King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMMS30 (CMMS30) RELATIVITY, MECHANICS AND QUANTUM THEORY

JANUARY 2009

TIME ALLOWED: TWO HOURS

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO TWO QUESTIONS FROM PART A AND TWO QUESTIONS FROM PART B - FOUR QUESTIONS IN TOTAL.
IF MORE THAN FOUR QUESTIONS ARE ANSWERED, ONLY THE BEST TWO QUESTIONS FROM PART A AND THE BEST TWO QUESTIONS FROM PART B WILL COUNT TOWARDS GRADES A AND B, BUT CREDIT WILL BE GIVEN FOR ALL WORK DONE FOR LOWER GRADES.

NO CALCULATORS ARE PERMITTED.

TURN OVER WHEN INSTRUCTED

PART A

1. (i) Give the proof of Noether's theorem and find the conserved charge for a system described by an action $S_R[q]$ which is invariant under the infinitesimal internal symmetry $\delta q^i = \epsilon X^i(q)$, where R is the time interval that the action is defined over, q^i are the positions, ϵ is the infinitesimal parameter and $X^i(q)$ are functions of q^i.

(ii) Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\frac{|\dot{q}|^2}{|q|^2} ,$$

where $|q|^2 = \sum_{i=1}^n (q^i)^2$ and similarly for $|\dot{q}|^2$.

Find the Lagrangian equations of motion.

This system is invariant under the infinitesimal internal symmetries

$$\delta q^i = \epsilon q^i ,$$

and also under

$$\delta q^i = \epsilon \omega_{ij} q^j ,$$

where (ω_{ij}) is a constant skew-symmetric matrix.

Find the conserved charges associated to both symmetries.

Verify that these charges are conserved subject to the equations of motion.

2. (i) Define the adjoint $A^\dagger$ of an operator A in a Hilbert space. Give the definition of a self-adjoint operator. Show that the eigenvalues of self-adjoint operators are real, and the eigenvectors of two distinct eigenvalues are orthogonal.

(ii) The Schrödinger equation is

$$i\partial_t \psi = \hat{H}\psi \ .$$

Describe the Schrödinger and Heisenberg pictures of quantum mechanics and give the transformation which relates the states and operators of one picture to the other.

Show that the matrix elements of operators do not depend on the picture.

Derive the time evolution of Heisenberg picture operators.

(iii) Quantize the classical system described by the Lagrangian

$$L = \frac{1}{2}\left[(\dot{\varphi})^2 + (\dot{x})^2\right] \ ,$$

where φ is an angle, $0 \leq \varphi \leq 2\pi$ and $-\infty < x < +\infty$. In particular, give the Hamiltonian operator in the position representation, and compute its eigenvalues assuming the wave functions are periodic in φ. You need not be concerned with the normalization of the eigenfunctions.

3. The Lagrangian of non-relativistic particle with mass m and charge e propagating in a manifold M with metric $ds^2 = g_{ij}(x)dx^i dx^j$ and coupled to a magnetic gauge potential $A_i(x)$ is

$$\mathcal{L} = \frac{m}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + eA_i(x)\dot{x}^i \ ,$$

where x^i are the coordinates of the manifold, $i = 1, \ldots, \dim M$, and $\dot{x}^i$ is the time derivative of the coordinate x^i.

(i) Give the equations of motion in the Lagrangian formalism. In particular express the equations of motion in terms of the Levi-Civita connection.

(ii) Find the canonical momentum and the Hamiltonian of the theory.

(iii) The Noether's charge for general (space)time symmetries is

$$Q = \xi\mathcal{L} + X^i(x)\frac{\partial\mathcal{L}}{\partial\dot{x}^i} \ .$$

Specialise this to time translations and use the invariance of the action under time translations to calculate the energy E of the charged particle.

Is E expressed in terms of the velocities dependent on A?

Verify that E is conserved subject to the equations of motion.

PART B

4. Generalities from group theory

(i) Let H be a subgroup of a group G. Give the definition of the left and of the right cosets of H. When is H called normal?

(ii) Let H be a normal subgroup of G, and denote by G/H the set of left cosets of H. Show that G/H has a group structure. (You will need that the product $S_1 \cdot S_2$ of two subsets $S_1,\ S_2 \subset G$ is defined to be the set of all products $s_1 \cdot s_2$ with $s_i \in S_i$.)

(iii) For $g \in G$, give the definition of the conjugacy class C_g.

(iv) Show that two conjugacy classes C_{g_1} and C_{g_2} either coincide or are disjoint.

(v) Show that if a conjugacy class has more than one element, then it cannot be a subgroup of G.

(vi) Conversely, if C_g consists of a single element only, does it follow that C_g is a subgroup? (Give reasons for your answer.)

5. Lie algebras and Lie groups

(i) Give the definition of an abstract Lie algebra.

(ii) Consider the linear differential operators

$$l_n := -z^{n+1}\,\frac{d}{dz}\ , \quad n \in \mathbb{Z}$$

acting on analytic functions $f(z)$ (for $z \neq 0$).
Show that the commutator $[\,l_n\,,\,l_m\,]$ can be written as a linear combination of l_k.
Then compute how the exponential $\exp\{-\alpha\, l_{-1}\}$, where α is constant, acts on an analytic function $f(z)$.

(iii) Compute the Lie algebra *Lie*(SU(n)) of the Lie group SU(n). (You can use the fact that all $g \in \mathrm{SU}(n)$ can be written as $g = \exp(\xi)$ for some Lie algebra element ξ, and that $\exp(t\,\xi)$ is in SU(n) for all real t.)
Show that the commutator of two elements of *Lie*(SU(n)) is again in *Lie*(SU(n).

(iv) Give three Lie groups $G_1,\ G_2,\ G_3$ which are pairwise non-isomorphic (i.e. $G_i \not\simeq G_j$ if $i \neq j$, for all $i, j \in \{1, 2, 3\}$) but whose Lie algebras are isomorphic. (Argue *briefly* in support of your examples.)

6. Group representations

(i) Give the definition of a representation π of a group G on a vector space V.
When is a group representation called unitary?
What is an invariant subspace of V (with respect to the representation π)?
When is π called irreducible?

(ii) Let π_i be representations of a group G on vector spaces V_i, $i = 1, 2$. Give a formula for the tensor product representation $\pi_1 \otimes \pi_2$ on $V_1 \otimes V_2$ and verify that it satisfies the representation property.

(iii) Let $\pi_{\frac{1}{2}}$ denote the fundamental representation of SU(2) on $\mathbb{C}^2$. Consider $\pi_{\frac{1}{2}} \otimes \pi_{\frac{1}{2}}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and show that it is not irreducible.
(To do this, look at the action $\pi_{\frac{1}{2}} \otimes \pi_{\frac{1}{2}}$ on the state $e_1 \otimes e_2 - e_2 \otimes e_1$, where the e_i are standard basis vectors of $\mathbb{C}^2$. Also, it may be advantageous to use the explicit form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ for elements in SU(2), with $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.)

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