Relativity, Mechanics and Quantum Theory

Problem Sheet 5

Problem 5.1

The one-dimensional simple harmonic oscillator with coordinate q and mass m is subject to a force $F = -kq$. Write down the Lagrangian for this system and solve Lagrange's equation to prove the mass m oscillates with frequency $\omega = \sqrt{\frac{k}{m}}$.

Problem 5.2

Consider the Lagrangian for electromagnetism (the Einstein summation convention is being applied for repeated indices):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$$

Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$(\mu, \nu = 0, 1, 2, 3)$ and the components of the two-form field strength are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

where E_i and B_i (i=1,2,3) are the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ respectively; J^μ are the components of the current four vector, specifically

$$J_\mu = \begin{pmatrix} \rho \\ J_1 \\ J_2 \\ J_3 \end{pmatrix}$$

Show that the equation of motion for the field A_μ derived from $\mathcal{L}$ contains the following pair of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho \qquad \text{and} \qquad \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}$$

Where $\mathbf{J}$ is the spatial part of the current 4-vector. Show that $\partial_{[\mu}F_{\nu\lambda]} = 0$ and that this identity contains the remaining two Maxwell equations:

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0$$

Problem 5.3

Noether's theorem associates a conserved charge to each symmetry of an action, $A_R[q]$ where,

$$A_R[q] \equiv \int_R dt\, \mathcal{L}(q, \dot{q})$$

$\mathcal{L}(q, \dot{q})$ is a Lagrangian and $R \equiv [t_1.t_2]$ is a time-interval over which the action is evaluated. Under a space-time symmetry of the action $A_R[q]$ is transformed into $A_{R'}[q']$ such that

$\delta A_R[q] = 0$. Show that the charge Q is conserved when such a symmetry of the action exists, where:

$$Q = \xi \mathcal{L} + X_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Specifically the transformations considered are:

$$q_i \to q_i' = q_i + \epsilon X_i(q) \qquad \text{and} \qquad t \to t' = t + \epsilon \xi(t)$$

Where R' is the image of R under the temporal transformation.