod \ p^M$. This implies $r^{j_0} b_{j_0} \equiv 0 \ mod \ p^M$ for all $M \geq 0$. Therefore $r^{j_0} b_{j_0} = 0$ and thus $b_{j_0} = 0$. This contradicts the choice of j_0 and proves the injectivity.

Next we prove the surjectivity. Let $\sum_{n \geq 0} f_n$ be such that $f_n \to 0$ as $n \to \infty$. By corollary 8.7 we have a decomposition of f_n:

$$f_n = \sum_{j=0}^{n} E_{p-1}^{n-j} b_j(n)$$

with $b_j(n) \in B^D(R_0, N, k, j)$. Again $f_n \equiv 0 \ mod \ p^M$ implies that $b_j(n) \equiv 0 \ mod \ p^M$ for all $0 \leq j \leq n$. Thus $b_j(n)$ tends to zero uniformly in j, as $n \to \infty$. Now f_n has the same image in $S^D(R_0, r, V_1(N), k)$ as:

$$f_n' = \sum_{j=0}^{n} r^{n-j} b_j(n)$$

which is obtained from the expression for f_n by replacing E_{p-1} with r. Define

$$b_j' := \sum_{n \geq j} r^n b_j(n).$$

The uniform convergence of $b_j(n)$ in j implies that b_j' tends to zero as $j \to \infty$. Now it is clear that $\sum_{j \geq 0} b_j'$ is an element of $B_{rig}^D(R_0, N, k)$ which maps to $\sum_{n \geq 0} f_n'$, which has the same image in $S^D(R_0, r, V_1(N), k)$ as $\sum_{n \geq 0} f_n$. This proves the surjectivity. □

This shows that if we choose a basis $\{b_n^{(j)}\}_j$ for each $B^D(R_0, N, k, n)$, then the set $\{b_n^{(j)} Y_r^n\}_{n,j}$ forms a Banach basis for $S^D(R_0, r, V_1(N), k)$. In other words any p-adic modular form $f \in S^D(R_0, r, V_1(N), k)$ can be written as $\sum_{n,j} \lambda_{n,j} b_n^{(j)} Y_r^n$ in which $\lambda_{n,j} \in R_0$ tends to zero as $n + j$ goes to ∞.

Corollary 8.9. *If $r_2 = r r_1$ in R_0, the canonical mapping:*

$$S^D(R_0, r_2, N, k) \to S^D(R_0, r_1, N, k)$$

defined from the transformation of functors:

$$(A, i, \alpha_N, Y) \mapsto (A, i, \alpha_N, rY)$$

is injective. In terms of the described Banach basis, it is given by:

$$\begin{array}{rcl} B_{rig}^D(R_0, N, k) & \to & B_{rig}^D(R_0, N, k) \\ \sum_{n \geq 0} b_n & \mapsto & \sum_{n \geq 0} r^n b_n \end{array}$$

Proof. This is immediate from looking at the induced endomorphism of $B_{rig}^D(R_0, N, k)$. □

Corollary 8.9 tells us that any element of $S^D(R_0, r, V_1(N), k)$ for $ord(r) > 0$, which we call an *overconvergent modular form*, determines a unique element of $S^D(R_0, 1, V_1(N), k)$ whose elements are called *convergent modular forms.* We will later see the reason behind this terminology. In fact for our purposes we are mostly interested in understanding the overconvergent modular forms. It turns out that it is often more difficult to prove similar results for overconvergent modular forms, compared to the convergent ones. It is interesting to have a criterion which can decide whether a convergent modular form is overconvergent. Indeed there is an easy answer to this question in terms of the Banach basis. We are still making the assumptions of proposition 8.4.

Corollary 8.10. *Assume r is not a unit in R_0. Let $f \in S^D(R_0, 1, V_1(N), k)$ and*

$$f = \sum_{n \geq 0} b_n Y_1^n$$

be its expansion in terms of our Banach basis. Then f is in $S^D(R_0, r, V_1(N), k)$ (or more precisely in the image of the inclusion) iff r^n divides b_n in $S^D(R_0, N, k + (p-1)n)$ for all $n \geq 0$ and $b_n / r^n \to 0$ as $n \to \infty$.

Proof. It is clear from theorem 8.8. □

9 p-adic Modular Forms in the Rigid Setting

In section 8, we derived a description of p-adic modular forms as sections of line bundles over certain schemes, when p is nilpotent in the ground ring. To give a similar description for more general ground rings we have to work with formal schemes and rigid analytic varieties.

9.1 Formal Schemes and Rigid Analytic Varieties

Throughout this section we assume R_0 to be a complete valuation ring of mixed characteristic $(0, p)$, with field of fractions K and the valuation normalized such that $ord(p) = 1$.

J. Tate in [17] introduced the notion of a rigid analytic space over K which is similar to the notion of an analytic space over $\mathbb{C}$. Every rigid space is equipped with a Grothendieck topology and a structure sheaf $\mathcal{O}^{an}$ whose sections on any open set U are the "holomorphic" functions on U. A basic example of the a rigid space is the unit ball $B_K^1 = \{x \in K \mid |x|_v \leq 1\}$. The global sections of $\mathcal{O}^{an}_{B_K^1}$ are exactly the convergent power series on B_K^1. In fact B_K^1 is the simplest example of an affinoid over K. Affinoids are a special class of rigid spaces. They are the building blocks of rigid analytic spaces as affine schemes are in algebraic geometry. One can define the notion of a coherent sheaf of modules as well as its cohomologies on any rigid space. Tate showed that one can equip any algebraic variety over K with an analytic structure and make it into a rigid analytic space so that any Zariski open set remains open and all the algebraic functions on it become holomorphic. In other words, there exists a functor:

$$\begin{array}{rcl} an : ((Algebraic\ Varieties/K)) & \to & ((Rigid\ Spaces/K)) \\ X & \mapsto & X^{an}. \end{array}$$

This functor takes any sheaf of modules $\mathcal{F}$ on X to a sheaf of modules $\mathcal{F}^{an}$ on X^{an}. It is known that there is a rigid GAGA theorem for this functor:

Theorem 9.1. *If X is a closed subscheme of $\mathbb{P}^n$*

i. *$\mathcal{F} \mapsto \mathcal{F}^{an}$ is an equivalence of categories between algebraic and rigid coherent sheaves.*

ii. *$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an})$ for each i.*

iii. *The functor "an" is a fully faithful functor.*

Therefore we can think of any algebraic variety over K as a rigid analytic space via *an*. Often we will let X denote its analyfication X^{an}. There are, of course, many rigid analytic spaces which are not algebraic. M. Raynaud has constructed a functor which associates to any admissible formal scheme over R_0, a rigid analytic space. A formal scheme is admissible if it is flat over R_0 and is locally topologically finitely generated. This functor is described in [3].

$$\begin{array}{rcl} rig : ((Admissible\ Formal\ Schemes/R_0)) & \to & ((Rigid\ Analytic\ Spaces/K)) \\ X & \mapsto & X^{rig} \end{array}$$

One can associate to any sheaf of modules $\mathcal{F}$ on X to a sheaf of modules $\mathcal{F}^{rig}$ on X^{rig}.

Theorem 9.2. **i.** *The functor "rig" is a faithful functor.*

ii. *Any rigid algebraic variety over K is in the image of this functor.*

iii. *If X is a variety over R_0 and $\tilde{X}$ is its completion in the maximal ideal of R_0, then there is an open immersion $\tilde{X}^{rig} \hookrightarrow (X \otimes K)^{an}$ which is an isomorphism when X is proper over R_0.*

9.2 Formal Schemes Setting

Throughout this section we assume R_0 to be a complete valuation ring of mixed characteristic $(0, p)$ and with field of fractions K. Recall that for $r \in R_0$ we defined $\mathbb{Y}_r^D \otimes R_0$ to be $Spec_{\mathbb{X}_1^D(N) \otimes R_0}(Symm(\underline{\omega}^{\otimes p-1})/<E_{p-1} - r>)$. If $r = 1$, this is an affine scheme.

Definition 9.3. *Let $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ denote the formal scheme over R_0 defined by the completion of $\mathbb{Y}_r^D \otimes R_0$ along the closed subscheme defined by $p = 0$. Denote the completion of $\mathbb{B}_r^D \otimes R_0$ (defined in section 8) and $\underline{\omega}$ on $\mathbb{Y}_r^D \otimes R_0$ by $\widetilde{\mathbb{B}}_r^D \otimes R_0$ and $\underline{\omega}_{\widetilde{\mathbb{B}}_r^D \otimes R_0 / \widetilde{\mathbb{Y}}_r^D \otimes R_0}$ or simply $\underline{\omega}$.*

Proposition 9.4. $S^D(R_0, r, V_1(N), k) = H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \underline{\omega}^{\otimes k})$

Proof.

$$\begin{aligned} H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \underline{\omega}^{\otimes k}) &= \varprojlim_{n\geq 0} H^0(\mathbb{Y}_r^D \otimes R_0/p^n, \underline{\omega}^{\otimes k}) \\ &= \varprojlim_{n\geq 0} S^D(R_0/p^n, r, N, k) \\ &= S^D(R_0, r, V_1(N), k) \end{aligned}$$

□

Let $r_1, r_2 \in R_0$ be such that $r_1 = rr_2$. There is a morphism

$$\widetilde{\mathbb{Y}}_{r_2}^D \otimes R_0 \to \widetilde{\mathbb{Y}}_{r_1}^D \otimes R_0$$

which on the moduli level is defined by $(A, i, \alpha_N, Y) \mapsto (A, i, \alpha_N, rY)$.

There is also a morphism

$$\widetilde{\mathbb{Y}}_r^D \otimes R_0 \to \widetilde{\mathbb{X}}_1^D(N)$$

which is obtained by completion of the natural map $\mathbb{Y}_r^D \otimes R_0 \to \mathbb{X}_1^D(N) \otimes R_0$.

9.3 Rigid Setting

In this section we assume that R_0 is a complete discrete valuation ring of mixed characteristic $(0, p)$, with field of fractions K and valuation normalized so that $ord(p) = 1$. Let K_∞ denote the completion of an algebraic closure of K and denote its ring of integers by R_∞

Let X be a reduced proper flat scheme of finite type over R_0 and $\mathcal{L}$ a line bundle on X. Let s be a global section of $\mathcal{L}$. In [7] Coleman explains how to associate to this data affinoid subdomains of $X \otimes K$. Let x be a closed point of $X \otimes K$ with residue field K_x, which is a finite extension of K and carries a unique extension of norm of K. Let R_x denote its ring of integers. Then since X is proper, the morphism $Spec(K_x) \to X$ corresponding to x extends to a morphism $f_x : Spec(R_x) \to X$. Now since R_x is a discrete valuation ring, $f_x^*(\mathcal{L})$ is a trivial line bundle generated by a section t. Let $f_x^*(s) = at$ with $a \in R_x$. Define $|s(x)| = |a|$, which is independent of choice of t. For any $r \in |K_\infty|$, there is a unique rigid subspace $X(r)$ of $(X \otimes K)^{an}$ which is a finite union of affinoids and whose closed points are points x of $X \otimes K$ such that $|s(x)| \geq r$. Coleman shows that if X is an irreducible curve and $r \in |K_\infty|$, then $X(r)$ is an affinoid subdomain of $X \otimes K$ unless $\mathcal{L} = \mathcal{O}_X$ and s is nowhere vanishing, in which case $X(r) = X$.

Definition 9.5. *Do the same construction as above with $X = \mathbb{X}_1^D(N) \otimes R_0$, $\mathcal{L} = \underline{\omega}^{\otimes p-1}$ and $s = E_{p-1}$ to obtain affinoid subdomains $X^D(r)$ of $\mathbb{X}_1^D(N) \otimes K$ for each $r \in |K_\infty|$. Denote the analyfication of $\underline{\omega}$ on $\mathbb{X}_1^D(N) \otimes K$ and its restrictions to any affinoid $X^D(r)$, again by $\underline{\omega}$.*

When $r = 1$, $X^D(r)$ contains all the points at which E_{p-1} has norm exactly 1. In other words it contains all the points x such that $\mathbf{H}$ does not vanish at the reduction of x modulo the maximal ideal of R_0. So $X^D(1)$ (*the ordinary part of* $\mathbb{X}_1^D(N) \otimes K$)) is obtained by removing from the Shimura curve, those false elliptic curves which have a supersingular reduction modulo the maximal ideal of R_0.. The complement of $X^D(1)$ is the union of the finitely many supersingular discs (each disc consists of all points whose reduction is a fixed supersingular false elliptic curve with level $V_1(N)$ structure in characteristic p). Allowing r to be less than 1, corresponds to removing smaller supersingular discs from $\mathbb{X}_1^D(N) \otimes K$. The ordinary part, $X^D(1)$, is an affinoid subdomain of $X^D(r)$ for each $r \leq 1$.

By rigid GAGA 9.1, modular forms of weight k and level $V_1(N)$ with respect to D over K are exactly the sections of $\underline{\omega}^{\otimes k}$ on $(\mathbb{X}_1^D(N) \otimes K)^{an}$. We will consider the sections of the same line bundle over the smaller affinoids $X^D(r)$.

We define a *convergent modular form* of weight k and level $V_1(N)$ over K to be an element of $H^0(X^D(1), \underline{\omega}^{\otimes k})$. An *overconvergent modular form* of weight k and level $V_1(N)$ and growth condition r over K is an element of $H^0(X^D(r), \underline{\omega}^{\otimes k})$. These modular forms are called overconvergent since they can be partially extended to supersingular discs. Not surprisingly, these modular forms are related to p-adic modular forms with respect to D.

Proposition 9.6. *Let $r \in R_0$. Then $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ is an admissible formal scheme and there is an isomorphism $(\widetilde{\mathbb{Y}}_r^D \otimes R_0)^{rig} \xrightarrow{\sim} X^D(|r|)$. If $r_1 = r.r_2$, then the following diagram is commutative:*

$$\begin{array}{ccc} (\widetilde{\mathbb{Y}}_{r_2}^D)^{rig} & \longrightarrow & X^D(|r_2|) \\ \downarrow & & \downarrow \\ (\widetilde{\mathbb{Y}}_{r_1}^D)^{rig} & \longrightarrow & X^D(|r_1|) \\ \downarrow & & \downarrow \\ (\widetilde{\mathbb{X}}_1^D(N))^{rig} & \longrightarrow & (\mathbb{X}_1^D(N) \otimes K)^{an} \end{array}$$

Proof. Let $\{Spec(C_j)\}_{j \in I}$ be a covering of $\mathbb{X}_1^D(N) \otimes R_0$, each element of which intersects the special fibre and such that the restriction of $\underline{\omega}$ to $Spec(C_j)$ is trivial and generated by section t_j. Let $E_{p-1} = a_j t_j$ over $Spec(C_j)$. Then $\mathbb{Y}_r^D \otimes R_0$ has a covering $\{Spec(C_j[t_j]/ < a_j t_j - r >)\}_{j \in I}$ with gluing data induced by those of the covering of $\mathbb{X}_1^D(N) \otimes R_0$. Completing along the $p = 0$ subscheme, we obtain $\widetilde{\mathbb{Y}}_r^D \otimes R_0$, in which the above covering becomes $\{Spf(\hat{C}_j << t_j >> / < a_j t_j - r >)\}_{j \in I}$ where $\hat{C}$ denotes the p-adic completion of C. This shows that $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ is locally topologically finitely generated. Also an argument like in 8.6, shows that $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ is flat over R_0. Therefore $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ is admissible. Now the the image of $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ under the Raynaud's functor is given by a covering $\{Sp(\hat{C}_j \otimes K << t_j >> / < a_j t_j - r >)\}_{j \in I}$ with the induced gluing data. On the other hand $\{Sp(\hat{C}_j \otimes K)\}_{j \in I}$ with the induced gluing data, gives us $(\mathbb{X}_1^D(N) \otimes R_0)^{an}$. But by proposition 7.2.3.4 of [4], $Sp(\hat{C}_j \otimes K << t_j >> / < a_j t_j - r >)$ is the affinoid subdomain of $Sp(\hat{C}_j \otimes K)$ on which $|a_j| \geq |r|$. To get the diagram first note that the lower isomorphism is given by 9.2 **iii.**, as $\mathbb{X}_1^D(N) \otimes R_0$ is proper over R_0. The commutativity of the diagram follows by looking at image of t_j in each chart. □

Lemma 9.7. *Let X be an admissible formal scheme. Let $\mathcal{F}$ be a locally free sheaf on X. Then:*

$$H^0(X^{rig}, \mathcal{F}^{rig}) = H^0(X, \mathcal{F}) \otimes K.$$

Proof. We only need to show this for $\mathcal{F} = \mathcal{O}_X$. Since the functor rig is faithful, there is an injection $H^0(X, \mathcal{O}_X) \otimes K \to H^0(X^{rig}, \mathcal{O}_X^{rig})$. We need to prove surjectivity. Cover X with a finite number of affinoid opens $\{Spf(C_i)\}$, such that each open intersects the special fibre. We get a finite covering of X^{rig} with affinoid opens $\{Sp(C_i \otimes K)\}$. This shows that over any of these affinoid subdomains, we can multiply the given section by a power of π (π a generator of maximal ideal of R_0), such that it comes from a section on X. Now because the cover is finite, multiplying the global section by a large enough power of π, it lies in the image of rig. We are done. □

We are now able to link the (over)convergent modular forms with p-adic modular forms with respect to D.

Corollary 9.8. *If $r \in R_0$ we have:*

$$S^D(R_0, r, V_1(N), k) \otimes K = H^0(X^D(|r|), \underline{\omega}^{\otimes k}).$$

Proof. By proposition 9.6 and lemma 9.2 we have:

$$\begin{aligned} H^0(X^D(|r|), \underline{\omega}^{\otimes k}) &= H^0((\widetilde{\mathbb{Y}}_r^D \otimes R_0)^{rig}, \underline{\omega}^{\otimes k}) \\ &= H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \underline{\omega}^{\otimes k}) \otimes K \\ &= S^D(R_0, r, V_1(N), k) \otimes K. \end{aligned}$$

□

From now on, if $r \in R_0$, we denote $X^D(|r|)$ by $X^D(r)$.

Corollary 9.9. *$S^D(R_0, r, V_1(N), 0) \otimes K$ is a K affinoid algebra and*

$$X^D(r) = Sp(S^D(R_0, r, V_1(N), 0) \otimes K).$$

Proof. Since $X^D(r)$ is an affinoid, it is equal to $Sp(H^0(X^D(r), \mathcal{O}_{X^D(r)}))$. But by the above corollary

$$H^0(X^D(r), \mathcal{O}_{X^D(r)}) = H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0}) \otimes K = S^D(R_0, r, V_1(N), 0) \otimes K.$$

This also shows that $S^D(R_0, r, V_1(N), 0)$ is a K affinoid algebra. □

The results of this section prove that our theory of p-adic modular forms is independent of lifting of $\mathbf{H}$ (at least after tensoring with K) as long as we assume that $ord(r) < 1$, because any two such liftings have the same absolute value and define the same rigid analytic spaces $X^D(r)$.

The description of p-adic modular forms as sections of certain line bundles over rigid analytic spaces, will allow us to use rigid geometry in our study of these objects. This rigid geometric description is essential in construction of the U operator of overconvergent modular forms and in the study of its eigenforms.

We end this section by defining the universal family of false elliptic curves over $X^D(r)$. We define $A^D(r)$ to be the image of $\widetilde{\mathbb{B}}_r^D \otimes R_0$ under the Raynaud's functor rig.

$$A^D(r) := (\widetilde{\mathbb{B}}_r^D \otimes R_0)^{rig}$$

This is a family of false elliptic curves over $X^D(r)$. We will study this more, later on.

10 p-divisible and Formal Group of a False Elliptic Curve

Let R be a p-adically complete $\mathbb{Z}_p$-algebra. Let (A, i) be a false elliptic curve over R. Let H be a finite flat subgroup scheme of A which is stable under the action of $\mathcal{O}_D$ on A and is killed by p^n for some $n \geq 1$. Since $\mathbb{Z}_p$ acts via $\mathbb{Z}/p^n\mathbb{Z}$ on H, we gain an action of $\mathcal{O}_D \otimes \mathbb{Z}_p \xrightarrow{\sim} M_2(\mathbb{Z}_p)$ on H. Recall our choice of idempotents e_1, e_2 in $M_2(\mathbb{Z}_p)$ from section 2 and fix g such that $g^{-1}e_1 g = e_2$. Let H^i denote $Ker(e_i : H \to H)$. It is easy to see that the inclusions $H^i \hookrightarrow H$ give an isomorphism $H^1 \times H^2 \xrightarrow{\sim} H$. Clearly g induces an isomorphism $H^1 \xrightarrow{\sim} H^2$.

For each $n \geq 1$, let $A[p^n]$ denote the kernel of multiplication by p^n in A. By the above discussion, for each $n \geq 1$, we have a decomposition:

$$A[p^n]^1 \times A[p^n]^2 \xrightarrow{\sim} A[p^n].$$

Also g gives us functorial isomorphisms:

$$g : A[p^n]^1 \xrightarrow{\sim} A[p^n]^2.$$

These isomorphisms are compatible for different n and therefore induce a similar decomposition of the p-divisible group of A:

$$A[p^\infty]^1 \times A[p^\infty]^2 \xrightarrow{\sim} A[p^\infty]$$
$$g : A[p^\infty]^1 \xrightarrow{\sim} A[p^\infty]^2.$$

Lemma 10.1. *Let H be a finite flat subgroup of A which is killed by p^n for some $n \geq 1$. H is stable under the action of $\mathcal{O}_D$ iff it can be written as $H^1 \times H^2$ where H^1 is a finite flat subgroup of $A[p^n]^1$ and H^2 is the image of H^1 under the isomorphism $g : A[p^n]^1 \xrightarrow{\sim} A[p^n]^2$.*

Proof. By discussions in the beginning of this section, we only need to show that any such H is stable under action of $\mathcal{O}_D$. The elements of $M_2(\mathbb{Z}_p)$, given by $e_1, e_2, ge_2, g^{-1}e_1$, generate $M_2(\mathbb{Z}_p)$ over $\mathbb{Z}_p$. Since H is stable under any of these actions, it is so under the action of $M_2(\mathbb{Z}_p)$ and hence of $\mathcal{O}_D$.

□

We now study the formal group associated to a false elliptic curve. The formal group associated to $A \to S$ which we denote by $G_{A/S}$ or simply by G_A, is the completion of A along the identity section. Assume G_A is given by a 2-dimensional formal group law . That is always possible after a localization on the base. As always e_1 and e_2 denote our choice of idempotents in $M_2(\mathbb{Z}_p)$.

Since R is a p-adically complete $\mathbb{Z}_p$-algebra, there is an action of $\mathbb{Z}_p$ on G_A. On the other hand any endomorphism of A induces an endomorphism of G_A and hence we obtain an action of $\mathcal{O}_D$ on G_A. As a result one sees that $\mathcal{O}_D \otimes \mathbb{Z}_p \xrightarrow{\sim} M_2(\mathbb{Z}_p)$ acts on G_A. Let G_A^i denote $Ker(e_i : G_A \to G_A)$ for $i = 1, 2$. Then the inclusions $G_A^i \hookrightarrow G_A$ induce an isomorphism

$$G_A^1 \times G_A^2 \xrightarrow{\sim} G_A.$$

Furthermore since e_1 and e_2 are conjugate by g we have an isomorphism

$$g : G_A^1 \xrightarrow{\sim} G_A^2.$$

We prove that over an integral base these formal group schemes are given by formal group laws.

Lemma 10.2. *Let $R \xrightarrow{\pi} S$ be a homomorphism of commutative rings. Let $S \xrightarrow{\pi'} R$ be a surjection such that $\pi' \circ \pi = id_R$. Let $I = Ker(\pi')$. Assume that S is complete in the I-adic topology and that I/I^2 is free of rank d as a module over $S/I = R$. Then there is a surjection $R[\![x_1, ...x_d]\!] \to S$.*

Proof. :

Choose $\delta_1, ..., \delta_d$ in I so that their images I/I^2 form a basis over R. Define a morphism $f : R[\![x_1, ...x_d]\!] \to S$ by sending x_i to δ_i for $i = 1, .., d$. This is possible because δ_i's are in I and S is I-adically complete. From this definition it is evident that the induced map $gr^k(R[\![x_1, ...x_d]\!]) \xrightarrow{f} I^k/I^{k+1}$ is surjective. Suppose $s \in S$ and $s_0 = P_0 = s \ mod \ I$ in R. Suppose we have chosen $s_i \in I^i$ and $P_i \in gr^i(R[\![x_1, ...x_d]\!])$ such that $f(P_i) = s_i$ for $i = 0, ..., k-1$ and that $s - \sum_{i=1}^{k-1} s_i \in I^k$. We can then choose $P_k \in gr^k(R[\![x_1, ...x_d]\!])$ such that $s_k := f(P_k) = s - \sum_{i=1}^{k-1} s_i \ mod \ I^{k+1}$. This way we can define s_i and P_i as above for all $i \geq 0$. Now it is clear from the above construction that $s = \sum_{i \geq 0} s_i$. Let $P := \sum_{i \geq 0} P_i$ which is convergent. Then we have $f(P) = s$. This proves surjectivity of f.

□

Proposition 10.3. *If R is a domain,then G_A^i for $i = 1, 2$ is given by a formal group law.*

Proof. Since G_A is given by a formal group law, we have $G_A \xrightarrow{\sim} Spf(R[\![x_1, x_2]\!])$. Let the action of e_1 and e_2 on G_A be given by $(h_1(x_1, x_2), h_2(x_1, x_2))$ and $(g_1(x_1, x_2), g_2(x_1, x_2))$ resepctively. Then $G_A^1 = Ker(e_1) = Spf(R[\![x_1, x_2]\!]/(h_1, h_2))$. Denote $R[\![x_1, x_2]\!]/(h_1, h_2)$ by S and let I be the ideal of S generated by x_1, x_2. S is complete in the I-adic topology. Since e_1 induces an idempotent endomorphism of G_A, we know $Jac(h_1, h_2)$ is an idempotent matrix. By a linear change of coordinates we can assume $Jac(h_1, h_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathbb{Z}_p)$. This means $h_1(x_1, x_2) = x_1 \; mod \; (x_1, x_2)^2$ and $h_2(x_1, x_2) = 0 \; mod \; (x_1, x_2)^2$. Now $I/I^2 = (x_1, x_2)/((x_1, x_2)^2 + (h_1, h_2)) = (x_1, x_2)/((x_1, x_2)^2 + (x_1)) = R.x_2$ which is clearly free of rank 1 over R. Applying the lemma gives a surjection $R[\![t]\!] \to R[\![x_1, x_2]\!]/(h_1, h_2)$ in which the image of t is x_2. We prove the injectivity. We have :

$$R[\![t]\!] \to R[\![x_1, x_2]\!]/(h_1, h_2) \to R[\![x_1, x_2]\!]$$

in which the second map corresponds to projection $G_A \to G_A^1$. Thus the image of (x_1, x_2) under this map is (g_1, g_2). Now if $l(t)$ is in the kernel of the above surjection, then the image of $l(t)$ in $R[\![x_1, x_2]\!]$ under the above composite is 0. But this image is $l(g_2(x_1, x_2))$. Now using that R is a domain and considering the factor of g_2 with minimum degree, we easily deduce that $l(t) = 0$ unless $g_2 = 0$. In that case since e_2 is idempotent we get $g_1(g_1, 0) = g_1$. A similar argument considering the term with minimum degree in $g_1(x, 0)$ implies $g_1 = 0$. But this means $e_2 = 0$ which is not the case. To get the final result we note that any formal group scheme of the form $Spf(R[\![t]\!])$ is given by a formal group law. □

Now we recall a general fact about formal group laws of dimension 1 from [11].

Proposition 10.4. *Let R be a $\mathbb{Z}_p$-algebra and G a 1-dimensional formal group law over R. There exists a coordinate x on G such that for each $(p-1)$'st root of unity $\zeta \in \mathbb{Z}_p$ we have $[\zeta](x) = \zeta x$. (where $[\zeta]$ denotes the power series in x giving the action of $\zeta \in \mathbb{Z}_p$ on G). In this coordinate multiplication by p takes a special form:*

$$[p](x) = px + ax^p + \sum_{m=2}^{\infty} c_m.x^{m(p-1)+1}$$

where $a, c_n (n \geq 2) \in R$ and $c_n \in pR$ unless $n \equiv 1 \; mod \; p$.

Proof. : This is proposition 3.6.6 of [11].

□

Now let (A, i) be a false elliptic curve over a $\mathbb{Z}_p$-algebra, R. Let x be a coordinate on G_A^1 as above. In particular we can write

$$[p](x) = px + a_A x^p + \sum_{m=2}^{\infty} c_m.x^{m(p-1)+1}.$$

Let ω denote the dual differential with respect to this coordinate. Then ω can be written as $\omega = (1 + \sum_{m=1}^{\infty} a_m x^m)dx$.

Proposition 10.5. *With notation as above, a_A is congruent to $E_{p-1}(A, i, \omega)$ mod pR.*

Proof. We will use $\bar{}$ to denote reduction mod p. Let $V : (\bar{G}_A^1)^{(p)} \to \bar{G}_A^1$ denote the Verschebung and $F : \bar{G}_A^1 \to (\bar{G}_A^1)^{(p)}$ denote the Frobenius morphism. Assume

$$V(x) = ax + terms\ of\ higher\ degree$$

in which a is determined by the relation $V^*(\bar{\omega}) = a\bar{\omega}^{(p)}$. But we have $F^*(\check{\bar{\omega}}^{(p)}) = \check{\bar{\omega}}^{(p)} \circ V^*$. Thus

$$\mathbf{H}(\bar{A}, \bar{i}, \bar{\omega}) = F^*(\bar{\omega}^{(p)})(\bar{\omega}) = \check{\bar{\omega}}^{(p)} \circ V^*(\bar{\omega}) = \check{\bar{\omega}}^{(p)}(a\bar{\omega}^{(p)}) = a.$$

But:

$$[\bar{p}](x) = V(F(x)) = V(x^p) = ax^p + terms\ of\ higher\ degree$$

which shows that $a = \bar{a}_A$. This means $\bar{a}_A = \mathbf{H}(\bar{A}, \bar{i}, \bar{\omega})$ or equivalently

$$E_{p-1}(A, i, \omega) \equiv a_A \quad mod\ p.$$

□

11 Canonical Subgroup Theorem

Throughout this section we assume R_0 to be a complete discrete valuation ring of mixed characteristic $(0, p)$ for which $ord(p) = 1$.

In this section we will construct the canonical subgroup of a false elliptic which is not "too supersingular". To motivate, let (A, i) be a false elliptic curve over R_0 whose associated formal group G_A is isomorphic to the 2-dimensional multiplicative formal group over R. The kernel of multiplication by p in G_A is therefore isomorphic to the group scheme $\mu_p \times \mu_p$ over R which can be thought of as a finite flat subgroup scheme of A. It has rank p^2 and is stable under the action of $\mathcal{O}_D$ and lifts the kernel of $Frob_A$. This is called the *Canonical Subgroup Scheme* of A. It is clear that the false elliptic curves of the form above have ordinary reduction mod p. In analogy to the classical case [12] we prove the existence of a canonical subgroup for any false elliptic curve which is not too supersingular.

Theorem 11.1. *(***Canonical Subgroup Theorem***)*

i. *Let $r \in R_0$ with $ord(r) < p/(p+1)$. There is a canonical way to associate to every r-test object (A, i, α_N, Y), where:*

- *(A, i, α_N) is a false elliptic curve with level $V_1(N)$ structure over a p-adically complete R_0-algebra R,*
- *Y is a section of $\underline{\omega}_{A/R}^{\otimes 1-p}$ which satisfies $YE_{p-1} = r$,*

a finite flat subgroup scheme H of A such that:

- *H has rank p^2 and is stable under the action of $\mathcal{O}_D$,*
- *H depends only on the R-isomorphism class of $(A/R, i, \alpha, Y)$,*
- *The formation of H commutes with arbitrary base change of p-adically complete R_0-algebras,*
- *If $p/r = 0$ in R, then H can be identified with the kernel of Frobenius $Frob_A : A \to A^{(p)}$.*

ii. *Let $r \in R_0$ with $ord(r) < 1/(p+1)$. There is a canonical way to associate to every $(A/R, i, \alpha, Y)$ as in part* **i.**, *an r^p-test object $(A'/R, i, \alpha', Y')$ where:*

- *(A', i, α'_N) is the quotient of (A, i, α_N) by H,*
- *Y' is a section of $\underline{\omega}_{A'/R}^{\otimes 1-p}$ which satisfies: $Y' E_{p-1} = r^p$,*

such that:

- *Y' depends only on the R-isomorphism class of $(A/R, i, \alpha, Y)$,*
- *The formation of Y' commutes with arbitrary base change of p-adically complete R_0-algebras,*
- *If $p/r = 0$ in R, then Y' is equal to $Y^{(p)}$ on $A^{(p)} = A/H$.*

Proof. We will construct the canonical subgroup by studying the formal group associated to a false elliptic curve. First we construct a precanonical subscheme for formal groups.

Construction of precanonical subscheme for formal groups over $\mathbf{R_0}$-algberas flat over $\mathbb{Z}_\mathbf{p}$:

Lemma 11.2. *Let R be a p-adically complete R_0-algebra which is flat over $\mathbb{Z}_p$ and $r \in R_0$ with $ord(r) < p/(p+1)$ and $r_1 = -p/r \in R_0$. Let G be a formal group scheme over R which is defined by a one dimensional formal group law. By 10.4 there is a coordinate x on G for which $[\zeta](x) = \zeta x$ for ζ any $(p-1)$'st root of unity in $\mathbb{Z}_p$, and we have $[p](x) = px + ax^p +$ terms of higher degree. Assume that there are $y, b \in R$ such that $(a+pb)y = r$. Then there is a subscheme H of G such that $H \otimes R/r_1$ can be identified with kernel of $Frob_{G \otimes R/r_1} : G \otimes R/r_1 \to G^{(p)} \otimes R/r_1$ and its formation commutes with arbitrary base change of p-adically complete R_0-algebras which are flat over $\mathbb{Z}_p$. Furthermore H is independent of choice of x.*

Proof of lemma as in [11]: By 10.4 $[p](x) = px + ax^p + \sum_{n \geq 2} c_n x^{n(p-1)+1}$. Define

$$f(T) := p + aT + \sum_{n \geq 2} c_n T^n$$

It is clear that $[p](x) = xf(x^{p-1})$. The desired subscheme will be defined by means of a canonical zero t_{can} of the power series $f(T)$. It will consist of "0 and the $p-1$ solutions" of $x^{p-1} = t_{can}$. Since $ord(r_1) > 0$, $1 + r_1 by$ is invertible in R. Let $t_0 := r_1 y/(1 + r_1 by)$ which satisfies: $p + at_0 = 0$. Now let $f_1(T) := f(t_0 T)$. Then $f_1(T) = p - pt + \sum_{n\geq 2} c_n t_0^n T^n$.

We study the coefficients of $f_1(T)$. By assumption $ord(r_1^{p+1}/p) > 0$. Let r_2 be a generator of ideal generated by r_1^{p+1}/p and r_1^2 in R_0. If $n \not\equiv_p 1$, by 10.4 $c_n \in pR$ and we have $(c_n/p)t_0^n \in t_0^2 R \subset r_1^2 R \subset r_2 R$. If $n \equiv_p 1$, $c_n t_0^n \in c_n t_0^{p+1} R \subset c_n r_1^{p+1} R$. But $r_1^{p+1}/p \in R_0$. Thus $c_n t_0^n/p \in (t_0^{p+1}/p)R \subset (r_1^{p+1}/p)R \subset r_2 R$. Clearly in each case $c_n t_0^n/p$ tends to zero as n tends to ∞. This shows that we can write $f_1(T) = pf_2(T)$ where $f_2(T) = 1 - T + \sum_{n\geq 2} d_n T^n$ with $d_n \in r_2 R$ and $d_n \to 0$ when $n \to \infty$.

Now let $f_3(T) = f_2(1-T)$. Then we can write $f_3(T) = e_0 + (1+e_1)T + \sum_{n\geq 2} e_n T^n$ where $e_n \in r_2 R$ for all $n \geq 0$. Let $I = r_2 R$. We now show that there is a unique element $t_\infty \in I$ such that $g(t_\infty) = 0$. If $t \in I$ then $f_3(t) \in I$ and $f_3'(t) \in 1 + I$ and hence $f_3'(t)$ is invertible. The Newton process of successive approximation: $t_1 = 0, ..., t_{n+1} = t_n - f_3(t_n)/f_3'(t_n)$ converges to a zero of f_3 which lies in I. If t and $t' = t + \Delta$ are two zeros of f_3 in I, we get: $-f_3'(t)\Delta = f_3(t+\Delta) - f_3(t) - f_3'(t)\Delta \in \Delta^2 R$. Since $f_3'(t)$ is a unit in R, $\Delta \in \Delta^2 R$. But $\Delta \in I$ and R is I-adically separated. This proves $\Delta = 0$ and $t = t'$.

Going backwards we obtain a zero $t_{can} = t_0(1 - t_\infty)$ of $f(T)$. Since $t_0 \in r_1 R$ is topologically nilpotent, we can expand $f(T)$ in terms of $(T - t_{can})$ and deduce that $f(T)$ is divisible by $T - t_{can}$ in $R[\![T]\!]$. Therefore $[p](x)$ is divisible by $x^p - t_{can}x$ in $R[\![x]\!]$. We define the precanonoical subscheme H of G to be the subscheme of $G[p]$ defined by $x^p - t_{can}x$. Thus $H = Spec(R[\![x]\!]/ < x^p - t_{can}x >)$ which is finite and flat of rank p over R. Furthermore $H/r_1 = Spec(R[\![x]\!]/<x^p, r_1>) = Spec((R/<r_1>)[\![x]\!]/x^p)$ which is nothing but $Ker(Frob_{G/r_1} : G/r_1 \to (G/r_1)^{(p)})$. This ends the proof of lemma 11.2.

Construction of precanonical subscheme H^1 for false elliptic curves:

It is enough to construct the canonical subgroup for false elliptic curves over bases in which p is nilpotent. Let R be an R_0-algebra in which p is nilpotent and (A, i, α_N, Y) be a test object over R. We will construct the precanonical subscheme locally on the base. Cover $\mathbb{Y}_r^D \otimes R_0$ with affine opens $\{Spec(C_i)\}_{i\in I}$ such that $G_{\mathbb{B}_r^D \otimes R_0}$ is given by a formal group law on each C_i, $i \in I$. Since each C_i is a domain by proposition 10.3 $G^1_{\mathbb{B}_r^D \otimes R_0}$ is also given by a formal group law for each $i \in I$. By 8.2 there is a map $h : Spec(R) \to \mathbb{Y}_r^D \otimes R_0$ such that $h^*((\mathbb{B}_r^D \otimes R_0, i, \alpha_N, Y_r)) = (A, i, \alpha_N, Y)$. Working locally on $Spec(R)$ we may assume that h lands in a member of the affine covering which we call $Spec(C)$. As a result G_A^1 will be given by a formal group law. Note that p is still nilpotent in the new base and thus the base is still p-adically complete. Therefore the induced map $C \xrightarrow{h^*} R$ factors through $\hat{C} :=$ p-adic completion of C. In other words there is a map $\phi : \hat{C} \to R$ such that $C \xrightarrow{\pi} \hat{C} \xrightarrow{\phi} R$ is equal to h^*.

Let x be a coordinate on $G^1_{\mathbb{B}_r^D \otimes R_0}|_{Spec(C)}$ as in proposition 10.4. Then we have $[p](x) =$

$px + ax^p + terms\ of\ higher\ degree$. Let ω be a basis of $\underline{\omega}$ on $Spec(C)$ which reduces to the dual differential with respect to x in $G^1_{\mathbb{B}^D_r \otimes R_0 | Spec(C)}$. Write $Y = y\omega^{\otimes 1-p}$. By proposition 10.4

$$E_{p-1}(A, i, \omega) \equiv a \ mod\ p$$

Thus $E_{p-1}(A, i, \omega) = a + pb$ for some $b \in C$ and $YE_{p-1} = r$ implies $(a + pb)y = r$. Now $\hat{C}$ is p-adically complete and flat over $\mathbb{Z}_p$ and therefore we can apply lemma 11.2 to obtain a precanonical subscheme of $G^1_{\mathbb{B}^D_r \otimes R_0}|_{Spec(C)} \otimes \hat{C}$, which is the formal group associated to the false elliptic curve $\mathbb{B}^D_r \otimes \hat{C}$. We denote this subscheme by $\mathbb{H}^1$, which is a finite flat subscheme of rank p of $G^1_{\mathbb{B}^D_r \otimes R_0}|_{Spec(C)} \otimes \hat{C}$ and hence of $\mathbb{B}^D_r \otimes \hat{C}$. Note that $A = \phi^*(\mathbb{B}^D_r \otimes \hat{C})$. We define the precanonical subscheme of (A, i, α_N, Y), denoted H^1, to be the inverse image of $\mathbb{H}^1$ in A.

$$H^1 = \phi^*(\mathbb{H}^1)$$

Clearly H^1 is a finite flat subscheme of rank p of $A[p]^1 \subset A$ which reduces to the kernel of $Frob_A$, modulo r_1.

The precanonical subscheme H^1, is a subgroup scheme of $A[p]^1$:

Let's first consider the case when $r = unit$ in R_0. This implies that $E_{p-1}(A, i)$ is nowhere vanishing. Hence in the notation of proposition 10.4, we have:

$$[p](x) = px + ax^p + ... \qquad a\ unit\ in\ R$$

Factorize $[p](x) = (x^p - t_{can}x)h(x)$. Writing $h(x) = \sum_{n \geq 0} h_i x^i$, we get

$$h_0 = t_{can}h_{p-1} + a$$

Since a is a unit in R and t_{can} is topologically nilpotent, we see that h_0 is a unit in R and therefore $h(x)$ is a unit in $R[\![x]\!]$. This implies that when r is a unit in R_0, the precanonical subscheme of an r-test object (A, i, α_N, Y) is equal to $Ker([p] : G^1_A \to G^1_A)$, which is a subgroup of $A[p]^1$.

In the general case we argue as follows. Let $G(x, y) \in R[\![x, y]\!]$ be the power series giving the addition in G^1_A. Now $H^1 \subset A[p]^1$ is a subgroup scheme if it is so in G^1_A. Note that $H^1 = Spec(R[\![x]\!]/{<}x^p - t_{can}x{>})$. Thus we need to prove that:

$$G(x, y)^p - t_{can}G(x, y) = 0 \qquad in\ \ R[\![x, y]\!]/{<}x^p{-}t_{can}x, y^p{-}t_{can}y{>}$$

Since t_{can} is topologically nilpotent in R, $R[\![x, y]\!]/{<}x^p{-}t_{can}x, y^p{-}t_{can}y{>}$ is free of rank p^2 with basis $\{x^i y^j\}_{0 \leq i,j \leq p-1}$. Express

$$G(x^p - t_{can}x, y^p - t_{can}y) = \sum_{0 \geq i,j \geq p-1} g_{ij} x^i y^j$$

in $R[\![x,y]\!]/<x^p - t_{can}x, y^p - t_{can}y>$ with $g_{ij} \in R$. Note that the formation of g_{ij} is functorial. Now H^1 is a subgroup scheme iff $g_{ij} = 0$ for all $0 \geq i, j \geq p-1$.

Let $\mathbb{H}_r^1$ denote the precanonical subscheme of $G^1_{\mathbb{B}_r^D \otimes R_0/\mathbb{Y}_r^D \otimes R_0} \otimes \widetilde{\mathbb{Y}}_r^D$. By construction of H^1, it suffices to show that $\mathbb{H}_r^1 \subset G^1_{\mathbb{B}_r^D \otimes R_0/\mathbb{Y}_r^D \otimes R_0} \otimes \widetilde{\mathbb{Y}}_r^D$ is a subgroup scheme. By the above discussion we have to show that the elements

$$g_{ij} \in H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0}) = S^D(R_0, r, V_1(N), 0)$$

vanish for all $0 \leq i, j \leq p-1$. Consider the natural map defined in 9.2.,

$$\widetilde{\mathbb{Y}}_1^D \otimes R_0 \to \widetilde{\mathbb{Y}}_r^D \otimes R_0.$$

We have already seen that $\mathbb{H}_1^1$ is a subgroup ($r = 1 = unit$ in this case). Therefore g_{ij}'s vanish on $\widetilde{\mathbb{Y}}_1^D \otimes R_0$. But by theorem 8.9

$$S^D(R_0, r, V_1(N), 0) \to S^D(R_0, 1, V_1(N), 0)$$

is injective. Thus g_{ij} vanish on $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ and $\mathbb{H}_r^1$ is a subgroup. This proves that for any (A, i, α_N, Y), H^1 is a group scheme.

Definition 11.3. *(Canonical Subgroup of a False Elliptic Curve)*
For any false elliptic curve (A, i) we can write $A[p] = A[p]^1 \times A[p]^2$ and there is a functorial isomorphism $A[p]^1 \xrightarrow{\sim} A[p]^2$. Let H^2 be the image of H^1 under this isomorphism. We define $H := H^1 \times H^2$ to be the canonical subgroup of (A, i, α_N, Y).

H is clearly of rank p^2. It is stable under the action of $\mathcal{O}_D$ by lemma 10.1. Other desired properties follow directly from similar properties proven for the precanonical subscheme in lemma 11.2. This ends the proof of part **i** of theorem 11.1.

Construction of Y' on $(A, i, \alpha_N, Y)/H$:

Let R be an R_0-algebra in which p is nilpotent and (A, i, α_N, Y) be an r-test object with $ord(r) < 1/(p+1)$. Let H be the canonical subgroup of (A, i, α_N, Y). Let (A', i, α_N') denote the quotient of (A, i, α_N) by H. As in the proof of part **i** we can assume there is an open affine subset $Spec(C)$ of $\mathbb{Y}_r^D \otimes R_0$, and a morphism $\phi : Spec(R) \to Spec(\hat{C})$ such that the pull back of $(\mathbb{B}_r^D \otimes \hat{C}, i, \alpha_N, Y_r)$ under ϕ is equal to (A, i, α_N, Y). So we only need to do the construction over $\mathbb{B}_r^D \otimes \hat{C}$. For simplicity we denote $\mathbb{B}_r^D \otimes \hat{C}$ by $\mathbb{B}$ and its canonical subgroup by $\mathbb{H}$. Let $(\mathbb{B}', i')$ be the quotient of $(\mathbb{B}, i)$ by $\mathbb{H}$. Let ω be a basis of $\underline{\omega}_{\mathbb{B}/\hat{C}}$ on $Spec(\hat{C})$. Write $Y_r = y\omega^{\otimes 1-p}$ on $Spec(\hat{C})$. Since $\mathbb{H}$ reduces to Kernel of $Frob_B$ mod $r_1\hat{C}$, $\mathbb{B}'$ reduces to $\mathbb{B}^{(p)}$ mod $r_1\hat{C}$. Let ω' be any basis of $\underline{\omega}_{\mathbb{B}'/\hat{C}}$ which reduces to $\omega^{(p)}$ on $\mathbb{B}^{(p)}$ mod $r_1\hat{C}$. We have:

$$E_{p-1}(\mathbb{B}', i', \omega') \equiv (E_{p-1}(\mathbb{B}, i, \omega))^p \quad mod r_1\hat{C}$$

This implies:

$$E_{p-1}(\mathbb{B}^{'},i^{'},\omega^{'}) = (E_{p-1}(\mathbb{B},i,\omega))^p + r_1.c$$

for some $c \in \hat{C}$. Since $ord(r) < 1/(p+1)$, r_1 is divisible by r^p in R_0 , and hence $r_3 = r_1/r^p$ lies in R. Define

$$y^{'} := y^p/(1 + r_3 c y^p)$$

Since $y.E_{p-1}(\mathbb{B},i,\omega) = r$, by calculation we deduce $y^{'}.E_{p-1}(\mathbb{B}^{'},i^{'},\omega^{'}) = r^p$. So we can define:

$$Y_r^{'} = y^{'}\omega^{'\otimes 1-p}$$

Uniqueness is clear from construction. This concludes the proof of part **ii** of theorem 11.1.

□

Let K denote the field of fractions of R_0. Let K_∞ denote the completion of an algebraic closure of K and R_∞ be its ring of integers. Assume that (A, i, α_N, Y) is an r-test object over R_0 which is supersingular. Denote its base extension to R_∞ with the same symbol. Choose a coordinate x as in 10.4 for G_A^1. We can write:

$$[p](x) = px + ax^p + \sum_{n=2}^{p} c_n x^{n(p-1)+1} + c_{p+1}x^{p^2} + \sum_{n\geq p+2} c_n x^{n(p-1)+1}$$

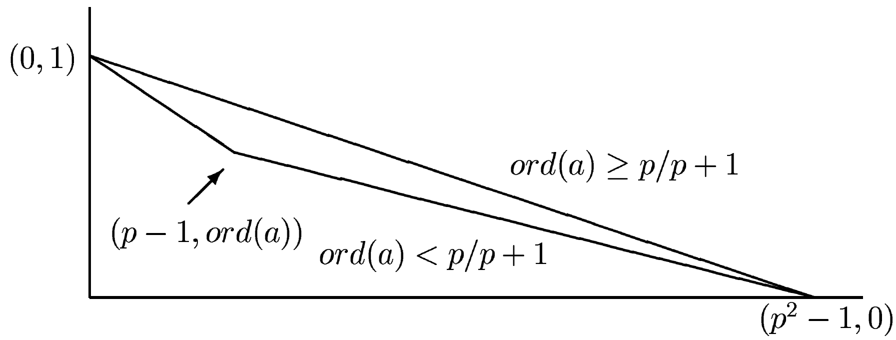
Since A has supersingular reduction, we have $ord(a) > 0$ and also $height(G_A^1) = 2$ which implies that $ord(c_{p+1}) = 0$. We also know that $ord(c_n) \geq 1$ if $n \not\equiv 1 \ mod\ p$. we can detect all the elements of the canonical subgroup of (A, i, α_N, Y) by looking at the valuations of zeroes of $[p](x)$.

Proposition 11.4. *Assume the notation is as above. Let H be the canonical subgroup of (A, i, α_N, Y). Write $H = H^1 \times H^2$ where $H^1 = H \cap G_A^1 = Ker(e_1 : H \to H)$ and H^2 is the image if H^1 under the isomorphism $A[p]^1 \xrightarrow{\sim} A[p]^2$. Then:*

1. *If $ord(a) < p/(p+1)$, then the valuation of any nonzero solution of $[p](x) = 0$ is either $ord(a)/(p^2 - p)$ or $(1 - ord(a))/(p-1)$ of which the greater is $(1 - ord(a))/(p-1)$. The precanonical subgroup H^1 consists of 0 and $p-1$ other solutions of $[p](x) = 0$ which have valuation $(1 - ord(a))/(p-1)$. The other solutions of $[p](x) = 0$ all have valuation $ord(a)/(p^2 - p)$.*

2. *If $ord(a) \geq p/(p+1)$, then all nonzero solutions of $[p](x) = 0$ have valuation $1/(p^2-1)$.*

Proof. From the conditions on coefficients of $[p](x)$, the Newton polygon of $[p](x)/x$ looks like:



which proves the claim about valuations of zeroes of $[p](x)$(see [10] for the theory of Newton polygons). On the other hand from the construction of precanonical subscheme of G^1_A, we can see that any nonzero element of H^1 satisfies $x^{p-1} = t_{can}$. This shows that:

$$ord(x) = ord(t_{can})/(p-1) = ord(t_0(1-t_\infty))/(p-1) = ord(t_0)/(p-1).$$

But $at_0 = -p$ and thus $ord(t_0) = 1 - ord(a)$. This proves the proposition. □

We conclude this section by an application of the canonical subgroup theorem.

Proposition 11.5. *Let $ord(r) < 1/(p+1)$. Any modular form (with respect to D) of weight k and level $V_1(N) \cap V_0(p)$ over R_0 determines an element of $S^D(R_0, r, V_1(N), k)$. In other words, modular forms of level Np with trivial character at p, are p-adically of level N.*

Proof. For an element f of $S^D(R_0, (N, p), k)$, simply define:

$$f(A, i, \alpha_N, Y) = f(A, i, \alpha_N, H),$$

where H denotes the canonical subgroup of (A, i, α_N, Y). This defines a p-adic modular form of growth condition r. □

12 The Frobenius Morphism of p-adic Modular Functions

The Hecke operators T_l, for $l \neq p$, over the space of p-adic modular forms with respect to D, can be easily defined in analogy with the classical case. Most of our work will be devoted to define and study the properties of the remaining Hecke operator which we will call the U operator.

In analogy with the case of modular curves, we will first define the Frobenius operator, $Frob$,which is the analogue of Atkin's V_p operator. The U operator will then essentially be defined as a trace of $Frob$. The existence of the canonical subgroup for certain false elliptic curves makes it possible to define the Frobenius operator for p-adic modular forms of weight 0, with respect to D.

Definition 12.1. *Let $r \in R_0$ with $ord(r) < 1/(p+1)$. The Frobenius Operator:*

$$\begin{array}{rcl} Frob: S^D(R_0, r^p, V_1(N), 0) & \rightarrow & S^D(R_0, r, V_1(N), 0) \\ f & \mapsto & Frob(f) \end{array}$$

is defined as follows:

$$Frob(f)(A, i, \alpha_N, Y) = f(A', i', \alpha'_N, Y')$$

where (A, i, α_N, Y) is an r-test object over an R_0-algebra R in which p is nilpotent and (A', i', α'_N, Y') is the r^p-test object obtained by dividing by its canonical subgroup as in theorem 11.1, part **ii**.

12.1 $Frob$ in the Rigid Setting

Define a morphism:

$$\begin{array}{rcl} Frob_n: \mathbb{Y}_r^D \otimes R_0/p^n & \rightarrow & \mathbb{Y}_{r^p}^D \otimes R_0/p^n \\ (A, i, \alpha_N, Y) & \mapsto & (A', i', \alpha'_N, Y'), \end{array}$$

where again (A', i', α'_N, Y') denotes the r^p-test object (over an R_0/p^n-algebra) obtained from (A, i, α_N, Y) by dividing by its canonical subgroup.

Let $\mathbb{H}_{r,n}$ denote the canonical subgroup of the r-test object $(\mathbb{B}_r^D \otimes R_0/p^n, i, \alpha_N, Y_r)$ over $\mathbb{Y}_r^D \otimes R_0/p^n$, for any $n \geq 1$. Let $\widetilde{\mathbb{H}}_r^D \subset \widetilde{\mathbb{B}}_r^D \otimes R_0$ be given by the inverse limit of the group schemes $\mathbb{H}_{r,n} \subset \mathbb{B}_r^D \otimes R_0/p^n$.

From the description of $Frob_n$ we have an isomorphism:

$$Frob_n^*(\mathbb{B}_{r^p}^D \otimes R_0/p^n) \xrightarrow{\sim} (\mathbb{B}_r^D \otimes R_0/p^n)/\mathbb{H}_{r,n}$$

Let $\pi_n : \mathbb{B}_r^D \otimes R_0/p^n \rightarrow (\mathbb{B}_r^D \otimes R_0/p^n)/\mathbb{H}_{r,n}$ denote the canonical projection and $\Phi_n : (\mathbb{B}_r^D \otimes R_0/p^n)/\mathbb{H}_{r,n} \rightarrow \mathbb{B}_{r^p}^D \otimes R_0/p^n$ denote the base extension map. We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{B}_r^D \otimes R_0/p^n & \xrightarrow{\mathbf{Frob_n} = \Phi_n \circ \pi_n} & \mathbb{B}_{r^p}^D \otimes R_0/p^n \\ \downarrow & & \downarrow \\ \mathbb{Y}_r^D \otimes R_0/p^n & \xrightarrow{Frob_n} & \mathbb{Y}_{r^p}^D \otimes R_0/p^n \end{array}$$

By passing to completion along $p = 0$, we get a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{B}}_r^D \otimes R_0 & \xrightarrow{\widetilde{\mathbf{Frob}}} & \widetilde{\mathbb{B}}_{r^p}^D \otimes R_0 \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{Y}}_r^D \otimes R_0 & \xrightarrow{\widetilde{Frob}} & \widetilde{\mathbb{Y}}_{r^p}^D \otimes R_0 \end{array}$$

Similarly we have $\widetilde{Frob}^*(\widetilde{\mathbb{B}}^D_{r^p} \otimes R_0) = (\widetilde{\mathbb{B}}^D_r \otimes R_0)/\widetilde{\mathbb{H}}^D_r$, and the map induced by $\widetilde{\mathbf{Frob}}$:

$$\widetilde{\mathbb{B}}^D_r \otimes R_0 \to \widetilde{Frob}^*(\widetilde{\mathbb{B}}^D_{r^p} \otimes R_0)$$

is the canonical projection $\tilde{\pi}$, so that we have:

$$\widetilde{\mathbf{Frob}} = \tilde{\Phi} \circ \tilde{\pi}.$$

From definition of $Frob_n$ and $Frob$ and the above construction, it is clear that we have the following diagram:

$$\begin{array}{ccc}
H^0(\widetilde{\mathbb{Y}}^D_{r^p} \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}^D_{r^p} \otimes R_0}) & \xrightarrow{\widetilde{Frob}^*} & H^0(\widetilde{\mathbb{Y}}^D_r \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}^D_r \otimes R_0}) \\
\| & & \| \\
S^D(R_0, r^p, V_1(N), 0) & \xrightarrow{Frob} & S^D(R_0, r, V_1(N), 0)
\end{array}$$

We have so far described the Frobenius morphism in the formal setting. This can be used to derive a description of $Frob$ in the rigid setting. We make a definition before stating the next theorem. The *canonical subgroup* of $A^D(r)$, is the image of $\widetilde{\mathbb{H}}^D_r$ under the functor rig and is denoted by $\mathbb{H}^D_r$. We will study $\mathbb{H}^D_r$ more closely in the next section.

Theorem 12.2. *Let $r \in R_0$ have $ord(r) < 1/(p+1)$. There exists a commutative diagram of rigid analytic spaces over K:*

$$\begin{array}{ccc}
A^D(r) & \xrightarrow{\mathbf{Frob}^{rig}} & A^D(r^p) \\
\downarrow & & \downarrow \\
X^D(r) & \xrightarrow{Frob^{rig}} & X^D(r^p)
\end{array}$$

in which the pull back of $A^D(r^p)$ under $Frob^{rig}$ is isomorphic to $A^D(r)/\mathbb{H}^D_r$ and the map induced by $\mathbf{Frob}^{rig}$ and the natural map $A^D(r) \to X^D(r)$:

$$A^D(r) \xrightarrow{\pi} Frob^{rig*}(A^D(r^p))$$

is the natural projection, so that:

$$\mathbf{Frob}^{rig} = \Phi^{rig} \circ \pi.$$

Furthermore the Frobenius morphism of (convergent and) overconvergent modular functions can be described as the pullback of $Frob^{rig}$. In other words, there is a commutative diagram:

$$\begin{array}{ccc} H^0(X^D(r^p), \mathcal{O}_{X^D(r^p)}) & \xrightarrow{(Frob^{rig})^*} & H^0(X^D(r), \mathcal{O}_{X^D(r)}) \\ \| & & \| \\ S^D(R_0, r^p, V_1(N), 0) \otimes K & \xrightarrow{Frob_K} & S^D(R_0, r, V_1(N), 0) \otimes K \end{array}$$

Proof. Apply Raynaud's functor rig to the construction we have done in the formal setting. □

12.2 $Frob$ on Points

We will study the action of $Frob^{rig}$ on points of $X^D(r)$. Let K_∞ denote the completion of an algebraic closure of K. First we prove a lemma:

Lemma 12.3. *Let X be a scheme which is flat and finite type over R_0. Let $\tilde{X}$ denote the completion along the subscheme $p = 0$ of X. Let X^{rig} be the rigid analytic space over K associated to $\tilde{X}$ under the Raynaud functor. For any L which is either K_∞ or a finite extension of K with ring of integers R, we have a one-to-one correspondence between:*

$$Hom_{R_0}(Spec(R), X) \leftrightarrow Hom_L(Sp(L), X^{rig}\hat{\otimes}L)$$

(Note that if $[L : K]$ is finite then $Hom_L(Sp(L), X^{rig}\hat{\otimes}L) = Hom_K(Sp(L), X^{rig})$.)

Proof. Any morphism $Spec(R) \to X$ induces a morphism $Spf(R) \to \tilde{X} \otimes R$ which by Raynaud's functor induces a morphism $Sp(L) \to X^{rig}\hat{\otimes}L$ in a one-to-one manner (the functor is faithful). So we only need to show that any morphism $Sp(L) \to X^{rig}\hat{\otimes}L$ is obtained from a morphism $Spec(R) \to X$ in the above way.

Let $V = Spec(C) \subset X$ be an affine open subset whose image in $X^{rig}\hat{\otimes}L$ (which is isomorphic to $Sp(\hat{C}\hat{\otimes}_{R_0}L)$ where $\hat{C}$ denotes the p-adic completion of C) contains the image of $Sp(L) \to X^{rig}\hat{\otimes}L$.. Hence we get a map of affinoid algebras over L, $\hat{C}\hat{\otimes}_{R_0}L \to L$. But by [4] any map of affinoid L-algebras is continuous for their canonical L-Banach topologies. So the map $\phi : C \to \hat{C}\hat{\otimes}L \to L$ is continuous for the p-adic topology on C. We show that ϕ factors through R.

Assume there is an $x \in C$ such that $\phi(x) \notin R$. Then $\phi(x) = \beta$ with $ord(\beta) < 0$. So for $\alpha = 1/\beta$ we have $\phi(\alpha x) = 1$. But $(\alpha x)^n$ goes to zero as n goes to ∞ because $ord(\alpha) > 0$. However $\phi((\alpha x)^n) = 1$ for all $n > 0$. This contradicts continuity of ϕ and proves that $\phi(C) \subset R$. Therefore ϕ gives a map $Spec(R) \to V \subset X$ which induces the given rigid point of $X^{rig}\hat{\otimes}L$. □

A closed point of $X^D(r)$ gives a map $Sp(L) \to X^D(r)$ for some finite extension L of K with ring of integers R. Thinking of $X^D(r)$ as an affinoid subdomain of $(\mathbb{X}_1^D(N))^{an}$ we get a map $Sp(L) \to (\mathbb{X}_1^D(N))^{an}$ which by rigid GAGA corresponds to a map $Spec(L) \to \mathbb{X}_1^D(N)$. This is nothing but (the analyfication of) a false elliptic curve with level $V_1(N)$ structure (A, i, α_N) over L. By lemma 12.3 the map $Sp(L) \to X^D(r)$ is obtained as the image of a map $Spec(R) \to \mathbb{Y}_r^D$ under the Raynaud functor. This gives an r-test object $(\mathbb{A}, i, \alpha_N, Y)$ over R. Now since the Raynaud functor agrees with *an* for proper schemes, we deduce that $(\mathbb{A}, i, \alpha_N)$ is a model for (A, i, α_N) over R. The existence of Y is clearly equivalent to the condition $|E_{p-1}(A,i)| \geq |r|$ and Y is uniquely determined from this inequality. Let's call a morphism $Sp(K_\infty) \to X^D(r)\hat{\otimes}K_\infty$ a K_∞-point of $X^D(r)$. Similarly by lemma 12.3 to give a K_∞-point of $X^D(r)$ is equivalent to giving (A, i, α_N) over R_∞ (the ring of integers of K_∞), such that $|E_{p-1}(A,i)| \geq |r|$.

Now we investigate the canonical subgroup of a closed or K_∞-point of $X^D(r)$. Let $x : Sp(L) \to X^D(r)\hat{\otimes}L$ be such a point giving (A, i, α_N) over R with $|E_{p-1}(A,i)| \geq |r|$. The fibre of $\mathbb{H}_r^D\hat{\otimes}L$ over $Sp(L)$ gives a finite flat subgroup scheme of $A\hat{\otimes}L$. By construction of $A^D(r)$ and $\mathbb{H}_r^D$, this subgroup is the image of H, the canonical subgroup of (A, i), under rig. But since H is finite and flat over R, this equals the (analyfication of the) generic fibre of H. Now the description of $\widetilde{\mathbf{Frob}}$ shows that the image of x under $Frob^{rig}$, is the point given by (A', i', α_N') which is obtained from (A, i, α_N) by dividing by H.

Let's summarize the above observations:

Corollary 12.4. *To give a closed point (respectively K_∞-point) of $X^D(r)$ is equivalent to giving a false elliptic curve with level $V_1(N)$ structure (A, i, α_N) over R, the ring of integers of a finite extension of K (respectively over R_∞), which satisfies: $|E_{p-1}(A, i, \alpha_N)| \geq |r|$. The fibre of $\mathbb{H}_r^D \subset A^D(r)$ over this point is the generic fibre of $H \subset A$ where H is the canonical subgroup of (A, i, α_N). The image of this point under $Frob^{rig}$ is determined by (A', i', α_N') which is the quotient of (A, i, α_N) by H.*

12.3 Fibre of $Frob$ over a point

Let R be the ring of integers in L a finite extension of K. Let (A, i, α_N) over R represent a closed point of $X^D(r^p)$. For an embedding of L in K_∞, let us denote the base extension of (A, i, α_N) to R_∞ with the same symbol. It represents a closed point of $X^D(r)\hat{\otimes}K_\infty$. The fibre of $Frob$ over this point consists of all (A', i', α_N') over R_∞ such that $|E_{p-1}(A,i)| \geq |r|$ for some nowhere vanishing ω and $(A', i', \alpha_N')/H' = (A, i, \alpha_N)$ where H' denotes the canonical subgroup of (A', i'). Lemma 10.1 implies that there are $p+1$ $\mathcal{O}_D$-stable subgroups of A of rank p^2. Let $H_0, H_1, ..., H_p$ denote these subgroups, of which H_0 is the canonical subgroup. Any such A' is isomorphic to $A^j = A/H_j$ for some $j \geq 0$. Now in order for this to give a point of the fibre we need to know that $A[p]/H_j$ is the canonical subgroup of A^j and $\alpha_N' = \alpha_N^j$ is $1/p$ times the induced level structure on A^j from α_N.

First let us assume $r = 1$. This means that A and hence A^j, $0 \leq j \leq p$ are ordinary. Hence from the proof of theorem 11.1 we know that a subgroup of A^j which has rank p^2 and is stable under action of $\mathcal{O}_D$ is the canonical subgroup iff it reduces to the kernel of multiplication by p in the formal group of A^j. Now since H_0 reduces to $Ker[p]$ in G_A, $A[p]/H_0$ reduces to 0 in G_{A^0} and hence can not be the canonical subgroup of A^0. On the other hand H_j for $j > 0$ reduces to 0 in G_A (since its image in G_A is stable under $\mathcal{O}_D$, has rank dividing p^2 and is not $Ker[p]$ which leaves 0). Thus $A[p]/H_j$ reduces to $Ker[p]$ in G_{A^j}

which means that $A[p]/H_j$ is the canonical subgroup of A^j. So we have shown that when $r = 1$ the points lying above (A, i, α_N) are exactly (A^j, i, α_N^j) for $1 \leq j \leq p$.

Now let us assume $ord(r) > 0$. In order to study the fibre of $Frob^{rig}$ over a supersingular point, we follow Lubin's idea to carefully analyse the formal group of a false elliptic curve and its canonical subgroup.

By lemma 10.4, G_A^1 has a coordinate x in which we can write:

$$[p](x) = px + ax^p + \sum_{n=2}^{p} c_n x^{n(p-1)+1} + c_{p+1}x^{p^2} + \sum_{n \geq p+2} c_n x^{n(p-1)+1}$$

Since A has supersingular reduction, we have $ord(a) > 0$ and also $height(G_A^1) = 2$ which implies that $ord(c_{p+1}) = 0$. We also know that $ord(c_n) \geq 1$ if $n \not\equiv 1 \ mod\ p$. If G is a formal group we will denote the power series giving addition by $G(x, y)$.

First we prove a lemma:

Lemma 12.5. *Let $ord(a) < p/(p+1)$, H_0 be the canonical subgroup of (A, i) and $A^0 = A/H_0$. $G_{A^0}^1$ has a coordinate $x^{(0)} = \prod_{\beta \in H_0^1} G_A^1(x, \beta)$ (where $H_0^1 = H_0 \cap G_A^1$) in which:*

$$[p](x^{(0)}) = px^{(0)} + a^{(0)}(x^{(0)})^p + ...$$

- *If $ord(a) < 1/(p+1)$, then $ord(a^{(0)}) = p.ord(a)$ and $A[p]/H_0$ is not the canonical subgroup of A^0.*
- *If $1/(p+1) < ord(a) < p/(p+1)$, then $ord(a^{(0)}) = 1 - ord(a)$ and $A[p]/H_0$ is the canonical subgroup of A^0.*
- *If $ord(a) = 1/(p+1)$ then $ord(a^{(0)}) \geq p/(p+1)$.*

Proof. By [14] for any finite subgroup H of a one parameter formal group G over a finite extension of R_0

$$x' = \prod_{\beta \in H} G(x, \beta)$$

is a coordinate on G/H and $x \to \prod_{\beta \in H} G(x, \beta)$ is the projection. Since (A, i) is defined over R, we can apply this to our situation to get a coordinate $x^{(0)} = \prod_{\beta \in H_0^1} G_A^1(x, \beta)$.

We calculate the valuations of elements $y^{(0)}$ of $Ker[p] = G_{A^0}^1[p]$. There are two cases:

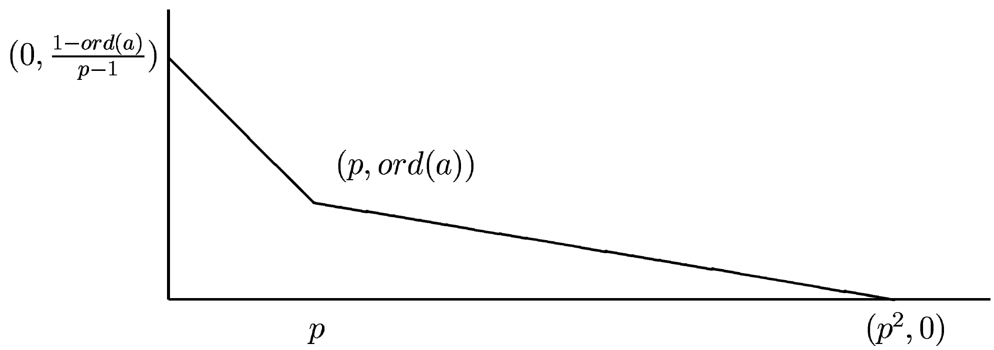
- $y^{(0)} = \prod_{\beta \in H_0^1} G_A^1(y, \beta) \qquad [p](y) = 0, y \notin H_0^1$
- $y^{(0)} = \prod_{\beta \in H_0^1} G_A^1(y, \beta) \qquad 0 \neq [p](y) \in H_0^1$

In the first case we have $ord(y) = ord(a)/(p^2 - p) < (1 - ord(a))/(p-1) = ord(\beta)$ by proposition11.4. Thus $ord(G_{A^0}^1(y, \beta)) = ord(y) = ord(a)/(p^2 - p)$. This implies that

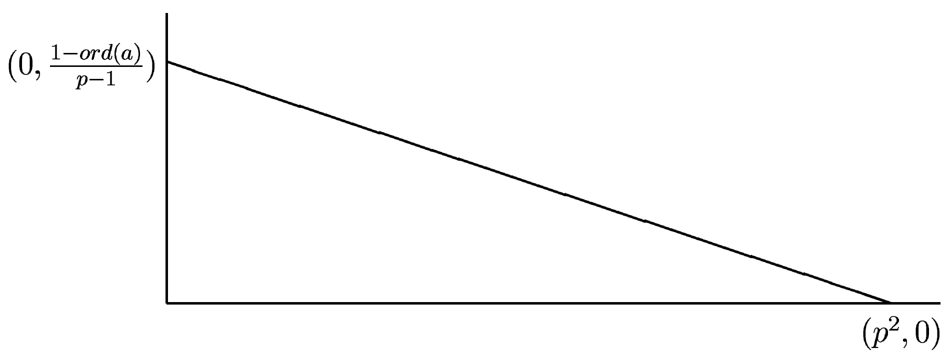
$$ord(y^{(0)}) = \sum_{\beta \in H_0^1} ord(G^1_{A^0}(y,\beta)) = p.ord(y) = ord(a)/(p-1).$$

In the second case y is a solution to $[p](z) - \beta = 0$ for some $\beta \in H_0^1 - \{0\}$. The Newton polygon of $f(z) = [p](z) - \beta$ looks like:

$ord(a) < 1/p + 1$:



$1/p + 1 \leq ord(a) < p/p + 1$:



If $ord(a) < 1/(p+1)$, then from the first Newton polygon, either $ord(y) = ord(a)/(p^2-p)$ or $ord(y) = (1 - p.ord(a))/(p^2 - p)$. So for all $\beta \in H_0^1$ we have $ord(y) < ord(\beta) = (1 - ord(a))/(p-1)$. Now:

$$ord(y^{(0)}) = \sum_{\beta \in H_0^1} ord(G^1_{A^0}(y,\beta)) = p.ord(y) = ord(a)/(p-1) \ or \ (1 - p.ord(a))/(p-1).$$

We deduce that in this case the elements of $G^1_{A^0}[p]$ have valuation either $ord(a)/(p-1)$ or $(1-p.ord(a))/(p-1)$, the greater of which is $(1-p.ord(a))/(p-1)$. So by proposition 11.4 $(1-p.ord(a))/(p-1) = (1-ord(a^{(0)}))/(p-1)$ and hence $ord(a^{(0)}) = p.ord(a)$. Also the points of $G^1_{A^0}[p]$ coming from $A[p]^1/H_0^1$ are of valuation $ord(a)/(p-1) < (1-ord(a^{(0)}))/(p-1)$ and hence $A[p]/H_0$ is not equal to the canonical subgroup of A^0.

If $1/(p+1) \leq ord(a) < p/(p+1)$, then the second Newton polygon shows that $ord(y) = (1-ord(a))/p^2(p-1)$ which satisfies $ord(y) < ord(\beta)$ for all $\beta \in H_0$. Thus

$$ord(y^{(0)}) = \sum_{beta \in H_0^1} ord(G^1_{A^0}(y,\beta)) = p.ord(y) = (1-ord(a))/)p(p-1))$$

So the numbers occurring among valuations of points of $G^1_{A^0}[p]$ are $(1-ord(a))/(p(p-1))$ and $ord(a)/(p-1)$. If $ord(a) = 1/(p+1)$ then these two are equal. So in this case all the elements of $Ker[p]$ have valuation $1/(p^2-1)$ and by lemma 11.4 we have $ord(a^(0)) \geq p/(p+1)$.

If $1/(p+1) < ord(a) < p/(p+1)$ then the greater of the above two valuations is $ord(a)/(p-1)$. Again by proposition 11.4 we get $ord(a)/(p-1) = (1-ord(a^{(0)}))/(p-1)$ or $ord(a^{(0)}) = 1-ord(a)$.

Since the points $y^{(0)}$ coming from $A[p]^1/H_0^1$ have valuation $(1-ord(a^{(0)}))/(p-1)$, we deduce that the canonical subgroup of A/H_0 is $A[p]/H_0$. This proves the lemma.

□

Theorem 12.6. *Let $s = (A, i, \alpha_N)$ over R with $|E_{p-1}(A, i, \omega)| \geq |r^p|$ represent a closed point of $X^D(r^p)$ of residue field L which is a finite extension of K. For an embedding of L in K_∞ let $\bar{s}$ denote the closed point of $X^D(r^p)\hat{\otimes}K_\infty$ obtained by base extension. There are exactly p closed points of $X^D(r)\hat{\otimes}K_\infty$ lying above $\bar{s}$ under $Frob^{rig}$. These points are (A^j, i, α^j_N) for $1 \leq j \leq p$ where $A^j = A/H_j$ and α^j_N is $1/p$ times the induced level $V_1(N)$ structure on A^j from α_N.*

Proof. We have already proven this for $r = 1$. Now assume $ord(r) > 0$. First we show that (A^0, i, α^0_N) does not lie above (A, i, α_N).

If $ord(a) < 1/(p+1)$ then by lemma 12.5 $A[p]/H_0$ is not the canonical subgroup of A^0. If $ord(a) = 1/(p+1)$ then lemma 12.5 shows that $ord(a^{(0)} \geq p/(p+1)$ and therefore (A^0, i, α^0_N) is not a point of $X^D(r)\hat{\otimes}K_\infty$. If $1/(p+1) < ord(a) < p/(p+1)$ then again lemma 12.5 implis that (A^0, i, α^0_N) is not a point of $X^D(r)\hat{\otimes}K_\infty$, as $ord(a^{(0)}) = 1-ord(a) \geq 1/(p+1)$. This excludes $j = 0$.

Now we study A^j for $1 \leq j \leq p$. Let $x^{(j)}$ be the coordinate on $G^1_{A^j}$ given by $x^{(j)} = \prod_{\beta \in H^1_j} G^1_A(x,\beta)$. We calculate the valuations of elements $y^{(j)}$ of $G^1_{A^j}[p]$.

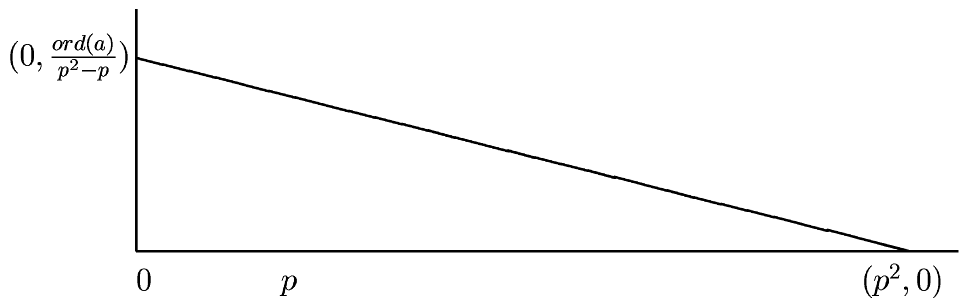
There are two possibilities:

- $y^{(j)} = \prod_{\beta \in H^1_j} G^1_A(y,\beta) \qquad y \in H^1_0 - \{0\}$
- $y^{(j)} = \prod_{\beta \in H^1_j} G^1_A(y,\beta) \qquad 0 \neq [p](y) \in H^1_j$

In the first case we have $ord(y) = (1-ord(a))/(p-1) > ord(a)/(p^2-p) = ord(\beta)$ for all $\beta \in H^1_j - \{0\}$ and hence for such β , $ord(G^1_{A^j}(y,\beta) = ord(\beta) = ord(a)/(p^2-p)$. Therefore:

$$
\begin{aligned}
ord(y^{(j)}) &= \textstyle\sum_{\beta\in H_j^1} ord(G^1_{A^j}(y,\beta)) &&= (1-ord(a))/(p-1)+(p-1)ord(a)/(p^2-p)\\
& &&= (p-ord(a))/(p(p-1)).
\end{aligned}
$$

In the second case y is a root of $f(z)=[p](z)-\beta$ for some $\beta \in H_j^1-\{0\}$. The Newton polygon of $f(z)$ is:



So $ord(y)=ord(a)/(p^2(p^2-p)) < ord(\beta)$ for all $\beta\in H_j^1$. This means:

$$ord(y^{(j)}) = \sum_{\beta\in H_j^1} ord(G^1_{A^j}(y,\beta)) = p.ord(y) = ord(a)/(p(p^2-p)).$$

So there are two numbers occurring among the valuations of elements of $G^1_{A^j}[p]$, which are $(p-ord(a))/(p(p-1))$ and $ord(a)/(p(p^2-p))$. The greater of these valuations is $(p-ord(a))/(p(p-1))$. By proposition11.4 we have

$$(1-ord(a^{(j)}))/(p-1) = (p-ord(a))/(p(p-1))$$

or $ord(a^{(j)}) = ord(a)/p < 1/(p+1)$. Furthermore since the points $y^{(0)}$ coming from $A[p]^1/H_j^1$ have valuation equal to $(1-ord(a^{(j)}))/(p-1)$ we conclude that the canonical subgroup of A/H_j is $A[p]/H_j$. This proves that (A^j,i,α_N^j) lies above (A,i,α_N) if $j\neq 0$.

□

12.4 Properties of *Frob*

We saw that the Frobenius morphism of p-adic modular functions can be described as the pullback of a morphism of rigid analytic varieties.This allows us to study the properties of *Frob*.

Theorem 12.7. *For $r\in R_0$ with $ord(r)<1/(p+1)$,*

$$Frob: S^D(R_0,r^p,V_1(N),0)\to S^D(R_0,r,V_1(N),0)$$

is a finite morphism. If $r=1$ it is finite and flat of rank p.

Proof. We start by a lemma:

Lemma 12.8. *Let $S_1 \to S_2$ be a homomorphism between two Noetherian R_0 algebras which are π-adically complete. Let M be a finite S_2-module over which π is not a zero divisor. Then M is a finitely generated S_1 module iff M/π is finitely generated over S_1/π. Furthermore if M/π is flat over S_1/π, then M is flat over S_1.*

The statement about finiteness is a consequence of Nakayama's lemma for complete rings. The other statement is the result of criterion for flatness. See lemma 1.6 of [3] and theorem $III.5.2.1$ of [1] for details.

Let π be a uniformizer in R_0. By lemma we need to show that:

$$Frob \otimes R_0/\pi : S^D(R_0, r^p, V_1(N), 0) \otimes R_0/\pi \to S^D(R_0, r, V_1(N), 0) \otimes R_0/\pi$$

is finite. By proposition 9.6, $\widetilde{\mathbb{Y}}_r^D \otimes R_0$ is flat over R_0 and hence we have a short exact sequence:

$$0 \to \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0} \xrightarrow{\times\pi} \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0} \to \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi} \to 0$$

which results in:

$$H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0}) \otimes R_0/\pi \hookrightarrow H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi}) \to H^1(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0})$$
$$\xrightarrow{\times\pi} H^1(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0}) \to H^1(\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi, \mathcal{O}_{\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi}) \to 0$$

which shows that the natural map $S^D(R_0, r, V_1(N), 0) \otimes R_0/\pi \to S^D(R_0/\pi, r, V_1(N), 0)$ is an injection with cokernel of finite dimension over R_0/π. In particular $S^D(R_0/\pi, r, V_1(N), 0)$ is finite over $S^D(R_0, r, V_1(N), 0) \otimes R_0/\pi$.

Now we show that $Frob : S^D(R_0/\pi, r^p, V_1(N), 0) \to S^D(R_0/\pi, r, V_1(N), 0)$ is finite. First note that when $r = 1$ both $S^D(R_0/\pi, r, V_1(N), 0)$ and $S^D(R_0/\pi, r^p, V_1(N), 0)$ are equal to $S^D(R_0/\pi, 1, V_1(N), 0)$. But if $ord(r) > 0$, then they are both equal to $S^D(R_0/\pi, 0, V_1(N), 0)$. Since $ord(r) < 1/(p+1)$, both p/r and p/r^p lie in πR_0. Hence the canonical subgroup over $\widetilde{\mathbb{Y}}_r^D \otimes R_0/\pi$ and $\widetilde{\mathbb{Y}}_{r^p}^D \otimes R_0/\pi$ is nothing but the kernel of Frobenius. This shows that in the following diagram:

$$\begin{array}{ccc} S^D(R_0/\pi, r^p, V_1(N), 0) & \xrightarrow{Frob} & S^D(R_0/\pi, r, V_1(N), 0) \\ \| & & \| \\ S^D(R_0/\pi, t, V_1(N), 0) & \xrightarrow{Frob} & S^D(R_0/\pi, t, V_1(N), 0) \end{array}$$

where $t = 1$ if $r = 1$, and $t = 0$ if $ord(r) > 0$, Frob is given by:

$$Frob(f)((A, i, \alpha_N, Y, \omega)) = f(A^{(p)}, i, \alpha_N^{(p)}, Y^p, \omega^{(p)})$$

But when $t = 0, 1$, we have $S^D(R_0/\pi, t, V_1(N), 0) = S^D(\mathbb{F}_p, t, V_1(N), 0) \otimes R_0/\pi$ and hence we only need to show that the above morphism is finite when the ground ring is $\mathbb{F}_p$. In that case $(A^{(p)}, i, \alpha_N^{(p)}, Y^{(p)}, \omega^{(p)})$ over an $\mathbb{F}_p$-algebra, R, is obtained from $(A, i, \alpha_N, Y, \omega)$ by base extension by the p-th power mapping $R \stackrel{p}{\longrightarrow} R$, of $\mathbb{F}_p$-algebras. This means that

$$Frob : S^D(R_0/\pi, t, V_1(N), 0) \to S^D(R_0/\pi, t, V_1(N), 0)$$

is the p-th power mapping. Now since $\widetilde{\mathbb{Y}}_t^D \otimes \mathbb{F}_p$ is a scheme of finite type over $\mathbb{F}_p$, $S^D(\mathbb{F}_p, t, V_1(N), 0)$ is a finitely generated $\mathbb{F}_p$-algebra. It is easy to see that the p-th power mapping makes any finitely generated $\mathbb{F}_p$-algebra a finite module over itself. This proves the claim.

Now in the diagram:

$$\begin{array}{ccc} S^D(R_0/\pi, r^p, V_1(N), 0) & \xrightarrow{Frob} & S^D(R_0/\pi, r, V_1(N), 0) \\ \uparrow & & \uparrow \\ S^D(R_0, r^p, V_1(N), 0) \otimes R_0/\pi & \xrightarrow{Frob \otimes R_0/\pi} & S^D(R_0, r, V_1(N), 0) \otimes R_0/\pi \end{array}$$

$S^D(R_0/\pi, r, V_1(N), 0)$ is finite over $S^D(R_0, r^p, V_1(N), 0) \otimes R_0/\pi$ and therefore its submodule $S^D(R_0, r, V_1(N), 0) \otimes R_0/\pi$ is so (note that all the rings are Noetherian here). This proves the finiteness of $Frob$.

Now assume $r = 1$. $S^D(\mathbb{F}_p, 1, V_1(N), 0)$ is the coordinate ring of the open of $\mathbb{X}_1^D(N) \otimes \mathbb{F}_p$ where E_{p-1} is not zero which is an affine smooth scheme of finite type over $\mathbb{F}_p$. Therefore the p-th power endomorphism of $S^D(\mathbb{F}_p, r, V_1(N), 0)$ makes it finite and flat of rank p over itself (one way to see this, is to prove it locally by using the local criterion of flatness and a regular sequence of the local ring of a closed point). By above discussion the Frobenius morphism of $S^D(R_0/\pi, r, V_1(N), 0)$ is a base extension of the p-th power endomorphism of $S^D(\mathbb{F}_p, 1, V_1(N), 0)$. This means that :

$$Frob : S^D(R_0/\pi, 1, V_1(N), 0) \to S^D(R_0/\pi, 1, V_1(N), 0)$$

is finite and flat of degree p. Now notice that if r=1, $\mathbb{Y}_r^D \otimes R_0$ is an affine scheme. Therefore in the above short exact sequence, we have $H^1(\mathbb{Y}_r^D \otimes R_0, \mathcal{O}_{\mathbb{Y}_r^D \otimes R_0}) = 0$ which shows that:

$$S^D(R_0, 1, V_1(N), 0) \otimes R_0/\pi \stackrel{\sim}{\longrightarrow} S^D(R_0/\pi, 1, V_1(N), 0).$$

Thus in the above diagram, the vertical arrows are isomorphisms when $r = 1$. This shows that

$$Frob : S^D(R_0, 1, V_1(N), 0) \otimes R_0/\pi \to S^D(R_0/\pi, 1, V_1(N), 0) \otimes R_0/\pi$$

is finite and flat of degree p. Applying lemma 12.8 with $M = S_1 = S_2 = S^D(R_0/\pi, 1, V_1(N), 0)$ proves the result.

□

We have seen that when $r = 1$ the Frobenius morphism is finite and flat of rank p. This is not true for general r. However the same result holds true if we tensor with K. This is proven in the next theorem. This will be the last step before we can define the U operator.

Theorem 12.9. *If $ord(r) < 1/p+1$, then $Frob^{rig} : X^D(r) \to X^D(r^p)$ is a finite étale map of rigid analytic spaces over K of degree p.*

Proof. We have already proven the finiteness of $Frob_K : S^D(R_0, r^p, V_1(N), 0) \otimes K \to S^D(R_0, r, V_1(N), 0) \otimes K$. First we prove that it is flat. Since $X^D(r)$ is an affinoid subdomain of $(\mathbb{X}_1^D(N) \otimes K)^{an}$, the completion of the rigid local ring of $X^D(r)$ at any closed point of $X^D(r)$ equals the completion of the corresponding closed point on $\mathbb{X}_1^D(N) \otimes K$. But $\mathbb{X}_1^D(N) \otimes K$ is smooth and hence the completion of the local ring of any of its closed points is regular. This shows that the local rings of $S^D(R_0, r, V_1(N), 0) \otimes K$ at its maximal ideals are all regular. Hence $S^D(R_0, r, V_1(N), 0)$ and $S^D(R_0, r^p, V_1(N), 0)$ are regular rings of dimension one. Now by [EGA IV,17.3.5.2] any finite morphism between regular rings of the same dimension is flat. Therefore $Frob_K$ is flat. We have already seen that over the affinoid $X^D(1)$, $Frob_K$ has degree p. Hence:

$$Frob_K : S^D(R_0, r^p, V_1(N), 0) \otimes K \to S^D(R_0, r, V_1(N), 0) \otimes K$$

is finite and flat of degree p.

Now we prove the étaleness of $Frob^{rig}$. We need to show that for any closed point x of $X^D(r)$ represented by a map $Sp(L) \to X^D(r)$, the fibre of $Frob^{rig}$ over x is finite and étale of rank p. This is equivalent to proving that the fibre of any base extension of x to a closed point $Sp(K_\infty) \to X^D(r)\hat{\otimes}K_\infty$ of $X^D(r)\hat{\otimes}K_\infty$ under $Frob^{rig}$ has exactly p distinct points. But this is the content of theorem 12.6. This proves étaleness of $Frob^{rig}$. □

13 U Operator

We are finally in a position to define the U operator. We have proven that if $ord(r) < 1/(p+1)$:

$$Frob_K : S^D(R_0, r^p, V_1(N), 0) \otimes K \to S^D(R_0, r, V_1(N), 0) \otimes K$$

is finite flat of rank p. We can therefore define the trace of $Frob_K$ which we denote by Tr_{Frob_K}:

$$Tr_{Frob_K} : S^D(R_0, r, V_1(N), 0) \otimes K \to S^D(R_0, r^p, V_1(N), 0) \otimes K.$$

Note that if $r = 1$, then $Frob$ is already finite and flat of rank p before tensoring with K and thus in this case Tr_{Frob} can be defined:

$$Tr_{Frob} : S^D(R_0, 1, V_1(N), 0) \to S^D(R_0, 1, V_1(N), 0)$$

and we have the equality $Tr_{Frob} \otimes K = Tr_{Frob_K}$ for $r = 1$.

Lemma 13.1. $(Frob^{rig})^* \underline{\omega}_{A^D(r^p)/X^D(r^p)} \xrightarrow{\sim} \underline{\omega}_{(A^D(r)/\mathbb{H}_r^D)/X^D(r)}$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} (Frob^{rig})^* A^D(r^p) & \xrightarrow{\Phi^{rig}} & A^D(r^p) \\ \downarrow & & \downarrow \\ X^D(r) & \xrightarrow{Frob^{rig}} & X^D(r^p) \end{array}$$

Recall that $\Phi^{rig} : (Frob^{rig})^* A^D(r^p) \to A^D(r^p)$ is the base extension of $Frob^{rig}$. Therefore we have a natural isomorphism $(\Phi^{rig})^* \Omega^1_{A^D(r^p)/X^D(r^p)} \xrightarrow{\sim} \Omega^1_{(Frob^{rig})^* A^D(r^p)/X^D(r^p)}$. But from theorem 12.2, $(Frob^{rig})^* A^D(r^p) \xrightarrow{\sim} A^D(r)/\mathbb{H}^D_r$. Applying e_2 and pulling back by the zero sections of $(A^D(r)/\mathbb{H}^D_r)/X^D(r)$ and $A^D(r^p)/X^D(r^p)$, we get the desired natural isomorphism.

□

The above lemma allows us to define Tr_{Frob_K} for overconvergent modular forms of arbitrary weight. It is defined as the trace function

$$\begin{array}{ccc} H^0(X^D(r), \underline{\omega}^{\otimes k}_{(A^D(r)/\mathbb{H}^D_r)/X^D(r)}) & \xrightarrow{Tr_{Frob^{rig}}} & H^0(X^D(r^p), \underline{\omega}^{\otimes k}_{A^D(r^p)/X^D(r^p)}) \\ \uparrow {\scriptstyle (\check{\pi}^*)^{\otimes k}} & & \| \\ H^0(X^D(r), \underline{\omega}^{\otimes k}_{A^D(r)/X^D(r)}) & & \\ \| & & \| \\ S^D(R_0, r, V_1(N), k) \otimes K & \xrightarrow{Tr_{Frob_K}} & S^D(R_0, r^p, V_1(N), k) \otimes K \end{array}$$

on the sections of the sheaf $\underline{\omega}^{\otimes k}$.

The upper horizontal morphism, $Tr_{Frob^{rig}}$, is the trace from global sections of pull back of $\underline{\omega}^{\otimes k}_{A^D(r^p)/X^D(r^p)}$ on $X^D(r)$, to global sections of $\underline{\omega}^{\otimes k}_{A^D(r^p)/X^D(r^p)}$ on $X^D(r^p)$.

Also $\check{\pi}^*$ is obtained in the following way. We have:

$$e_2 H^0(A^D(r), \Omega^1_{A^D(r)/X^D(r)}) \xrightarrow{\check{\pi}^*} e_2 H^0(A^D(r)/\mathbb{H}^D_r, \Omega^1_{(A^D(r)/\mathbb{H}^D_r)/X^D(r)}),$$

where

$$\pi : A^D(r) \to (Frob^{rig})^* A^D(r^p) = A^D(r)/\mathbb{H}^D_r$$

is the natural projection (see theorem 12.2). Now $\check{\pi}$ is the dual morphism of π.

Note that when $k = 0$, we get the same definition as before. We now define the U operator:

Definition 13.2. *Let $r \in R_0$ be such that $ord(r) < 1/(p+1)$. We define the $U_{(k)}$ operator of $S^D(R_0, r^p, V_1(N), k) \otimes K$:*

$$U_{(k)} : S^D(R_0, r^p, V_1(N), k) \otimes K \to S^D(R_0, r^p, V_1(N), k) \otimes K$$

to be the following composite:

$$S^D(R_0, r^p, V_1(N), k) \otimes K \hookrightarrow S^D(R_0, r, V_1(N), k) \otimes K \stackrel{(1/p).Tr_{Frob_K}}{\longrightarrow} S^D(R_0, r^p, V_1(N), k) \otimes K$$

Here the first arrow is the natural inclusion. Note that the following diagram is commutative:

$$\begin{array}{ccc} S^D(R_0, r^p, V_1(N), k) \otimes K & \xrightarrow{U_{(k)}} & S^D(R_0, r^p, V_1(N), k) \otimes K \ . \\ \downarrow & & \downarrow \\ S^D(R_0, 1, V_1(N), k) \otimes K & \xrightarrow{U_{(k)}} & S^D(R_0, 1, V_1(N), k) \otimes K \end{array}$$

Therefore we can think of $U_{(k)}$ as an operator on $S^D(R_0, 1, V_1(N), k) \otimes K$, the space of convergent modular forms, which preserves the subspace of overconvergent modular forms. We will usually drop the subscript k when the weight is understood and simply refer to this operator as the U operator.

We now define the Frobenius morphism for overconvergent modular forms of general weight.

$$Frob_K : S^D(R_0, r^p, V_1(N), k) \otimes K \to S^D(R_0, r, V_1(N), k) \otimes K$$

is defined as follows:

$$\begin{array}{ccc} H^0(X^D(r), \underline{\omega}^{\otimes k}_{(A^D(r)/\mathbb{H}^D_r)/X^D(r^p)}) & \xrightarrow{(\pi^*)^{\otimes k}/p^k} & H^0(X^D(r), \underline{\omega}^{\otimes k}_{A^D(r)/X^D(r)}) \ . \\ \uparrow {\scriptstyle Frob^{rig*}} & & \| \\ H^0(X^D(r^p), \underline{\omega}^{\otimes k}_{A^D(r^p)/X^D(r^p)}) & & \\ \| & & \| \\ S^D(R_0, r^p, V_1(N), k) \otimes K & \xrightarrow{Frob_K} & S^D(R_0, r, V_1(N), k) \otimes K \end{array}$$

Note that when $k = 0$, we have $1/p^k = 1$ and we get the same definition as before. To conclude this section, we prove a projection formula:

Lemma 13.3. *Let $r \in R_0$, $ord(r) > 1/(p+1)$. Assume that $f \in S^D(R_0, r^p, V_1(N), k) \otimes K$ and $g \in S^D(R_0, r, V_1(N), k') \otimes K$. Then we have:*

$$U(Frob_K(f).g) = f.U(g)$$

Proof. We write:

$$\begin{aligned} U(Frob_K(f).g) &= (1/p).Tr_{Frob^{rig}}(\check{\pi}^*)^{\otimes k+k'}(1/p^k.(\pi^*)^{\otimes k}(Frob^{rig})^*(f).g) \\ &= 1/p.Tr_{Frob^{rig}}((Frob^{rig})^*(f).(\check{\pi}^*)^{\otimes k'}(g)) \\ &= 1/p.f.Tr_{Frob^{rig}}((\check{\pi}^*)^{\otimes k'}(g)) \\ &= f.U(g) \end{aligned}$$

Here we use the following easy general fact. Let $\phi : X' \to X$ be a finite flat morphism of affinoids. Let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent sheaves of $\mathcal{O}_X$ modules and $f \in H^0(X, \mathcal{F})$ and $g \in H^0(X', \phi^*\mathcal{G})$. Then $Tr_\phi(\phi^*(f).g) = f.Tr_\phi(g)$. Also note that the false degree of π is equal to p, which means $\pi \circ \check{\pi} = p$ and therefore $\check{\pi}^*\pi^* = p$ on sections of $(\Omega^1_{(A^D(r)/\mathbb{H}^D_r)/X^D(r)})$. ∎

14 Continuity Properties of U

In this section we study continutiy properties of U. This paves the way to use the Fredholm theory of U and rigid analytic techniques to study the eigenforms of U.

As we have seen $S^D(R_0, r, V_1(N), 0) \otimes K$ is an affinoid K-algebra and thus carries a canonical K-Banach space topology. There are quite a lot of norms which induce this canonical topology on $S^D(R_0, r, V_1(N), 0) \otimes K$. Since $S^D(R_0, r, V_1(N), 0) \otimes K$ is reduced, the supremum semi-norm on $S^D(R_0, r, V_1(N), 0) \otimes K$ is a norm and is equivalent to any other Banach norm on this affinoid K algebra (theorem 6.2.4.1 of [4]). We will denote this norm by $| \ |_{sup}$.

An element $f \in S^D(R_0, r, V_1(N), 0) \otimes K$ is called *power bounded* if the set $\{f^n \mid n \geq 0\}$ is bounded. By proposition 6.2.3.1 of [4], the set of all power bounded elements of $S^D(R_0, r, V_1(N), 0) \otimes K$ equals

$$\{f \in S^D(R_0, r, V_1(N), 0) \otimes K \mid |f|_{sup} \leq 1\}.$$

In the following theorem we will show that when $ord(r) > 0$, this is in fact equal to $S^D(R_0, r, V_1(N), 0)$.

Theorem 14.1. *Assume that $ord(r) > 0$. We have:*

$$S^D(R_0, r, V_1(N), 0) = \{f \in S^D(R_0, r, V_1(N), 0) \otimes K \mid |f|_{sup} \leq 1\}$$

Proof. Denote by B the finitely generated R_0-algebra

$$\bigoplus_{n \geq 0} S^D(R_0, n(p-1), N) \ / \langle E_{p-1} - r \rangle .$$

It is finitely generated over R_0 since $\underline{\omega}$ is an ample line bundle. By proposition 8.4, the p-adic completion of B, which we denote by $\hat{B}$ is equal to $S^D(R_0, r, V_1(N), 0)$. In fact via the isomorphism of theorem 8.8, B corresponds to the R_0-submodule of $B^D_{rig}(R_0, n, 0)$ consisting of all finite sums. This shows that B is a free R_0-module.In particular the map $\hat{B} \to \hat{B} \otimes K$ is injective. We show that $\hat{B}/\pi\hat{B} = B/\pi B$ is reduced. Since $p-1 > 3$ we can use lemma 6.1 to deduce that $B/\pi B$ is equal to

$$H^0(\mathbb{X}_1^D(N) \otimes R_0/\pi, \bigoplus_{n\geq 0} \underline{\omega}^{\otimes n(p-1)}) \ /\mathbf{H}$$

Let f be a nilpotent element of $B/\pi B$. Choose a representation of f, $f = \sum_{n=0}^{m} f_n$, with $f_n \in H^0(\mathbb{X}_1^D(N) \otimes R_0/\pi, \underline{\omega}^{\otimes n(p-1)})$, such that the number of summands is minimal. The relation $f^M = 0$ in $B/\pi B$ implies that $\mathbf{H}|f_m^M$. By lemma 5.2, $\mathbf{H}$ has simple zeroes and therefore $\mathbf{H}|f_M$. This results in a representation of f with fewer summands which is not possible. Therefore $\hat{B}/\pi\hat{B}$ is reduced. Let $f \in S^D(R_0, r, V_1(N), 0) \otimes K = \hat{B} \otimes K$ be power bounded. Then by [17], f is integral over $S^D(R_0, r, V_1(N), 0) = \hat{B}$. Now, if $g/\pi \in \hat{B} \otimes K$ satisfies an integral equation

$$g^n/\pi^n + \lambda_{n-1} g^{n-1}/\pi^{n-1} + ... = 0,$$

clearing the denominators in the equation and reducing $mod\ \pi$ we see that $\bar{g}$ is nilpotent in $\hat{B}/\pi\hat{B}$. Therefore $g \in \pi\hat{B}$, and $f \in \hat{B}$. This ends the proof of the theorem. □

Proposition 14.2. *For any $r \in R_0$ with $ord(r) < 1/(p+1)$ we have:*

$$U(S^D(R_0, r^p, V_1(N), 0)) \subset (1/p).(S^D(R_0, r^p, V_1(N), 0))$$

Therefore $U = U_0$ is a continuous operator of norm $|U| \leq p$.

Proof. We only need to show that $Tr_{Frob_K}(S^D(R_0, r, V_1(N), 0)) \subset S^D(R_0, r^p, V_1(N), 0)$. By [17], for any finite flat morphism $\phi : C_1 \to C_2$ of affinoid K algebras we always have:

$$Tr_\phi(\{power\ bounded\ elements\ of\ B\}) \subset \{power\ bounded\ elements\ of\ A\}.$$

By theorem 14.1, we deduce that for $ord(r) > 0$:

$$Tr_{Frob_K}(S^D(R_0, r, V_1(N), 0)) \subset S^D(R_0, r^p, V_1(N), 0),$$

which is the desired result in this case. For $r = 1$ this holds because $Tr_{Frob_K} = Tr_{Frob} \otimes K$. □

The continuity of U for general weight is a corollary of proposition 14.2.

Corollary 14.3. *For each k, the operator $U = U_k$ is a continuous operator of the K-Banach space $S^D(R_0, r^p, V_1(N), k)$.*

Proof. For general k we argue as follows. We only need to show that Tr_{Frob_K} is continuous. For simplicity let Ω_r denote $S^D(R_0, r, V_1(N), k)$ and S_r denote $S^D(R_0, r, V_1(N), 0)$. Let Ω denote $H^0(\widetilde{\mathbb{Y}}_r^D \otimes R_0, \underline{\omega}^{\otimes k}_{(\widetilde{\mathbb{B}}_r^D \otimes R_0/\widetilde{\mathbb{H}}_r^D)/\widetilde{\mathbb{Y}}_r^D \otimes R_0})$. By lemma 13.1 we have

$$\Omega \otimes K = H^0(X^D(r), \underline{\omega}^{\otimes k}_{(A^D(r)/\mathbb{H}_r^D)/X^D(r)}).$$

So Tr_{Frob_K} is the following:

$$\Omega_r \otimes K \xrightarrow{(\tilde{\pi}^*)^{\otimes k}} \Omega \otimes K \xrightarrow{Tr_{Frob^{rig}}} \Omega_{r^p} \otimes K.$$

We can endow $\Omega_r \otimes K$ (respectively $\Omega \otimes K$) with a norm such that Ω_r (respectively Ω) is the unit ball. To prove continuity of U_k, we need to show that the image of Ω_r in $\Omega_{r^p} \otimes K$ under Tr_{Frob_K} is bounded. The first arrow in the above diagram is induced from $\Omega_r \xrightarrow{(\tilde{\tilde{\pi}}^*)^{\otimes k}} \Omega$ by tensoring with K (here $\tilde{\pi}$ denotes the natural projection in the formal setting). Therefore we see that the image of Ω_r under Tr_{Frob_K} is contained in the image of Ω under $Tr_{Frob^{rig}}$. We show that this image is bounded.

By lemma 13.1, we know

$$(\Omega_{r^p} \otimes K) \otimes_{S_{r^p} \otimes K} S_r \otimes K \xrightarrow{\sim} \Omega \otimes K$$

and by definition, the trace function $Tr_{Frob^{rig}} : \Omega \otimes K \to \Omega_{r^p} \otimes K$ is obtained from

$$Tr_{Frob_K} : S_r \otimes K \to S_{r^p} \otimes K$$

by tensoring with $\Omega_{r^p} \otimes K$ over $S_{r^p} \otimes K$. From the description of Tr_{Frob_K} and by proposition 14.2, we deduce that

$$Tr_{Frob_K}(\Omega_{r^p} \otimes_{S_{r^p}} S_r) \subset \Omega_{r^p}.$$

So we only need to show that for some $M \geq 0$,

$$t^M \Omega \hookrightarrow \Omega_{r^p} \otimes_{S_{r^p}} S_r,$$

where t is a uniformizer in R_0. But this is true since

$$\Omega \otimes K \xrightarrow{\sim} (\Omega_{r^p} \otimes_{S_{r^p}} S_r) \otimes K$$

and Ω is a finitely generated S_r-module.

□

The operator theory of U heavily depends on whether it acts on the full space of convergent modular forms or just on the subspace of overconvergent ones. It turns out that in the latter case, apart from the kernel of U, the eigenspaces are all finite dimensional, whereas this does not hold in the first case. This is because when $ord(r) > 0$, the U operator is a completely continuous operator of K-Banach spaces. An operator of K-Banach spaces is called completely continuous, if it is a limit of operators whose images are finite dimensional over K.

We recall some facts about completely continuous operator from [16]. An operator $L : M_1 \to M_2$ is completely continuous if it takes any bounded subspace of M_1 to a relatively compact subset (i.e. with compact closure) of M_2.

If $L : M_1 \to M_2$ is a completely continuous operator of K-Banach spaces and$u : N_1 \to M_1$ and $v : M_2 \to N_2$ are continuous operators then $v \circ L \circ u$ is also completely continuous.

Proposition 14.4. *Assume $0 < ord(r) < 1/(p+1)$ and $k = 0$ or $k \geq 3$. Furthermore assume that the residue field of R_0 is finite. Then U is a completely continuous operator of $S^D(R_0, r, V_1(N), k)$.*

Proof. By definition, U is the following composite:

$$S^D(R_0, r^p, V_1(N), k) \otimes K \hookrightarrow S^D(R_0, r, V_1(N), k) \otimes K \overset{(1/p).Tr_{Frob_K}}{\longrightarrow} S^D(R_0, r^p, V_1(N), k) \otimes K.$$

We have seen that $S^D(R_0, r, V_1(N), k) \otimes K \overset{(1/p).Tr_{Frob_K}}{\longrightarrow} S^D(R_0, r^p, V_1(N), k) \otimes K$ is continuous. So to prove the claim we only need to show that:

$$S^D(R_0, r^p, V_1(N), k) \otimes K \hookrightarrow S^D(R_0, r, V_1(N), k) \otimes K$$

is a completely continuous homomorphism of K Banach spaces. So we need to show that the image of the unit ball in $S^D(R_0, r^p, V_1(N), k) \otimes K$, which by our choice of norm is $S^D(R_0, r^p, V_1(N), k)$, is relatively compact. Corollary 8.9 describes this inclusion in terms of the chosen Banach basis. From this description we see that the image of $S^D(R_0, r^p, V_1(N), k)$ is finite modulo any power of p (because R_0/p^n is finite for any $n > 0$, and this image is finitely generated over R_0/p^n). Now since the image is also bounded, as it lies in the bounded set $S^D(R_0, r, V_1(N), k)$, we deduce that it is relatively compact. □

This implies that there is a a Fredholm theory for U acting on $S^D(R_0, r, V_1(N), k)$ when $ord(r) > 0$. In the next section we will use this to study the eigenforms of U.

15 Eigenforms of U

Coleman has used rigid geometry to study the eigenforms of U acting on the space of overconvergent modular forms over modular curves. We will follow the same idea to obtain some results about the eigenforms of U in this setting. As was explained earlier, the fact that U is a completely continuous operator over the space of overconvergent modular forms, allows one to use the Fredholm theory of U operator. Coleman [6] has developed a Fredholm theory for Banach modules over Banach algebras, which is a generalization of Serre's theory. In fact with his results, he is able to study the spectral theory of a family of completely continuous operators. We briefly mention some facts from [6] and [9] which we will use here.

Let A be a Banach algebra and M be a Banach module over A. We say that M is *orthonormizable* if it has a Banach basis over A. In other words there is a set $\{e_i : i \in I\}$ of elements of M, for some index set I, such that every element $e \in M$ can be uniquely written as $\sum_{i \in I} a_i e_i$ with $a_i \in A$ such that $lim_{i \to \infty} |a_i| = 0$.

Let $L : M \to N$ be a continuous operator where M and N are Banach modules over a Banach algebra A. As in the case of K-Banach spaces, we can define a completely continuous operator to be one which is a limit of operators whose images are finitely generated over A.

Assume $L : M \to N$ be a completely continuous operator of orthonormizable Banach modules over a Banach algebra A. Let $a \in A$ be a multiplicative element (i.e. $|ab| = |a|.|b|$ for all $b \in A$). Assume that $|L|$ is at most $|a|$. Let A^0 denote the set of all elements of A of norm at most 1. Then there is a power series $P_L(T) \in A^0[\![aT]\!]$ which is the called the Fredholm determinant of L. It is entire in T (i.e. if $P_L(T) = \sum_{m \geq 0} c_m T^m$, $|c_m| M^m \to 0$ for any real number M).

Let $L : M \to M$ be a continuous operator of K-Banach spaces, where K is a field, complete with respect to a non-archimedian valuation. Let α be a rational number. An element m of M is said to be a generalized eigenform of slope α of L, if there exists a polynomial $f(T)$ in $K[T]$ such that $f(L)(m) = 0$ and all the roots of $f(T)$ have valuation α. If $f(T) = (T - \lambda)^n$, we call m a generalized eigenform with eigenvalue λ.

If $A = K$, Coleman's Fredholm determinant agrees with that of Serre. In that case, $\lambda \neq 0$ is an eigenvalue of L if and only if $P_L(\lambda^{-1}) = 0$ and the dimension of the generalized eigenspace corresponding to λ is the multiplicity of λ^{-1} as a root of $P_L(T)$.

We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$ with normalized valuation such that $ord(p) = 1$.

Let $r = 1$. Then $Frob(E_{p-1}) \in H^0(X^D(1), \underline{\omega}^{\otimes p-1})$ is a nowhere vanishing section. In fact by definition of $X^D(1)$, E_{p-1} is a nowhere vanishing section of $\underline{\omega}^{\otimes p-1}$ on $X^D(1)$, and $Frob(E_{p-1}) = (\pi^*/p)^{\otimes p-1}(Frob^{rig})^*(E_{p-1})$ is nowhere vanishing because π is étale in this case. We define an element $\mathbf{e} \in H^0(X^D(1), \mathcal{O}_{X^D(1)}) = S^D(R_0, 1, V_1(N), 0)$ as follows:

$$\mathbf{e} := E_{p-1}/Frob(E_{p-1}).$$

In the context of modular curves, where we have q-expansions, the q-expansion of $\mathbf{e}$ is $E_{p-1}(q)/E_{p-1}(q^p)$ and the following result is a simple consequence of the fact that $E_{p-1}(q) \equiv 1 \ mod \ p$.

Proposition 15.1. *Let $|\ |_{X^D(1)}$ denote the supremum norm on $X^D(1)$. We have:*

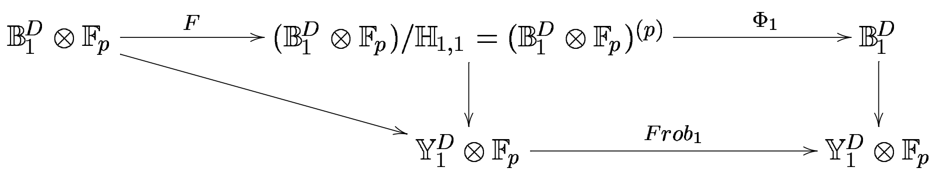
$$|\mathbf{e} - 1|_{X^D(1)} \leq |p|.$$

Proof. We may assume $K = \mathbb{Q}_p$ throughout this proof. First we show that $Frob(E_{p-1}) \in H^0(\widetilde{\mathbb{Y}}_1^D, \underline{\omega}^{\otimes p-1}) \subset H^0(X^D(1), \underline{\omega}^{\otimes p-1})$. We know that

$$p^{p-1}.Frob(E_{p-1}) = (\tilde{\pi}^*)^{\otimes p-1}(\widetilde{Frob}^*)(E_{p-1}) \in H^0(\widetilde{\mathbb{Y}}_1^D, \underline{\omega}^{\otimes p-1}).$$

So we have to show that $(\tilde{\pi}^*)^{\otimes p-1}(\widetilde{Frob}^*)(E_{p-1})$ is divisible by p^{p-1} in $H^0(\widetilde{\mathbb{Y}}_1^D, \underline{\omega}^{\otimes p-1})$. Since $(\widetilde{Frob}^*)(E_{p-1})$ can be locally written as a tensor product of $p-1$ differential forms on $\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{H}}_1^D$, it suffices to show that for any $\eta \in H^0(\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{H}}_1^D, \Omega^1_{(\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{H}}_1^D)/\widetilde{\mathbb{Y}}_1^D})$, the pullback $\tilde{\pi}^*(\eta)$ is divisible by p in $H^0(\widetilde{\mathbb{B}}_1^D, \Omega^1_{\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{Y}}_1^D})$, noting that $\widetilde{\mathbb{B}}_1^D$ is flat over $\mathbb{Z}_p$.

Since $\widetilde{\mathbb{H}}_1^D$ modulo p is the kernel of Frobenius morphism of $\mathbb{B}_1^D \otimes \mathbb{F}_p$, we have the following diagram from discussions of section 9.



In which $Frob_1$ is the p-th power morphism of $\mathbb{Y}_1^D$ over $\mathbb{F}_p$, $\Phi_1 : (\mathbb{B}_1^D \otimes \mathbb{F}_p)^{(p)} \to \mathbb{B}_1^D \otimes \mathbb{F}_p$ is the base extension morphism and F is the relative Frobenius morphism of $\mathbb{B}_1^D \otimes \mathbb{F}_p$ (which is the reduction of $\tilde{\pi}$ modulo p). Now if we denote the reduction mod p of η by $\bar{\eta}$, we know that:

$$F^*(\bar{\eta}) = 0.$$

This shows that $\tilde{\pi}^*(\eta)$ is divisible by p in $H^0(\widetilde{\mathbb{B}}_1^D, \Omega^1_{\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{Y}}_1^D})$ as desired. Now, since $\tilde{\pi}\check{\tilde{\pi}} = p$ we know

$$\check{\tilde{\pi}}^*(Frob(E_{p-1})) = (\widetilde{Frob})^*(E_{p-1})$$

Which shows that $\check{\tilde{\pi}}^*(Frob(E_{p-1}))$ is nowhere vanishing. Since $\check{\tilde{\pi}}$ is étale (since $r = 1$), we deduce that $Frob(E_{p-1})$ is a nowhere vanishing section of $\underline{\omega}^{\otimes p-1}$ on $\widetilde{\mathbb{Y}}_1^D$.

Since the result we are proving is local on $X^D(1)$, we will henceforth work locally on $\widetilde{\mathbb{Y}}_1^D$, so that we can assume $\underline{\omega}$ to be a trivial line bundle. Fix generators ω and ω' of $\underline{\omega}_{\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{Y}}_1^D}$ and $\underline{\omega}_{(\widetilde{Frob})^*\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{Y}}_1^D}$ such that ω' reduces to $\omega^{(p)}$ modulo p. Notice that this makes sense, because the reduction of $(\widetilde{Frob})^*\widetilde{\mathbb{B}}_1^D/\widetilde{\mathbb{Y}}_1^D$ mod p is $(\mathbb{B}_1^D \otimes \mathbb{F}_p)^{(p)}$.

Write $E_{p-1} = \lambda.\omega^{\otimes p-1}$. let $(\widetilde{Frob})^*(E_{p-1}) = \lambda'.(\omega')^{\otimes p-1}$. We show that

$$\lambda' \equiv \lambda^p \quad mod\ p.$$

Since the reduction modulo p of $\widetilde{Frob}$ is the p-th power morphism, we have

$$\bar{\lambda}^p.(\bar{\omega}^{(p)})^{\otimes p-1} = \bar{\lambda}'.(\bar{\omega'})^{\otimes p-1}$$

where by $^-$ we denote reduction mod p. Now by choice of ω' we have $\bar{\omega'} = \bar{\omega}^{(p)}$ and hence $\lambda' \equiv \lambda^p$ *mod p*.

Next we prove that $\check{\tilde{\pi}}^*(\omega) = \beta.\omega^{'}$ with $\beta \equiv \lambda \ mod \ p$. This is essentially shown in the proof of proposition 10.5. Reducing mod p, $\check{\tilde{\pi}}$ becomes the Verschebung morphism:

$$V : (\mathbb{B}_1^D \otimes \mathbb{F}_p)^{(p)} \to (\mathbb{B}_1^D \otimes \mathbb{F}_p).$$

From the proof of proposition 10.5, we can write $V^*(\bar{\omega}) = \gamma.\bar{\omega}^{(p)}$, where:

$$\gamma = \mathbf{H}(\mathbb{B}_1^D \otimes \mathbb{F}_p, i, \bar{\omega}).$$

By choice of $\omega^{'}$, we have $\bar{\beta} = \gamma$ and hence:

$$\beta \equiv E_{p-1}(\mathbb{B}_1^D \otimes \mathbb{F}_p, i, \omega) \quad mod \ p.$$

Now by definition $E_{p-1}(\mathbb{B}_1^D \otimes \mathbb{F}_p, i, \omega) = \lambda$, which proves the claim. We can now write:

$$\begin{aligned} \check{\tilde{\pi}}^*(E_{p-1}) &= \check{\tilde{\pi}}^*(\lambda.\omega^{\otimes p-1}) \\ &= \lambda.\check{\tilde{\pi}}^*(\omega)^{\otimes p-1} \\ &= \lambda.\beta^{p-1}.(\omega^{'})^{\otimes p-1} \end{aligned}$$

On the other hand from the definition of $Frob(E_{p-1})$ we see that:

$$\check{\tilde{\pi}}^*(Frob(E_{p-1})) = (\widetilde{Frob})^*(E_{p-1}) = \lambda^{'}.(\omega^{'})^{\otimes p-1}.$$

And thus:

$$\check{\tilde{\pi}}^*(E_{p-1}) = \check{\tilde{\pi}}^*(\mathbf{e}Frob(E_{p-1})) = \mathbf{e}.\lambda^{'}.(\omega^{'})^{\otimes p-1}.$$

This shows that $\lambda\beta^{p-1} = \mathbf{e}\lambda^{'}$. However by the congruences mod p which we have so far obtained, this means:

$$\mathbf{e}.\lambda^p \equiv \lambda^p \quad mod \ p.$$

Since E_{p-1} is nowhere vanishing, we deduce that λ is a unit and therefore:

$$\mathbf{e} \equiv 1 \quad mod \ p.$$

This proves that $|\mathbf{e} - 1|_{X^D(1)} \leq |p|$. We are done.

□

Let $r \in K$ be a p-th power such that $0 < ord(r) < 1/(p+1)$. Consider the isomorphism of K Banach spaces defined by multiplication by E_{p-1}:

$$\begin{array}{rcl} S^D(R_0, r, V_1(N), k) \otimes K & \xrightarrow{\sim} & S^D(R_0, r, V_1(N), k+p-1) \otimes K \\ h & \mapsto & hE_{p-1} \end{array}$$

It is an isomorphism because E_{p-1} is nowhere vanishing on $X^D(r)$. The pullback of the operator $U_{(k+p-1)}$ via this isomorphism is an operator on $S^D(R_0, r, V_1(N), k)$ given by:

$$h \mapsto E_{p-1}^{-1} U_{(k+p-1)}(hE_{p-1}).$$

However from lemma 13.3 we have:

$$E_{p-1}^{-1} U_{(k+p-1)}(hE_{p-1}) = E_{p-1}^{-1} U_{(k+p-1)}(Frob(E_{p-1}).\mathbf{e}.h) = U_{(k)}(\mathbf{e}.h).$$

In other words the pullback of $U_{(k+p-1)}$ via the above isomorphism can be written as:

$$U_{(k)} \circ m_{\mathbf{e}},$$

where $m_{\mathbf{e}}$ denotes multiplication by $\mathbf{e}$. This proves the following lemma.

Lemma 15.2. *Let notation be as above. For each $n \geq 0$, we have:*

$$P_{U_{(k+n(p-1))}} = P_{U_{(k)} \circ m_{\mathbf{e}^n}}.$$

So studying operator theory of U in weight $k+p-1$ is equivalent to studying the operator theory of $U \circ m_{\mathbf{e}^n}$ in weight k. Coleman's idea is to put all the $U \circ m_{\mathbf{e}^n}$ together in a family of operators, producing a completely continuous operator of a Banach space. The study of the latter will give us information about each of the original operators. We will follow this idea to produce a family of operators parametrized by a rigid analytic disc, acting on a space whose elements are families of overconvergent modular forms with respect to D, parametrized by the same rigid space.

Since $|\mathbf{e}-1|_{X^D(1)} = lim_{t \to 1^-} |\mathbf{e}-1|_{X^D(t)}$, there exists a $t \in |\mathbb{C}_p|$ such that $1 > t > |p|^{1/(p+1)}$ and $|\mathbf{e}-1|_{X^D(t)} \leq |p|^{1/(p-1)}$. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers R, such that there is an element $r \in R$ with $|r| = t$ which is a p-th power. Note that we have $ord(r) > 0$.

It is straightforward to check that if $s \in \mathbb{C}_p$ has $|s| \leq 1$ and $|x| < |p|^{1/(p-1)}$, then $(1+x)^s = \sum_{n \geq 0} C_s^n x^n$ is convergent. Let us fix an integer k_0 throughout the discussion. For any $s \in \mathbb{C}_p$ with $|s| \leq 1$, define:

$$\begin{array}{rcl} u_s : H^0(X^D(r), \underline{\omega}^{\otimes k_0}) & \to & H^0(X^D(r), \underline{\omega}^{\otimes k_0}) \\ h & \mapsto & U_{(k_o)}(h.\mathbf{e}^s) \end{array}$$

Let $B = B_K[0, \delta]$ denote the affinoid subset of the rigid space $\mathbb{A}^1_K$ given by $|x| \leq \delta$. Define the affinoid rigid space $Z := B \times X^D(r)$. Let us denote the pullback of $\underline{\omega} = \underline{\omega}_{A^D(r)/X^D(r)}$ to Z under the second projection, again by $\underline{\omega}$. Then we have:

$$H^0(Z, \underline{\omega}^{\otimes n}) = H^0(B, \mathcal{O}_B) \hat{\otimes}_K H^0(X^D(r), \underline{\omega}^{\otimes n}).$$

Now consider the operator :

$$id \hat{\otimes} U_{k_0} : H^0(Z, \underline{\omega}^{\otimes k_0}) \to H^0(Z, \underline{\omega}^{\otimes k_0}).$$

This is obtained by base extension of $U_{(k_0)}$ under the map $K \to H^0(B, \mathcal{O}_B)$. Since we can think of $\mathbf{e}^s$ as a rigid function on Z, there is a continuous operator:

$$m_{\mathbf{e}^s} : H^0(Z, \underline{\omega}^{\otimes k_0}) \to H^0(Z, \underline{\omega}^{\otimes k_0})$$

which is given by multiplication by $\mathbf{e}^s$. Now we define:

$$\mathbb{U}_{k_0} := id\hat{\otimes} U_{k_0} \circ m_{\mathbf{e}^s}.$$

This is a continuous operator of the $H^0(B, \mathcal{O}_B)$-Banach space $H^0(Z, \underline{\omega}^{\otimes k_0})$. It is clear that the restriction of $\mathbb{U}_{k_0}$ to the fibre of Z over s_0 is the already defined operator u_{s_0}. We show that $\mathbb{U}_{k_0}$ is completely continuous. We have seen in proposition 14.4 $U_{(k_0)} : H^0(X^D(r), \underline{\omega}^{\otimes k_0}) \to H^0(X^D(r), \underline{\omega}^{\otimes k_0})$ is completely continuous. It is easy to seethe base extension of a completely continuous operator under a contractive map of Banach algebras is again completely continuous [6]. This shows that $id\hat{\otimes} U_{k_0}$ is completely continuous. But it is easily shown that the composition of a continuous operator and a completely continuous operator is always completely continuous [6]. Since $m_{\mathbf{e}^s}$ is continuous, we deduce that $\mathbb{U}_{k_0}$ is completely continuous. We can therefore study its Fredholm theory as described in [6] §A. Indeed to do so we also need to check that the Banach module in question has a Banach basis. Since we have already proven this for $H^0(X^D(r), \underline{\omega}^{\otimes k_0})$ in proposition 8.8, we can deduce it for its base extension, $H^0(Z, \underline{\omega}^{\otimes k_0})$ by proposition A.1.3 of [6].

By our notation $P_{\mathbb{U}_{k_0}}(s, T) \in K[\![s, T]\!]$ denotes the Fredholm determinant of $\mathbb{U}_{k_0}$. Now letting $s = n$ corresponds to a base extension $H^0(B, \mathcal{O}_B) \to K$. Under this base extension our operator becomes $u_n : H^0(X^D(r), \underline{\omega}^{\otimes k_0}) \to H^0(X^D(r), \underline{\omega}^{\otimes k_0})$. Since the formation of Fredholm determinant commutes with contractive base change ([6], lemma A.2.5) and by lemma 15.2, we deduce that:

$$P_{\mathbb{U}_{k_0}}(n, T) = P_{u_n}(T) = P_{U_{(k_0+n(p-1))}}(T).$$

This means that the number of zeroes of $P_{\mathbb{U}_{k_0}}(n, T)$ over $\mathbb{C}_p$ (counting multiplicities) which have valuation $-\alpha$ is the same as the dimension of the generalized eigenforms of $U_{(k_0+n(p-1))}$ on $H^0(X^D(r), \underline{\omega}^{\otimes k_0+n(p-1)})$ of slope α.

We study the zero locus of the entire series $P = P_{\mathbb{U}_{k_0}}(s, T)$ as in [6] A.5. Suppose α is a real number in $|K|$. Let us call A_α the affinoid subdomain of $B \times \mathbb{A}^1_K$ determined by $|T| = |p|^{-\alpha}$. The subspace of this affinoid determined by $P(T) = 0$ is an affinoid over B which we call Z_α. The projection map $f : Z_\alpha \to B$ is quasi-finite as U is completely continuous. For any closed point x of B, $f^{-1}(x)$ is a scheme of dimension 0 over the residue field of x. Denote the dimension of its ring of functions over the residue field of x by $deg(f^{-1}(x))$. Coleman proves in [6], proposition $A.5.5$:

Proposition 15.3. *Let the notation be as above. For each integer $i \geq 0$, the set of closed points x of B such that $deg(f^{-1}(x)) \geq i$ is the set of closed points of an affinoid subdomain B_i of B. Moreover, B_i is empty for large i.*

If m is the biggest integer for which B_m is not empty, then the set of closed points of B for which $deg(f^{-1}(x)) = m$ is the set of closed points of an affinoid subdomain in B. This shows that if $deg(f^{-1}(x)) = m$ for some x which corresponds to $n \in \mathbb{Z}_p$, then there is a γ such that if $|n - n^{'}| \leq \gamma$, the same holds for $n^{'} \in \mathbb{Z}_p$. Now looking at the intersection of the complement of B_m with B_{m-1}, over which $deg(f^{-1}(x)) = m - 1$, we obtain the same result for integers for which the fibre has degree $m - 1$. Continuing like this we see that for any $n \in \mathbb{Z}_p$ there is a positive number γ such that $|n - n^{'}| \leq \gamma$ implies that the degree of the fibres over these points are equal. Since $\mathbb{Z}_p$ is compact γ can be chosen uniformly. Now degree of $f^{-1}(x)$ is the dimension of the space of generalized eigenforms of slope α of u_x. But when x is given by an integer n, by the above discussions, this is the same as the dimension of the space of generalized eigenforms of slope α of $U_{(k_0+n(p-1))}$. For any finite extension of $\mathbb{Q}_p$ whose ring of integers contains a non-unit element r (which is a p-th power) and has absolute value at least $t > |p|^{1/(p+1)}$ (t was described earlier), denote the dimension of the space of generalized eigenforms of $U_{(k)}$ in $H^0(X^D(r), \underline{\omega}^{\otimes k})$ by $d(k, N, \alpha)$. Varying k_0 in the above argument, we have proven the following:

Theorem 15.4. *There exists an $M > 0$ depending only on D, α and N such that if $k \equiv k^{'} \bmod p^M(p-1)$, then:*

$$d(k, N, \alpha) = d(k^{'}, N, \alpha).$$

Moreover $d(k, N, \alpha)$ is uniformly bounded for all $k \in \mathbb{Z}$.

We expect the following to be true as in the case of modular curves:

Guess: If the slope is small enough, any overconvergent generalized eigenform form with respect to D is classical (i.e. it is a modular form of level Np with trivial character at p with respect to D).

Note that in the case of modular curves Coleman [7] proves that indeed if $\alpha < k - 1$, any generalized eigenform of slope α is classical. If the guess is true, then we can deduce theorem 15.4 for classical quaternionic modular forms of small enough slope.

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