Relativity, Mechanics and Quantum Theory

Problem Sheet 7

Problem 7.1

The Schrödinger equation is:

$$i\hbar\frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t)$$

Let the inner product between two states $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$ in the Hilbert space be:

$$\langle\,\psi\mid\phi\,\rangle = \int_{\mathbb{R}^k} d^k q\,\psi^*\phi$$

The probability of finding a system in some eigenstate is normalised by the quantity $\langle\psi\;\mid\psi\,\rangle$. Evidently,

$$\frac{\partial}{\partial t}\langle\,\psi\mid\psi\,\rangle = \frac{\partial}{\partial t}\int_{\mathbb{R}^k} d^k q\rho(q,t)$$

Where $\rho(q,t) \equiv \psi^*\psi$ is the probability density.

(i.) Show that

$$\frac{\partial}{\partial t}(\rho(q,t)) + \sum_i \frac{\partial}{\partial q_i}J^i = 0$$

Where J^i is the probability current given by:

$$J^i \equiv \frac{i\hbar}{2m_i}(\frac{\partial\psi^*}{\partial q_i}\psi - \psi^*\frac{\partial\psi}{\partial q_i})$$

(ii.) Prove, with suitable assumptions, that the inner product $\langle\,\psi\mid\phi\,\rangle$ is time-independent.

Throughout this problem let

$$\hat{H} \equiv -\frac{\hbar^2}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial q_i^2} + \sum_i V(q_i)$$

Problem 7.2

Describe the Schrödinger and Heisenberg pictures of quantum mechanics and give the transformation which relates the states and operators of one picture to the other. Show that the matrix elements of operators do not depend on the picture. Derive the time evolution of Heisenberg picture operators.

Problem 7.3

Quantize the classical system described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}[\dot{\theta}^2 + \dot{x}^2]$$

where ϕ is an angle, $0 \leq \theta \leq 2\pi$ and $-\infty < x < +\infty$. In particular, give the Hamiltonian operator in the position representation, and compute its eigenvalues assuming the wave functions are periodic in θ. You need not be concerned with the normalisation of the eigenfunctions.

Problem 7.4

The annihilation and creation operators are given by

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} + \frac{i}{m\omega}\hat{p}) \qquad \text{and} \qquad \alpha^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} - \frac{i}{m\omega}\hat{p})$$

(i.) Show that

$$[\alpha, \alpha^\dagger] = 1$$

(ii.) The Hamiltonian operator of the quantum harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$$

Show that

$$\hat{H} = \hbar\omega(\frac{1}{2} + \alpha^\dagger\alpha)$$

(iii.) Compute both $[\hat{H}, \alpha]$ and $[\hat{H}, \alpha^\dagger]$.

(iv.) The Hamiltonian is a self-adjoint operator and has eigenvectors which form a basis of the Hilbert space. Denote these eigenvectors by $\mid n \rangle$ so that

$$\hat{H} \mid n \rangle = E_n \mid n \rangle$$

where E_n is the energy eigenvalue. Show that

$$(\alpha)^k \mid n \rangle \qquad \text{and} \qquad (\alpha^\dagger)^k \mid n \rangle$$

are also eigenvectors for the Hamiltonian and find expressions for their eigenvectors in terms of E_n.

(v.) Argue that there is a minimum energy eigenvalue and give its value. Let this ground-state be denoted $\mid 0 \rangle$ and note that it satisfies $\alpha \mid 0 \rangle = 0$.

(vi.) Show that $\alpha^\dagger\alpha \mid n \rangle = n \mid n \rangle$.

(vii.) Suppose that $\langle n \mid n \rangle = 1$ and prove that

$$\alpha \mid n \rangle = \sqrt{n} \mid n - 1 \rangle$$

$$\alpha\dagger \mid n \rangle = \sqrt{n+1} \mid n + 1 \rangle$$