Relativity, Mechanics and Quantum Theory

Problem Sheet 4

Problem 4.1

Consider the space of rank $(0,2)$-tensors $T_{\mu\nu}$ which is a representation of the Lorentz group - but not an irreducible one. Show that the subspaces of symmetric tensors and of anti-symmetric tensors form two invariant subspaces. Furthermore show that within the first invariant subspace there is an invariant subspace formed by the traceless ($\eta^{\mu\nu}T_{\mu\nu} = T^{\mu}{}_{\mu} = 0$ symmetric tensors. (Hint: Simply take a tensor having one of the proscribed properties and apply to it a Lorentz transformation. Using index notation then show that the resulting tensor has the same properties (e.g. symmetric, antisymmetric or symmetric and traceless) as the original tensor.)

Problem 4.2

Let

$$A = \exp\left(i\underline{\alpha} \cdot \underline{\sigma}\right)$$

Where $\underline{\alpha}$ is a vector in $\mathbb{R}^3$ and

$$\underline{\sigma} \equiv \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

and where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.
Show that A gives an element of $SU(2)$ for each $\underline{\alpha}$.
Compute the commutators of the Pauli matrices and give a general expression for $[\sigma_i, \sigma_j]$.

Problem 4.3

Let $X = x^\mu \sigma_\mu$ and show that the Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ induced by $X' = AXA^\dagger$ has:

$$\Lambda^\mu{}_\nu(A) = \frac{1}{2} Tr(A\sigma^\mu A^\dagger \sigma_\nu)$$

thus defining a map $A \to \Lambda(A)$ from $SL(2,\mathbb{C})$ into $SO(1,3)$. Where σ_0 is the two-by-two identity matrix and σ_i are the Pauli matrices as defined in question 4.2. (Method: show first that $Tr(X\sigma_\nu) = 2x_\nu$, then find the expression for the Lorentz transform of $x_\nu \to x'_\nu$ associated to $X \to X'$. Finally set x to be the 4-vector with all components equal to zero apart from the x_μ component which is equal to one.)
By considering a further transformation $X'' = BX'B^\dagger$ show that:

$$\Lambda(BA) = \Lambda(B)\Lambda(A)$$

so that the mapping is a group homomorphism. Identify the kernel of the homomorphism as the centre of $SU(2)$ i.e. $A = \pm\mathbb{I}$, thus showing that the map is two-to-one.

Problem 4.4

Let R_i be the generators of spatial rotations and B_i be generators of pure Lorentz boosts in the Lie algebra of $SO(1,3)$, where $i = 1, 2, 3$. Use the identity:

$$-iX = \left(\frac{d\Lambda}{d\theta}\right)_{\theta=0}$$

where $\Lambda(\theta) \in SO(1,3)$ and X is an unspecified infinitesimal generator to find R_i and B_i. Show the following commutator relations:

$$[R_i, R_j] = i\epsilon_{ijk}R_k, \quad [R_i, B_j] = i\epsilon_{ijk}B_k \quad \text{and} \quad [B_i, B_j] = -i\epsilon_{ijk}R_k$$

Use the definition

$$X^{\pm} = \frac{1}{2}(R_i \pm iB_i)$$

to show that the algebra may be rewritten as:

$$[X_i^+, X_j^+] = i\epsilon_{ijk}X_k^+, \quad [X_i^-, X_j^-] = i\epsilon_{ijk}X_k^- \quad \text{and} \quad [X_i^+, X_j^-] = 0$$

Hence giving two commuting copies of the Lie algebra of $SU(2)$.