Relativity, Mechanics and Quantum Theory

Problem Sheet 6

Problem 6.1

Show that if time translation ($t \to t + \epsilon b$) is a symmetry of the action that the conserved quantity is the energy function, the precursor to the Hamiltonian:

$$h = \sum_i \dot{q}_i p_i - \mathcal{L}$$

Problem 6.2

The Lagrangian of non-relativistic particle with mass m and charge e propagating in a manifold $\mathcal{M}$ with metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and coupled to a magnetic gauge potential $A_\mu(x)$ is

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e A_\mu(x) \dot{x}^\mu$$

where x^μ are the coordinates of the manifold, $\mu = 1, ..., \dim(\mathcal{M})$ and $\dot{x}^\mu$ is the time derivative of the coordinate x^μ.

(i.) Give the equations of motion in the Lagrangian formalism. In particular express the equations of motion in terms of the Levi-Civita connection.

(ii.) Find the canonical momentum and the Hamiltonian of the theory.

(iii.) Noether's charge for general (space)time symmetries is

$$Q = \xi \mathcal{L} + X^\mu(x) \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$$

Specialise this to time translations and use the invariance of the action under time translations to calculate the energy E of the charged particle. Is E expressed in terms of the velocities dependent on A? Verify that E is conserved subject to the equations of motion.

Problem 6.3

The Lagrangian for the harmonic oscillator in two dimensions may be written:

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$$

(a.) Rewrite the Lagrangian in terms of the complex variable $z = x + iy$ and its complex conjugate as well as their time derivatives.

(b.) Show that the transformation

$$z \to z' = e^{i\omega} z$$

is a symmetry of the Lagrangian, where ω is a constant.

(c.) By writing

$$\delta z = i\omega z \qquad \text{and} \qquad \delta \bar{z} = -i\omega \bar{z}$$

deduce that the conserved charge associated to this symmetry is

$$Q = i\frac{m}{2}(z\dot{\bar{z}} - \bar{z}\dot{z})$$

(d.) Use the equations of motions to prove that $\frac{dQ}{dt} = 0$.