King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

ATTACH THIS PAPER TO YOUR SCRIPT **USING THE STRING PROVIDED**

Candidate No: .... **Desk No:** ....

MSC EXAMINATION

7CCMMS30 (CMMS30) RELATIVITY, MECHANICS AND QUANTUM THEORY

JANUARY 2010

TIME ALLOWED: TWO HOURS

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO TWO QUESTIONS FROM PART A AND TWO QUESTIONS FROM PART B - FOUR QUESTIONS IN TOTAL.
IF MORE THAN FOUR QUESTIONS ARE ANSWERED, ONLY THE BEST TWO QUESTIONS FROM PART A AND THE BEST TWO QUESTIONS FROM PART B WILL COUNT TOWARDS GRADES A AND B, BUT CREDIT WILL BE GIVEN FOR ALL WORK DONE FOR LOWER GRADES.

YOU ARE PERMITTED TO USE A CALCULATOR.
ONLY CALCULATORS APPROVED BY THE COLLEGE MAY BE USED.

TURN OVER WHEN INSTRUCTED

PART A

1. Consider a system with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}(p_i)^2 + V(q^i) \ ,$$

where V is a scalar potential, and p_i and q^i are the momenta and positions of the system.

(i) Define the Poisson bracket $\{F, G\}$ of two functions $F(q,p)$ and $G(q,p)$ on phase space. What is the Poisson bracket $\{q^i, p_j\}$?

(ii) Quantize the system described by the Hamiltonian H above. Write the momentum and Hamiltonian operators in the position representation.

(iii) Give the definition of a Hilbert space, $\mathcal{H}$ including that of the inner product on $\mathcal{H}$. State the definition of the adjoint $A^\dagger$ of an operator A. Give also the definition of a self-adjoint operator.

(iv) Show that the eigenvalues of a self-adjoint operator are real. Show that the eigenstates of a self-adjoint operator for two different eigenvalues are orthogonal.

(v) Assume that the Hamiltonian operator, $\hat{H}$, is self-adjoint to show that the inner product $\langle\psi_1, \psi_2\rangle$ of two solutions of the Schrödinger equation, $i\partial_t\psi_1 = \hat{H}\psi_1$ and $i\partial_t\psi_2 = \hat{H}\psi_2$, is time independent.

2. The Hamiltonian operator of a Harmonic oscillator with frequency ω is

$$H = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{x}^2 ,$$

where $\hat{p}$ is the momentum and $\hat{x}$ is the position operators, and $[\hat{x}, \hat{p}] = i$. Consider the dilatation type of operators

$$D = \frac{\omega}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) , \quad S = \frac{1}{2}\hat{p}^2 - \frac{\omega^2}{2}\hat{x}^2 .$$

(i) Compute the commutators

$$[\hat{x}, H] , \quad [\hat{x}, D] , \quad [\hat{x}, S] , \quad [\hat{p}, H] , \quad [\hat{p}, D] , \quad [\hat{p}, S] .$$

You may use without proof the operator identity

$$[A, BC] = [A, B]C + B[A, C] .$$

(ii) Compute the commutators

$$[H, D] , \quad [H, S] , \quad [D, S] .$$

(iii) Express H, D and S in terms of the creation, $\alpha^\dagger$, and annihilation, α, operators

$$\alpha^\dagger = \frac{\omega\hat{x} - i\hat{p}}{\sqrt{2\omega}} , \quad \alpha = \frac{\omega\hat{x} + i\hat{p}}{\sqrt{2\omega}} .$$

(iv) Express the states $H|0\rangle$, $D|0\rangle$, $S|0\rangle$, $H\alpha^\dagger|0\rangle$, $D\alpha^\dagger|0\rangle$ and $S\alpha^\dagger|0\rangle$ in terms of the basis $|n\rangle = \alpha^{\dagger n}|0\rangle$, where $|0\rangle$ is the vacuum state, ie $\alpha|0\rangle = 0$. You may use without proof that $[\alpha, \alpha^\dagger] = 1$.

3. The Lagrangian of a non-relativistic particle with mass m (propagating in the 5-brane geometry) is

$$\mathcal{L} = \frac{m}{2}\Big(1 + \frac{1}{x^2}\Big) \sum_{i,j=1}^{4} \delta_{ij}\dot{x}^i\dot{x}^j \ ,$$

where x^i are the real coordinates, $x^2 = \sum_{i=1}^{4}(x^i)^2$, and $\dot{x}^i$ is the time derivative of x^i.

(i) Give the Lagrangian equations of motion of the system.

(ii) Find the Hamiltonian of the system and give the Hamiltonian equations of motion.

(iii) Show that the Lagrangian is invariant under orthogonal $O(4)$ transformations, ie transformations

$$x^i \to x'^i = \sum_{j=1}^{4} A^i{}_j x^j$$

such that

$$\delta_{ij} A^i{}_k A^j{}_l = \delta_{kl} \ ,$$

or equivalently in matrix notation $A^t A = 1_{4\times 4}$, where A^t is the transposed of A.

(iv) Find the infinitesimal transformation generated by the orthogonal transformations in (iii) and use without proof Noether's theorem to calculate the conserved charge associated with the symmetry.

(iv) Verify that the above Noether charge is conserved subject to the equations of motion.

PART B

4. Cosets:

(i) Let G be a group and H a subgroup of G. Give the definitions of the left-cosets $g\,H$ and of the right-cosets $H\,g$ of H.

(ii) Let $g_1,\ g_2 \in G$. Show that $g_2 \in g_1\,H$ implies $g_1 \in g_2\,H$.

(iii) When is H called a normal subgroup? Show that all subgroups of an abelian group are normal. Give an example for a finite group G and a normal subgroup H such that all cosets of H have half as many elements as G.

(iv) Now let G_1 and G_2 be two non-trivial groups (meaning that they have other elements besides their identity elements). The cross product $G_1 \times G_2$ is the set of all pairs $G_1 \times G_2 := \{\ (g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\ \}$, and it is itself a group using the natural composition $(g_1, g_2) \circ (h_1, h_2) := (g_1 h_1, g_2 h_2)$.

Find two proper normal subgroups in G ("proper" meaning: other than $G_1 \times G_2$ or the subgroup consisting of the identity element only). You should show that they are normal subgroups.

(v) In the special case $G_1 = G_2$, there is another subgroup, the so-called diagonal subgroup $\Delta(G_1) := \{\ (g, g) \mid g \in G_1\ \}$. Show that this subgroup is not normal if G_1 is not abelian.

5. Group representations:

(i) Give the definition of a representation π of a group G on a vector space V.

(ii) The group SU(2) of unitary complex 2×2 matrices with determinant 1 acts on $\mathbb{C}^2$ in the fundamental representation $\pi_{\mathrm{f}}(g) = g$ for all $g \in \mathrm{SU}(2)$. One can show that π_{f} is equivalent to its complex conjugate representation $\overline{\pi_{\mathrm{f}}}$. Use this to show that π_{f} is also equivalent to its contragredient (or dual) representation π_{f}^c .

(iii) Let π_i be representations of a group G on vector spaces V_i for $i = 1, 2$. Give the definition of the tensor product representation $\pi_\otimes$ on $V_1 \otimes V_2$, and show that $\pi_\otimes$ is indeed a representation of G.

(iv) Consider the special case $G = \mathrm{SU}(2)$ and $\pi_1 = \pi_2 = \pi_{\mathrm{fund}}$. Show that the tensor product representation on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is not irreducible.

Hint: Study how the tensor product representation acts on the "Bell" vector $e_1 \otimes e_2 - e_2 \otimes e_1$, where the e_k are the standard basis vectors of $\mathbb{C}^2$. Also, it will be advantageous to derive a concrete form of the matrices in SU(2); to find this, you may use that for any complex 2×2 matrix one has

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

6. Lie groups and Lie algebras:

(i) The unit spheres $S^n := \{ x \in \mathbb{R}^{n+1} \mid x^2 = 1 \}$ are manifolds for all $n = 0, 1, 2, 3, \ldots$ – a fact that you may use throughout Question 6. Find composition laws that make S^0 and S^1 into groups. (You need not show associativity, but should give the identity elements and inverses.)

(ii) Show that the group SU(2) of unitary complex 2×2 matrices with determinant 1 is a manifold.

Hint: It will be advantageous to derive a concrete form of the matrices in SU(2); to find this, you may use that for any complex 2×2 matrix one has

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

(iii) Compute the Lie algebra $Lie(\mathrm{SO}(n))$ of the special orthogonal group $\mathrm{SO}(n)$. (Hint: you may use that any $g \in \mathrm{SO}(n)$ can be written as $g = \exp(\xi)$ for a suitable $\xi \in Lie(\mathrm{SO}(n))$, and also that $\exp(\xi) \in \mathrm{SO}(n)$ implies $\exp(t\,\xi) \in \mathrm{SO}(n)$ for all $t \in \mathbb{R}$.)

(iv) Let G be Lie group G whose elements are real matrices, i.e. $G \subset \mathrm{GL}(n, \mathbb{R})$ for some n. Then one can define its Lie algebra $Lie(G) \subset M(n, \mathbb{R})$ in a differerential geometry way, via derivatives of curves through the identity e of G as follows:

$$Lie(G) := \left\{ \xi = \frac{dg(t)}{dt}|_{t=0} \;\Big|\; g(t) \text{ a curve in } G \text{ with } g(0) = e \right\} .$$

Use this definition to show that G acts on $Lie(G)$ in the adjoint representation.
(You should give the definition of the adjoint action $\pi_{\mathrm{adj}}(g)$ for $g \in G$, point out why each $\pi_{\mathrm{adj}}(g)$ is a linear map on the Lie algebra, and show that π_{adj} satisfies the representation property.)